Lecture 8

# Power rules

## What are powers?

In Lecture 7, we learned about

powers with **positive** exponents: $x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}}$

powers with **negative** exponents: $x^{-n} = \dfrac{1}{x^n}$

powers with exponent $0$: $x^0 = 1.$

In this lecture, we study the **properties** of powers (a.k.a. "power **rules**").

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## Multiplication of powers with the same base

$$\boxed{x^n \cdot x^m = x^{n+m}}$$

This formula is valid for **any** integers $n, m$. To **prove** the formula, we consider 4 cases.

**Case 1.** If $n, m$ are both **positive**, then

$$x^n \cdot x^m = (\underbrace{x \cdot x \cdots x}_{n \text{ times}}) \cdot (\underbrace{x \cdot x \cdots x}_{m \text{ times}}) = \underbrace{x \cdot x \cdots x}_{(n+m) \text{ times}} = x^{n+m}.$$

**Case 2.** If $n, m$ are both **negative**, then $-n, -m$ are positive and

$$x^n \cdot x^m = \frac{1}{x^{-n}} \cdot \frac{1}{x^{-m}} = \frac{1}{x^{-n}x^{-m}} = \frac{1}{x^{-n-m}} = x^{-(-n-m)} = x^{n+m}.$$

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## Multiplication of powers with the same base

**Case 3:** one of the integers is **positive** and the other one is **negative**.

Say, if $n = 5$ and $m = -3$, then

$$x^5 \cdot x^{-3} = \frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x^2 = x^{5+(-3)}\,.$$

If $n = -5$ and $m = 3$, then

$$x^{-5} \cdot x^3 = \frac{x^3}{x^5} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \frac{1}{x^2} = x^{-2} = x^{-5+3}\,.$$

For **any other** values of $n$ and $m$, having opposite signs, the reasoning is the **same** as above.

**Case 4:** if one of the integers (say, $m$) is **zero**. Then

$$x^n \cdot x^m = x^n \cdot \underbrace{x^0}_{1} = x^n \cdot 1 = x^n = x^{n+0}\,.$$

We see that in **all** cases, $x^n \cdot x^m = x^{n+m}$.

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## Examples

**Example 1.** $2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 128$

$(-1)^9 \cdot (-1)^7 = (-1)^{9+7} = (-1)^{16} = 1$

$3^5 \cdot 3^{-8} = 3^{5-8} = 3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

$\left(\frac{2}{3}\right)^{-5} \cdot \left(\frac{2}{3}\right)^{7} = \left(\frac{2}{3}\right)^{-5+7} = \left(\frac{2}{3}\right)^{2} = \left(\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right) = \frac{4}{9}$

$10^{12} \cdot 10^{-12} = 10^{12-12} = 10^0 = 1$

**Example 2.** Simplify the expression $x^3 \cdot x^{-8} \cdot x^{-4}$.

**Solution.**

$$x^3 \cdot x^{-8} \cdot x^{-4} = x^{3-8-4} = x^{-9} = \frac{1}{x^9}\,.$$

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## Division of powers with the same base

$$\frac{x^n}{x^m} = x^{n-m}$$

This formula is valid for **any** integers $n, m$.

Indeed, $\dfrac{x^n}{x^m} = x^n \cdot \dfrac{1}{x^m} = x^n \cdot x^{-m} = x^{n-m}$.

**Example 1.** Find the value of the expression $\dfrac{5^4}{5^6}$.

**Solution.** $\dfrac{5^4}{5^6} = 5^{4-6} = 5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$.

**Example 2.** Simplify the expression $\dfrac{x^4\, y^{-3}}{x^{-2}\, y^2}$.

**Solution.**

$$\frac{x^4\, y^{-3}}{x^{-2}\, y^2} = \frac{x^4}{x^{-2}} \cdot \frac{y^{-3}}{y^2} = x^{4-(-2)} \cdot y^{-3-2} = x^6\, y^{-5} = \frac{x^6}{y^5}.$$



## A power of a power

$$(x^n)^m = x^{nm}$$

This formula is valid for **any** integers $n, m$.
It is proven by cases depending on the signs of the integers.

If $n, m$ are both **positive**, then

$$(x^n)^m = \underbrace{(x^n) \cdot (x^n) \cdot \dots \cdot (x^n)}_{m} =$$

$$\underbrace{\underbrace{(x \cdots x)}_{n} \cdot \underbrace{(x \cdots x)}_{n} \cdot \dots \cdot \underbrace{(x \cdots x)}_{n}}_{m} =$$

$$\underbrace{x \cdot x \cdot \dots \cdot x}_{nm} = x^{nm}.$$

All **other** cases can be reduced to this case using $x^{-n} = \dfrac{1}{x^n}$.

