Lecture 7

Powers

Multiplying repeated factors

Abbreviations are common and useful.
For example, instead of $2+2+2+2+2$ we can write $2 \cdot 5$:

$$2 \cdot 5 = 2+2+2+2+2 .$$

The sum of several equal numbers can be abbreviated to a product.

The product of several equal numbers can be **abbreviated** similarly:
instead of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, we can write 2^5:

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 .$$

In general,

$$x^n = \underbrace{x \cdot x \cdot x \cdots x \cdot x}_{n \text{ times}} .$$

Here x is any number and n is a positive integer.

Examples. $x^4 = x \cdot x \cdot x \cdot x$

$$10^2 = 10 \cdot 10 = 100$$

$$2^{10} = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{10 \text{ times}} = 1024.$$

Exponential notation

$$x^n$$

base → x; exponent → n

We read x^n as "x **to the** nth."

$n=2$ and $n=3$ are special:

x^2 is read as "x **squared**",

x^3 as "x **cubed**".

Do you see why $x^1 = x$ for any x?

and why $1^n = 1$ for any positive integer n?

When the base is negative

Example 1. $(-1)^2 = (-1)\cdot(-1) = 1$

$(-1)^3 = (-1)\cdot(-1)\cdot(-1) = -1\,.$

In general, if n is **even** ($n = 0, 2, 4, 6, \dots$) then $(-1)^n = 1$ and
if n is **odd** ($n = 1, 3, 5, 7, \dots$) then $(-1)^n = -1\,.$

Example 2. $(-2)^3 = (-2)\cdot(-2)\cdot(-2) = -8$
$(-2)^4 = (-2)\cdot(-2)\cdot(-2)\cdot(-2) = 16$

In general,

$$(-x)^n = \underbrace{(-x)\cdot(-x)\dots(-x)}_{n\text{ times}} = \underbrace{(-1)x\cdot(-1)x\dots(-1)x}_{n\text{ times}}$$
$$= \underbrace{(-1)\cdot(-1)\dots(-1)}_{n\text{ times}}\cdot\underbrace{x\cdot x\dots x}_{n\text{ times}} = (-1)^n x^n.$$

Recall: if n is **even**, then $(-1)^n = 1$, and if n is **odd**, then $(-1)^n = -1$.
Therefore, $(-x)^n = x^n$ if n is **even**, and $(-x)^n = -x^n$ if n is **odd**.

Warning: $(-x)^n \neq -x^n$ when n is even.

Zero and negative exponents

What is 2^0, or 2^{-1}, 2^{-2}, $2^{-3}, \dots$?

To answer this question, let us have a look at the process of consecutive multiplication by 2:

$$\dots \xrightarrow{\times 2} \frac{1}{8} \xrightarrow{\times 2} \frac{1}{4} \xrightarrow{\times 2} \frac{1}{2} \xrightarrow{\times 2} 1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} \dots$$

We understand this as an infinite sequence of **powers** of 2:

$$\dots \xrightarrow{\times 2} \underbrace{2^{-3}}_{\frac{1}{8}} \xrightarrow{\times 2} \underbrace{2^{-2}}_{\frac{1}{4}} \xrightarrow{\times 2} \underbrace{2^{-1}}_{\frac{1}{2}} \xrightarrow{\times 2} \underbrace{2^{0}}_{1} \xrightarrow{\times 2} \underbrace{2^{1}}_{2} \xrightarrow{\times 2} \underbrace{2^{2}}_{4} \xrightarrow{\times 2} \underbrace{2^{3}}_{8} \xrightarrow{\times 2} \underbrace{2^{4}}_{16} \xrightarrow{\times 2} \dots$$

We see that $2^0 = 1$, $2^{-1} = \frac{1}{2} = \frac{1}{2^1}$, $2^{-2} = \frac{1}{4} = \frac{1}{2^2}$, $2^{-3} = \frac{1}{8} = \frac{1}{2^3}$, and so on.

Zero and negative exponents

We define $x^0 = 1$ for any non-zero x and

$$x^{-n} = \frac{1}{x^n}$$ for any non-zero x and any **positive integer** n.

Examples. $7^0 = 1$, $\left(\frac{2}{3}\right)^0 = 1$, $(-5)^0 = 1$, $(-1)^0 = 1$

$$3^{-1} = \frac{1}{3^1} = \frac{1}{3}, \quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}, \quad x^{-2} = \frac{1}{x^2}.$$

Observe that the formula $x^{-n} = \frac{1}{x^n}$ means that x^n and x^{-n} are **reciprocals**. Therefore, $x^n = \frac{1}{x^{-n}}$. A power can be **moved** from numerator to denominator (or the other way around) with the **opposite** exponent.

Example. $\frac{3^{-1}}{2^{-4}} = \frac{2^4}{3^1} = \frac{16}{3}$.

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Drill

Here are the exponential **rules** we have learned so far:

$$x^n = \underbrace{x \cdot x \cdot x \cdot \cdots \cdot x \cdot x}_{n \text{ times}}$$

$$x^0 = 1$$

$$x^{-n} = \frac{1}{x^n}, \quad \frac{1}{x^{-n}} = x^n$$

Let us master these rules.

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

$$(-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$$

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}, \quad \frac{1}{2^{-3}} = 2^3 = 8$$

$$(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$$

$$2^0 = 1, \quad (-2)^0 = 1$$

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Summary

In this lecture, we have learned about

- ☑ powers with **positive** exponents: $x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}}$
- ☑ powers with **negative** exponents: $x^{-n} = \frac{1}{x^n}$
- ☑ reciprocals of powers with negative exponent: $\frac{1}{x^{-n}} = x^n$
- ☑ powers with exponent 0: $x^0 = 1$

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