Work with Araceli Bonifant:
arXiv:math/1603.09018

It is possible to write endlessly on elliptic curves.
(This is not a threat.)

Serge Lang
1704: Enumeratio Linearum Terti Ordinis
Newton corrected
During the next 140 years cubic curves (and elliptic integrals) were studied by many mathematicians:

Colin Maclaurin,

Jean le Rond d’Alembert,

Leonhard Euler,

Adrien-Marie Legendre,

Niels Henrik Abel,

Carl Gustav Jacobi
1844: Otto Hesse

\[ x^3 + y^3 + z^3 = 3k\,xyz. \]

\[(x, y, z) \mapsto k = \frac{x^3 + y^3 + z^3}{3\,xyz}. \]
The Hesse Pencil in $\mathbb{P}^2(\mathbb{R})$

(Singular) foliation by curves $\frac{x^3+y^3+z^3}{3xyz} = \text{constant} \in \mathbb{R} \cup \{\infty\}$. 
The Hessian determinant of $\Phi(x, y, z)$.

$$H_\Phi(x, y, z) = \det \begin{pmatrix} \Phi_{xx} & \Phi_{xy} & \Phi_{xz} \\ \Phi_{yx} & \Phi_{yy} & \Phi_{yz} \\ \Phi_{zx} & \Phi_{zy} & \Phi_{zz} \end{pmatrix}$$

**Theorem.** If $C \subset \mathbb{P}^2$ is a smooth curve with defining equation $\Phi(x, y, z) = 0$, then $(x : y : z) \in C$ is a flex point if and only if $H_\Phi(x, y, z) = 0$.

It follows (with some work) that:

*Every smooth complex cubic curve has exactly nine flex points.*

**Example:**

$$\Phi(x, y, z) = x^3 + y^3 + z^3, \quad H_\Phi(x, y, z) = 6^3 x y z.$$
The Fermat Curve in the real affine plane $\{(x : y : 1)\}$:
Projective equivalence

Every nonsingular linear transformation

\[(x, y, z) \mapsto (X, Y, Z)\]

of \(\mathbb{C}^3\) induces a \textit{projective automorphism}

\[(x : y : z) \mapsto (X : Y : Z)\]

of the projective plane \(\mathbb{P}^2(\mathbb{C})\).

Two algebraic curves \(C_1\) and \(C_2\) in \(\mathbb{P}^2\) are called \textit{projectively equivalent} if there is a projective automorphism of \(\mathbb{P}^2\) which maps one onto the other.

\textbf{Theorem.} Every smooth cubic curve \(C \subset \mathbb{P}^2(\mathbb{C})\) is projectively equivalent to one in the Hesse normal form

\[x^3 + y^3 + z^3 = 3kxyz.\]

(But \(k\) is not unique!)
The chord-tangent map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

For a flex point: $p \ast p = p$. 
The additive group structure $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

Choose a flex point $o \in \mathbb{C}$ as base point, and set $p + q = (p \ast q) \ast o$.

Lemma. $p \ast q = r \iff p + q + r = o$.

Note also that $p \ast o = -p$. 
Theorem.

The line between two flex points always intersects $C$ in a third flex point.
Nine flex points and 12 lines between them.
Other Fields.

Let $F$ be any field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.

If the curve $C$ has defining equation $\Phi(x, y, z) = 0$ with coefficients in $F$, let $C(F) = C \cap \mathbb{P}^2(F)$ be the set of all points $(x : y : z) \in C$ with coordinates $x, y, z \in F$. Then $p, q \in C(F) \implies p \ast q \in C(F)$.

If $o \in C(F)$, it follows that $C(F)$ is a subgroup of $C$.

In particular, for any $n \in \mathbb{Z}$ we can define multiplication by $n$ as a map $m_n : C(F) \to C(F)$.

The construction is inductive:

$$m_0(p) = o, \quad \text{and} \quad m_{n+1}(p) = m_n(p) + p.$$ 

These maps form a semigroup, with composition:

$$m_n \circ m_k = m_{nk}, \quad \text{and} \quad m_n(p) + m_k(p) = m_{n+k}(p).$$
Extending $m_n$ to a map from $\mathbb{P}^2$ to itself.

Consider the “foliation” of $\mathbb{P}^2$ by the curves

$$C_k = \{(x : y : z) ; x^3 + y^3 + z^3 = 3xyz \}$$

in the Hesse pencil.

**Theorem.** The various maps $m_n : C_k \rightarrow C_k$ fit together to yield a rational map

$$m_n : \mathbb{P}^2 \rightarrow \mathbb{P}^2.$$
Examples (taking \((0 : -1 : 1)\) as base point).

\[
\begin{align*}
\mathbf{m}_{-2}(x : y : z) &= (x(y^3 - z^3) : y(z^3 - x^3) : z(y^3 - x^3)) \\
&\quad \quad \text{(Desboves, 1886)}
\end{align*}
\]

\[
\begin{align*}
\mathbf{m}_{-1}(x : y : z) &= (x : z : y) \\
\mathbf{m}_0(x : y : z) &= (0 : -1 : 1) \\
\mathbf{m}_1(x : y : z) &= (x : y : z) \\
\mathbf{m}_2(x : y : z) &= (x(y^3 - z^3) : z(y^3 - x^3) : y(z^3 - x^3)) \\
\mathbf{m}_3(x : y : z) &= (xyz(x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3) : \\
&\quad (x^2y + y^2z + z^2x)(x^4y^2 + y^4z^2 + z^4x^2 - xy^2z^3 - yz^2x^3 - zx^2y^3) : \\
&\quad x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3).
\end{align*}
\]
Weierstrass Normal Form and the $J$-invariant

**Theorem (Nagel 1928).** Every smooth cubic curve is projectively equivalent to one in the normal form

\[ y^2 = (x - r_1)(x - r_2)(x - r_3) \quad \text{with} \quad r_1 + r_2 + r_3 = 0 \]

\[ = x^3 + s_2 x - s_3, \]

where the $s_j$ are elementary symmetric functions.

Furthermore: two such curves are projectively equivalent if and only if they have the same invariant

\[ J = \frac{4s_2^3}{4s_2^3 + 27s_3^2} \in \mathbb{C}. \]
Triangles and the $J$-invariant.

The $J$-invariant characterizes the “shape” of the triangle with vertices $r_1, r_2, r_3$.
The function $k \rightarrow J$

$$J = \frac{k^3(k^3 + 8)^3}{64(k^3 - 1)^3}.$$
Real Cubic Curves

**Theorem.** Every smooth real cubic curve is real projectively equivalent to a curve $C_k$ in the Hesse normal form for one and only one real value of $k$, with $k \neq 1$. 