Real and Complex Cubic Curves

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April 29, 2016

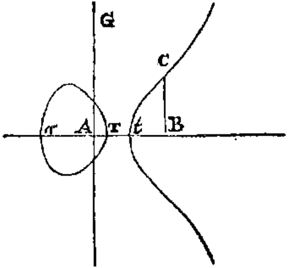
Dynamics Seminar

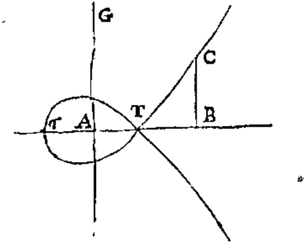
Work with Araceli Bonifant:
arXiv:math/1603.09018

It is possible to write endlessly on elliptic curves.
(This is not a threat.)

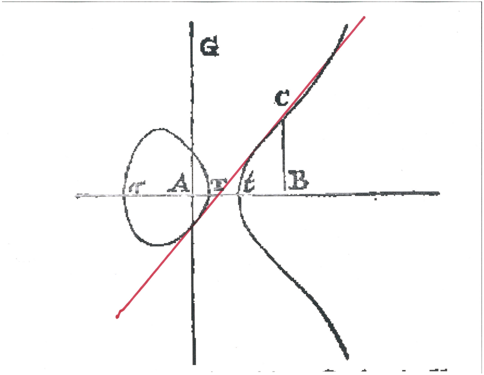
Serge Lang

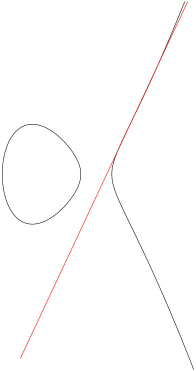
1704: Enumeratio Linearum Terti Ordinis





Newton corrected





A Rich Literature

During the next 140 years cubic curves (and elliptic integrals) were studied by many mathematicians:

Colin Maclaurin,

Jean le Rond d'Alembert,

Leonhard Euler,

Adrien-Marie Legendre,

Niels Henrik Abel,

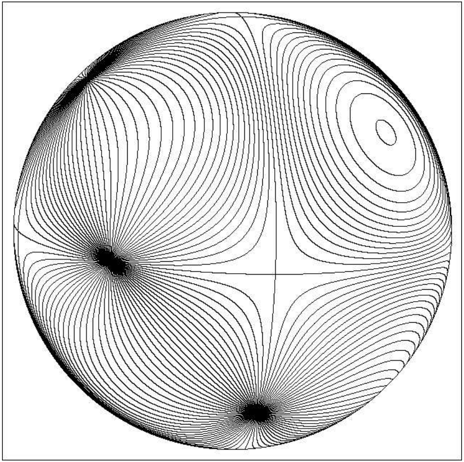
Carl Gustav Jacobi

1844: Otto Hesse

$$x^3 + y^3 + z^3 = 3\,k\,xyz\,.$$

$$(x, y, z) \mapsto k = \frac{x^3 + y^3 + z^3}{3\,xyz}\,.$$

The Hesse Pencil in $\mathbb{P}^2(\mathbb{R})$



(Singular) foliation by curves $\frac{x^3+y^3+z^3}{3\,xyz} = \text{constant} \in \mathbb{R} \cup \{\infty\}$.

The Hessian determinant of $\Phi(x,y,z)$.

$$\mathcal{H}_\Phi(x,y,z) \;=\; \det\begin{pmatrix}\Phi_{xx} & \Phi_{xy} & \Phi_{xz}\\ \Phi_{yx} & \Phi_{yy} & \Phi_{yz}\\ \Phi_{zx} & \Phi_{zy} & \Phi_{zz}\end{pmatrix}$$

Theorem. *If $\mathcal{C} \subset \mathbb{P}^2$ is a smooth curve with defining equation $\Phi(x,y,z) = 0$, then $(x:y:z) \in \mathcal{C}$ is a* ***flex point*** *if and only if $\mathcal{H}_\Phi(x,y,z) = 0$.*

