Lecture 6

Distributivity

Properties of operations

There are five important **properties** of the basic arithmetic operations of addition and multiplication.

These properties are

- **commutativity** of addition and multiplication
- **associativity** of addition and multiplication
- **distributivity** of multiplication over addition.

Commutativity and associativity (which we studied in Lecture 4)
refer either to addition or multiplication.

Distributivity connects addition and multiplication.

Distributivity of multiplication over addition

Multiplication **distributes** over addition :

$$a(b+c) = ab + ac \quad \text{for any } a, b \text{ and } c$$

Example 1. If $a = 2,\ b = 3,\ c = 4,$ then the distributive property reads

$$2(3+4) = 2 \cdot 3 + 2 \cdot 4.$$

Distributivity means that
we may calculate the value of the expression $2(3+4)$ in **two** different ways:

$$2(3+4) = 2 \cdot 7 = 14 \quad \text{or} \quad 2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14.$$

Which way is better (easier, faster)? The first one!

Example 2. Calculate the value of the expression $25(4+10)$.
Direct calculation gives

$$25(4+10) = 25 \cdot 14 = ? \text{ (need calculator?)}$$

If we use distributivity instead, then

$$25(4+10) = 25 \cdot 4 + 25 \cdot 10 = 100 + 250 = 350.$$

Distributivity gives us a choice. Use it!

Distributivity with variables

Example 3. If in the distributive formula $a(b+c) = ab + ac$ we put $a = 2,\ b = x,\ c = 3y,$ then we get

$$2(x + 3y) = 2x + 2 \cdot 3y = 2x + 6y.$$

Example 4. Eliminate the parentheses in the expression $x(1 - 2y)$.

In order to eliminate the parentheses, we need to distribute x over $1 - 2y$:

$$x(1 - 2y) = x(1 + (-2y)) =$$
$$x \cdot 1 + x(-2y) = x + x(-2)y = x + (-2)xy = x - 2xy.$$

This problem may be solved faster! Because multiplication distributes over subtraction, too.

Distributivity over subtraction

Distributivity is valid for **subtraction** also:

$$a(b - c) = ab - ac \quad \text{for any } a,\ b \text{ and } c$$

Indeed, since $b - c = b + (-c)$, we have

$$a(b - c) = a(b + (-c)) = ab + a(-c) = ab - ac.$$

Example (the same as before).

Get rid of the parentheses in the expression $x(1 - 2y)$.

$$x(1 - 2y) = x \cdot 1 - x(2y) = x - 2xy.$$

Example. Clear parentheses in the expression $x(-1 - 2y)$.

$$x(-1 - 2y) = x \cdot (-1) - x(2y) = -x - 2xy.$$

Another look at distributivity

$$(a+b)c = ac + bc \quad \text{for any } a\,,\ b \text{ and } c$$

Indeed, by **commutativity** of multiplication,

$$(a+b)c = c(a+b)\,.$$

By **distributivity**,

$$c(a+b) = ca + cb\,.$$

By commutativity,

$$ca + cb = ac + bc\,.$$

Overall,

$$(a+b)c = ac + bc\,.$$

Example. Clear parentheses in the expression $(2x+3y)z$.

Solution. $(2x+3y)z = 2xz + 3yz$.

Similarly, $(a-b)c = ac - bc$ for any a , b and c .

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Distribution of a negative quantity

Be careful in application of the formula $a(b+c) = ab + ac$, when a is **negative**!

Example 1. Clear parentheses in the expression $-2(x+y)$.

Solution.

$$-2(x+y) = (-2)x + (-2)y = -2x - 2y\,.$$

Example 2. Clear parentheses in the expression $-2(x-y)$.

Solution.

$$-2(x-y) = -2(x+(-y)) = (-2)x + (-2)(-y) = -2x + 2y\,.$$

Example 3. Clear parentheses in the expression $-2(-x-y)$.

Solution.

$$-2(-x-y) = -2((-x)+(-y)) = (-2)(-x) + (-2)(-y) = 2x + 2y\,.$$

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Negative sign in front of parentheses

What is the meaning of the expression $-x$? It represents a quantity **opposite** to x.
For example, if $x = 3$, then $-x = -3$.
If $x = -3$, then $-x = -\underbrace{(-3)}_{x} = 3$.

