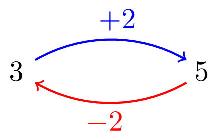
Lecture 5

Subtraction and Division

Subtraction is the opposite of addition

Subtraction is the operation which is **opposite** to addition:



This means that $(3+2)-2=3$ and $(5-2)+2=5$.

Recall that numbers a and $-a$ are called **opposite** to each other.
For example, -2 is opposite to 2, and 2 is opposite to -2.

Subtraction of a number is addition of its opposite:

$$5-2=5+(-2)=3 \quad \text{and} \quad 5-(-2)=5+2=7\,.$$

Therefore, we can **express** any subtraction as addition of the opposite quantity:

$$a-b=a+(-b) \quad \text{for any } a,\ b\,.$$

No commutativity for subtraction

We know that addition is **commutative**: $a+b=b+a$ for any $a,\ b$.

Subtraction is **not** commutative: it is **not** true that $a-b=b-a$ unless $a=b$.

Indeed, take $a=1$ and $b=2$. Then $a-b=1-2=-1$,
but $b-a=2-1=1$.

In general, $a-b$ and $b-a$ are opposite to each other: $b-a=-(a-b)$.

So subtraction is **not** commutative.
But expressing subtraction $a-b$ in terms of addition $a+(-b)$, we may apply the commutativity of addition to get:

$$a-b=a+(-b)=-b+a \quad \text{for any } a,\ b\,.$$

No associativity for subtraction

We know that addition is **associative**:

$$(a+b)+c = a+(b+c) \quad \text{for any } a,\ b,\ c\,.$$

Subtraction is **not** associative:

$$(a-b)-c \neq a-(b-c)\,.$$

For example, if $a=3$, $b=1$ and $c=1$, then

$$(a-b)-c = (3-1)-1 = 2-1 = 1\,,$$
$$\text{but } a-(b-1) = 3-(1-1) = 3-0 = 3\,.$$

So subtraction is **not** associative.

But expressing subtraction $(a-b)-c$ in terms of addition $(a+(-b))+(-c)$, we may apply the associativity of addition to get:

$$(a-b)-c = (a+(-b))+(-c) = a+((-b)+(-c)) = a+(-b-c)\,.$$

Recall that $a-b-c$ has to be understood as $(a-b)-c$.

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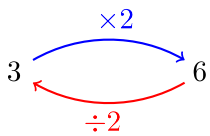
Division is the opposite of multiplication

Division is the operation which is **opposite** to multiplication:



This means that $(3\cdot 2)\div 2 = 3$ and $(6\div 2)\cdot 2 = 6.$

Recall that numbers a and $1/a$ are called **reciprocals**.

For example, 2 and $1/2$ are reciprocals.

Division by a **non-zero** number is multiplication by its **reciprocal**:

$$6\div 2 = 6\cdot\frac{1}{2} = 3 \quad \text{and} \quad 6\div\frac{1}{2} = 6\cdot 2 = 12\,.$$

(Keep in mind that the reciprocal of $\frac{1}{2}$ is 2.)

In general: $a\div b = a\cdot\frac{1}{b}$ for any a and non-zero b.

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Negative one

The reciprocal of -1 is -1, that is $\dfrac{1}{-1} = -1$. Indeed, $(-1)(-1) = 1$.

Sometimes negative one is slightly hidden: $-a = (-1)a$.

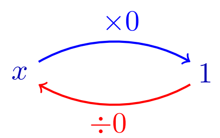
It is helpful to keep this in mind.

For example, $\dfrac{-a}{-b} = \dfrac{a}{b}$, because $\dfrac{-a}{-b} = \dfrac{(-1)a}{(-1)b} = \dfrac{a}{b}$.

Another example: $\dfrac{a}{-b} = \dfrac{a}{(-1)b} = \dfrac{1}{-1}\dfrac{a}{b} = (-1)\dfrac{a}{b} = -\dfrac{a}{b} = \dfrac{-a}{b}$.

Why division by zero does not make sense

Let us try to divide some number, say 1, by 0.
We do not know what result will be. Let us call it x: $1 \div 0 = x$.



If $1 \div 0 = x$, then x is a number such that $x \cdot 0 = 1$.
Which is **impossible** since $x \cdot 0 = 0$ for any x.

Never divide by zero! It doesn't make sense.

No commutativity for division

We know that multiplication is **commutative**: $ab = ba$ for any a, b.

Division is **not** commutative:

in general, it is **not** true that $a \div b = b \div a$.

For example, if $a = 2$ and $b = 1$, then $a \div b = 2 \div 1 = 2$,

but $b \div a = 1 \div 2 = \frac{1}{2}$.

The expressions $a \div b$ and $b \div a$ are **reciprocal** to each other.

Indeed, $a \div b = a \cdot \frac{1}{b}$ and $b \div a = b \cdot \frac{1}{a}$. Therefore

$$(a \div b)(b \div a) = \left(a \cdot \frac{1}{b}\right) \cdot \left(b \cdot \frac{1}{a}\right) = a\left(\frac{1}{b} \cdot b\right)\frac{1}{a} = a \cdot 1 \cdot \frac{1}{a} = a \cdot \frac{1}{a} = 1$$

In fractional notation, this may be written as $\frac{b}{a} = \frac{1}{a/b}$.

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No associativity for division

We know that multiplication is **associative**:

$$(ab)c = a(bc) \quad \text{for any } a, b, c.$$

Division is **not** associative: $(a \div b) \div c \neq a \div (b \div c)$.

Or, in fractional notation, $\frac{a/b}{c} \neq \frac{a}{b/c}$.

For example, if $a = 8$, $b = 4$ and $c = 2$, then

$(a \div b) \div c = (8 \div 4) \div 2 = 2 \div 2 = 1$,

but $a \div (b \div c) = 8 \div (4 \div 2) = 8 \div 2 = 4$.

So division is **not** associative.

But expressing division $(a \div b) \div c$ in terms of multiplication $\left(a \cdot \frac{1}{b}\right) \cdot \frac{1}{c}$,

we may apply the associativity of multiplication to get:

$$(a \div b) \div c = \left(a \cdot \frac{1}{b}\right) \cdot \frac{1}{c} = a \cdot \left(\frac{1}{b} \cdot \frac{1}{c}\right) = a \cdot \frac{1}{b \cdot c} = a \div (b \cdot c).$$

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Summary

In this lecture, we have learned that

- ☑ subtraction is the **opposite** of addition
- ☑ subtraction can be **expressed** as addition of the opposite: $a - b = a + (-b)$
- ☑ subtraction is **neither** commutative **nor** associative
- ☑ division is the **opposite** of multiplication
- ☑ division can be **expressed** as multiplication by the reciprocal: $a \div b = a \cdot \frac{1}{b}$
- ☑ division by zero **does not make sense**
- ☑ division is **neither** commutative **nor** associative

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