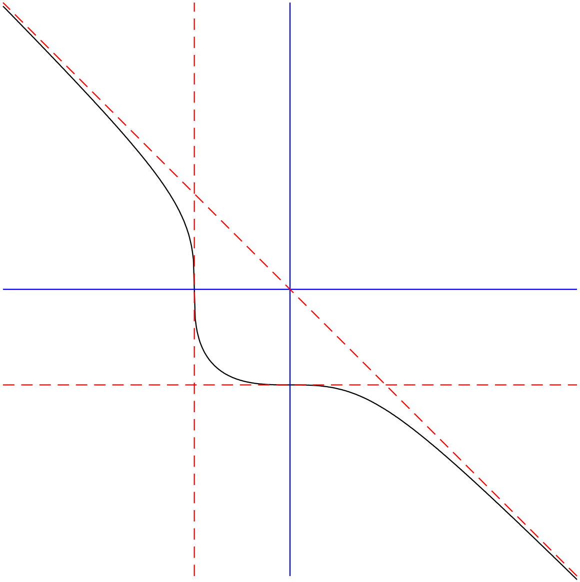
It follows (with some work) that:

Every smooth complex cubic curve has exactly nine flex points.

Example:

$$\Phi(x,y,z) = x^3 + y^3 + z^3 \;, \qquad \mathcal{H}_\Phi(x,y,z) = 6^3\, x\, y\, z \;.$$

The Fermat Curve in the real affine plane $\{(x : y : 1)\}$:



Projective equivalence

Every nonsingular linear transformation

$$(x, y, z) \mapsto (X, Y, Z)$$

of $\mathbb{C}^3$ induces a ***projective automorphism***

$$(x : y : z) \mapsto (X : Y : Z)$$

of the projective plane $\mathbb{P}^2(\mathbb{C})$.

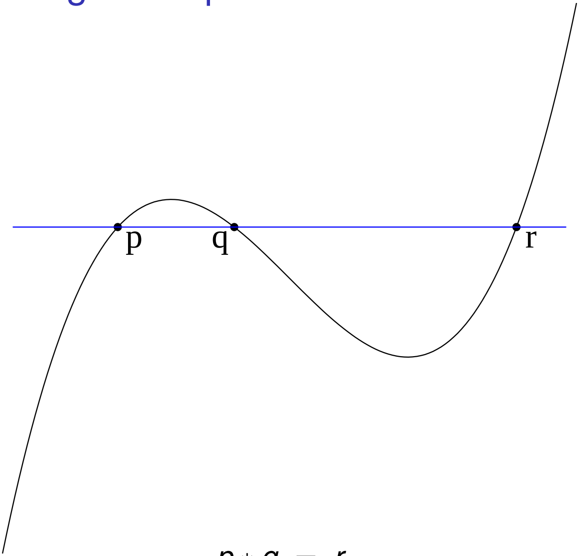
Two algebraic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{P}^2$ are called ***projectively equivalent*** *if there is a projective automorphism of $\mathbb{P}^2$ which maps one onto the other.*

Theorem. Every smooth cubic curve $\mathcal{C} \subset \mathbb{P}^2(\mathbb{C})$ is projectively equivalent to one in the Hesse normal form

$$x^3 + y^3 + z^3 = 3\,k\,xyz\,.$$

(But k is not unique!)

The chord-tangent map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.



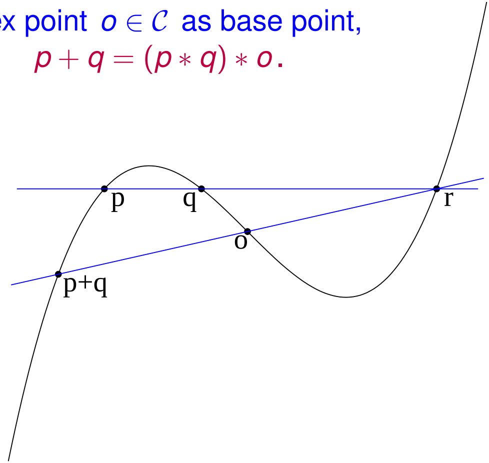
$$p * q = r .$$

For a flex point: $p * p = p$.