## Examples

**Example 1.** $(2^3)^4 = 2^{3\cdot 4} = 2^{12} = 4096.$

**Example 2.** $(2^{-3})^4 = 2^{(-3)\cdot 4} = 2^{-12} = \dfrac{1}{2^{12}} = \dfrac{1}{4096}.$

**Example 3.** $\left((-2)^{-3}\right)^{-4} = (-2)^{(-3)\cdot(-4)} = (-2)^{12} = 2^{12} = 4096.$

**Example 4.** $((-1)^{-1})^{-1} = (-1)^{(-1)\cdot(-1)} = (-1)^1 = -1.$

**Example 5.** Simplify the expression $(x^3)^2 \cdot x^{-4}$.

**Solution.** $(x^3)^2 \cdot x^{-4} = x^{3\cdot 2} \cdot x^{-4} = x^6 \cdot x^{-4} = x^{6-4} = x^2.$



## Multiplication of powers with the same exponent

$$x^n \cdot y^n = (xy)^n$$

This formula is valid for **any** integer $n$.

Indeed, if $n$ is **positive**, then

$$x^n \cdot y^n = (\underbrace{x \cdot x \cdot \dots \cdot x}_{n}) \cdot (\underbrace{y \cdot y \cdot \dots \cdot y}_{n}) = \underbrace{(xy) \cdot (xy) \cdot \dots \cdot (xy)}_{n} = (xy)^n.$$

If $n$ is **negative**, then $-n$ is positive and

$$x^n \cdot y^n = \frac{1}{x^{-n}} \cdot \frac{1}{y^{-n}} = \frac{1}{x^{-n}y^{-n}} = \frac{1}{(xy)^{-n}} = (xy)^n.$$

## Examples

**Example 1.** Simplify the expression $(-x)^9$.

**Solution.** $(-x)^9 = ((-1)\cdot x)^9 = (-1)^9 \cdot x^9 = (-1)\cdot x^9 = -x^9.$

**Example 2.** Simplify the expression $(10^{-5}x^2)^{-3}$.

**Solution.**

$$(10^{-5}x^2)^{-3} = (10^{-5})^{-3}\cdot (x^2)^{-3} = 10^{(-5)\cdot(-3)}\cdot x^{2\cdot(-3)} = 10^{15}x^{-6}.$$

**Example 3.** Simplify the expression $(5x)^2(-3x)^3$.

**Solution.**

$$(5x)^2(-3x)^3 = 5^2x^2\cdot(-3)^3x^3 = \underbrace{5^2\cdot(-3)^3}_{\text{numbers}}\cdot\underbrace{x^2\cdot x^3}_{\text{variables}} = 25\cdot(-27)x^{2+3} = -675x^5.$$

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## Division of powers with the same exponent

$$\frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n$$

This formula is valid for **any** integer $n$.

Indeed, if $n$ is **positive**, then

$$\frac{x^n}{y^n} = \frac{\overbrace{x\cdot x\cdot x\cdots x}^{n}}{\underbrace{y\cdot y\cdot y\cdots y}_{n}} = \underbrace{\frac{x}{y}\cdot\frac{x}{y}\cdots\frac{x}{y}}_{n} = \left(\frac{x}{y}\right)^n.$$

If $n$ is **negative**, then $-n$ is positive and

$$\frac{x^n}{y^n} = \frac{y^{-n}}{x^{-n}} = \left(\frac{y}{x}\right)^{-n} = \left(\frac{x}{y}\right)^n.$$

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## Examples

**Example 1.** $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$.

**Example 2.** $\left(\frac{2}{3}\right)^{-1} = \frac{2^{-1}}{3^{-1}} = \frac{3^1}{2^1} = \frac{3}{2}$.

In general, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

In particular, $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.



## Summary

In this lecture, we have learned

- ☑ how to **multiply** powers with the **same base**: $x^n \cdot x^m = x^{n+m}$
- ☑ how to **divide** powers with the **same base**: $\frac{x^n}{x^m} = x^{n-m}$
- ☑ how to calculate a **power** of a power: $(x^n)^m = x^{nm}$
- ☑ how to **multiply** powers with the **same exponent**: $x^n \cdot y^n = (xy)^n$
- ☑ how to **divide** powers with the **same exponent**: $\frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n$