You can always check if one number is the opposite of another: their sum must be zero.

In some cases, it may be convenient to represent $-x$ as $(-1)x$.

Example 1. Clear parentheses in the expression $-(x+y)$.
Solution.

$$-(x+y) = (-1)(x+y) = (-1)x + (-1)y = -x - y\,.$$

Example 2. Clear parentheses in the expression $-(x-y)$.
Solution.

$$-(x-y) = (-1)(x-y) = (-1)x - (-1)y = -x + y\,.$$

Expansion

Problem. Clear parentheses in the expression $(a+b)(c+d)$.

Solution. How to distribute $a+b$ over $c+d$?

We may think of $c+d$ as a **single** entity.

For this, denote $c+d$ by x. Then

$$(a+b)\underbrace{(c+d)}_{x} = (a+b)x = ax + bx = a(c+d) + b(c+d)$$
$$= ac + ad + bc + bd\,.$$

Expansion

Our result in distribution of $a+b$ over $c+d$ is

$$(a+b)(c+d) = ac + ad + bc + bd\,.$$

It is convenient to understand this formula in the following way:

First, we distribute a over $(c+d)$, the result is $ac+ad$.
Then, we distribute b over $(c+d)$, the result is $bc+bd$.
Overall, $(a+b)(c+d) = ac+ad+bc+bd$.

Observe that the right hand side contains **no** parentheses.

This procedure is called **expansion** or **clearing the parentheses**.

Similar arguments are valid when the parentheses contain **any** number of terms.

For example,

$$(a+b)(x+y+z) = ax + ay + az + bx + by + bz\,.$$

Examples of expansion

Example 1. Expand the expression $(2+x)(3+y)$.

Solution. Expand means clear parentheses using distribution.

$$(2+x)(3+y) = 2\cdot 3 + 2y + x\cdot 3 + xy = 6 + 2y + 3x + xy\,.$$

Example 2. Clear parentheses in the expression $(1-2x)(-3+y)$.

Solution. In this example, we have to be careful about the **negative** signs in the expression.

For this reason, we rewrite the expression as follows

$$(1-2x)(-3+y) = (1+(-2x))((-3)+y)\,.$$

Now we distribute:

$$\begin{aligned}(1+(-2x))((-3)+y) &= 1\cdot(-3) + 1\cdot y + (-2x)\cdot(-3) + (-2x)y \\ &= -3 + y + 6x - 2xy\,.\end{aligned}$$

Factoring

Rewriting the distributivity formula $a(b+c) = ab + ac$ backwards, we get

$$ab + ac = a(b+c)\,.$$

This formula is called **factoring**.

When we add two terms, ab and ac, containing a **common factor** of a,

we may **factor out** a from the parentheses.

Example. Factor the expression $6x + 9xy$.

Solution. Both terms, $6x$ and $9xy$, have a **common factor** of $3x$:

$$6x = 3x \cdot 2, \quad 9xy = 3x \cdot 3y.$$

Factoring out $3x$, we get

$$6x + 9xy = 3x \cdot 2 + 3x \cdot 3y = 3x(2 + 3y).$$

Combining similar terms

Distributivity and factoring helps us to **combine similar terms**:

$$2x + 3x = (2+3)x = 5x.$$

Example. Simplify the expression $2x + 3y + x + 4y$.

Solution. First, we use **commutativity and associativity of addition**:

$$2x + 3y + x + 4y = \underbrace{(2x + x)}_{x-\text{terms}} + \underbrace{(3y + 4y)}_{y-\text{terms}}\,.$$

Then we combine **similar terms**:

$$(2x + x) + (3y + 4y) = 3x + 7y\,.$$

Summary

In this lecture, we have learned

- ☑ what distributivity is: $a(b+c) = ab + ac$
  $a(b-c) = ab - ac$
- ☑ how to clear parentheses (expand expressions)
- ☑ what factoring is
- ☑ how to factor expressions using distributivity
- ☑ how to combine similar terms