The additive group structure $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Choose a flex point $o \in \mathcal{C}$ as base point,

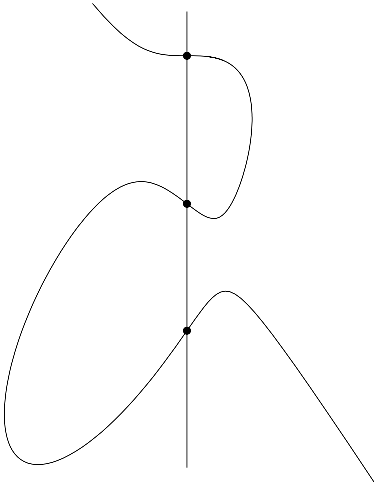
and set $p + q = (p * q) * o$.



Lemma. $p * q = r \iff p + q + r = o$.

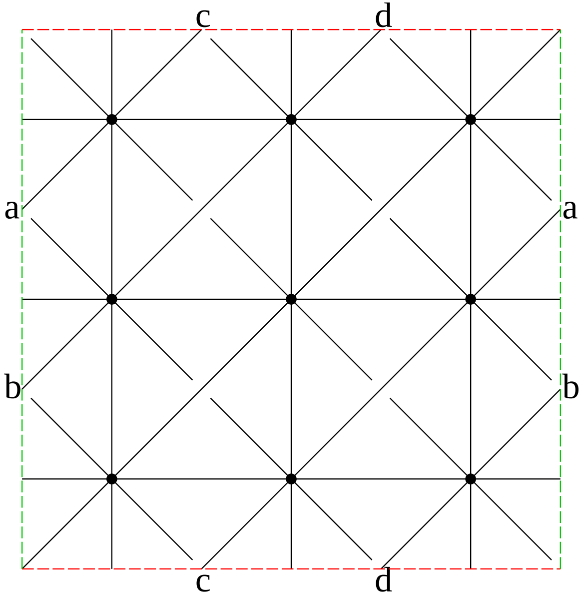
Note also that $p * o = -p$.

Theorem.



The line between two flex points always intersects $\mathcal{C}$ in a third flex point.

Nine flex points and 12 lines between them.



Other Fields.

Let F be any field with

$$\mathbb{Q} \subset F \subset \mathbb{C}\,.$$

If the curve $\mathcal{C}$ has defining equation $\Phi(x,y,z) = 0$ with coefficients in F, let $\mathcal{C}(F) = \mathcal{C} \cap \mathbb{P}^2(F)$ be the set of all points $(x:y:z) \in \mathcal{C}$ with coordinates $x,\ y,\ z \in F$.
Then $\quad p,\, q \in \mathcal{C}(F) \quad \Longrightarrow \quad p * q \in \mathcal{C}(F)$.

If $o \in \mathcal{C}(F)$, it follows that $\mathcal{C}(F)$ is a subgroup of $\mathcal{C}$.

In particular, for any $n \in \mathbb{Z}$ we can define multiplication by n as a map $\quad \mathbf{m}_n : \mathcal{C}(F) \to \mathcal{C}(F)$.

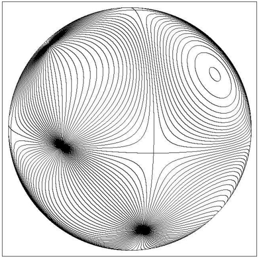
The construction is inductive:

$$\mathbf{m}_0(p) = o\,, \qquad \text{and} \qquad \mathbf{m}_{n+1}(p) = \mathbf{m}_n(p) + p\,.$$

These maps form a semigroup, with composition:

$$\mathbf{m}_n \circ \mathbf{m}_k = \mathbf{m}_{nk}\ , \quad \text{and with} \quad \mathbf{m}_n(p) + \mathbf{m}_k(p) = \mathbf{m}_{n+k}(p)\ .$$

Extending $\mathbf{m}_n$ to a map from $\mathbb{P}^2$ to itself.



Consider the "foliation" of $\mathbb{P}^2$ by the curves

$$\mathcal{C}_k = \{(x : y : z) \; ; \; x^3 + y^3 + z^3 \, = \, 3\, x\, y\, z\}$$

in the Hesse pencil.

Theorem. The various maps $\mathbf{m}_n : \mathcal{C}_k \to \mathcal{C}_k$ fit together to yield a rational map

$$\mathbf{m}_n : \mathbb{P}^2 - -\!\!-\!\!> \mathbb{P}^2 \; .$$

Examples (taking $(0 : -1 : 1)$ as base point).

$$\begin{aligned}
\mathbf{m}_{-2}(x : y : z) &= \Big(x(y^3 - z^3) : y(z^3 - x^3) : z(y^3 - x^3)\Big) \\
&\quad (\textit{Desboves}, 1886) \\
\mathbf{m}_{-1}(x : y : z) &= (x : z : y) \\
\mathbf{m}_{0}(x : y : z) &= (0 : -1 : 1) \\
\mathbf{m}_{1}(x : y : z) &= (x : y : z) \\
\mathbf{m}_{2}(x : y : z) &= \Big(x(y^3 - z^3) : z(y^3 - x^3) : y(z^3 - x^3)\Big)
\end{aligned}$$

$$\mathbf{m}_3(x : y : z) = \Big(xyz(x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3) :$$

$$(x^2y + y^2z + z^2x)(x^4y^2 + y^4z^2 + z^4x^2 - xy^2z^3 - yz^2x^3 - zx^2y^3) :$$

$$x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3\Big) .$$

Weierstrass Normal Form and the J-invariant

Theorem (Nagel 1928). Every smooth cubic curve is projectively equivalent to one in the normal form

$$y^2 = (x - r_1)(x - r_2)(x - r_3) \qquad \text{with} \qquad r_1 + r_2 + r_3 = 0$$

$$= x^3 + s_2 x - s_3 \ ,$$

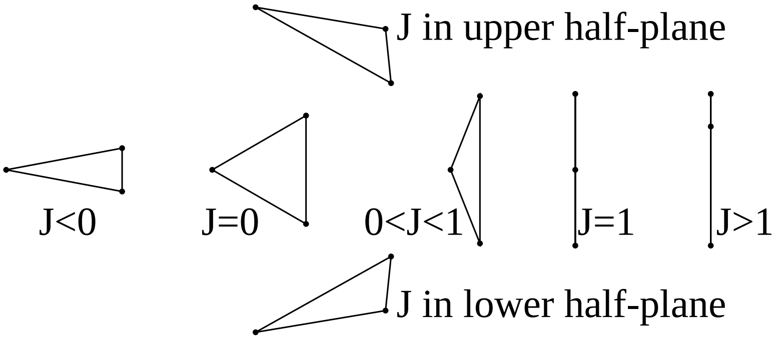
where the s_j are elementary symmetric functions.

Furthermore: two such curves are projectively equivalent if and only if they have the same invariant

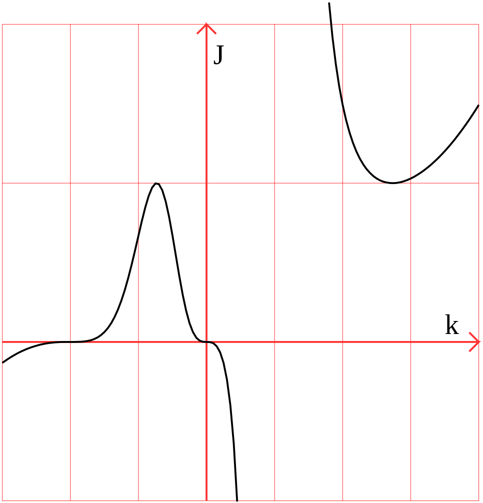
$$J = \frac{4s_2^3}{4s_2^3 + 27s_3^2} \in \mathbb{C} \, .$$

Triangles and the J-invariant.

The J-invariant characterizes the "shape" of the triangle with vertices r_1, r_2, r_3.



The function $k \to J$



$$J = \frac{k^3(k^3+8)^3}{64(k^3-1)^3}\,.$$

Real Cubic Curves

Theorem. Every smooth real cubic curve is real projectively equivalent to a curve $\mathcal{C}_k$ in the Hesse normal form for one and only one real value of k, with $k \neq 1$.