Hyperbolic Component Boundaries: **Nasty or Nice ?**

John Milnor

Stony Brook University

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A Theorem and a Conjecture.

Let $\mathcal{P}_n \cong \mathbb{C}^{n-1}$ be the space of monic centered polynomials of degree $n \geq 2$, and let $H \subset \mathcal{P}_n$ be a hyperbolic component in its connectedness locus.

Theorem. *If each $f \in H$ has exactly $n-1$ attracting cycles (one for each critical point), then the boundary ∂H and the closure $\overline{H}$ are semi-algebraic sets.*

Non Local Connectivity Conjecture. In all other cases, the sets ∂H and $\overline{H}$ are not locally connected.

Semi-algebraic Sets

Definition. A ***basic semi-algebraic set*** S in $\mathbb{R}^n$ is a subset of the form

$$S = S(r_1, \ldots, r_k ; s_1, \ldots, s_\ell)$$

consisting of all $\mathbf{x} \in \mathbb{R}^n$ satisfying the inequalities

$$r_1(\mathbf{x}) \geq 0 \ \ldots, r_k(\mathbf{x}) \geq 0 \quad \text{and} \quad s_1(\mathbf{x}) \neq 0, \ldots, s_\ell(\mathbf{x}) \neq 0 .$$

Here the $r_i : \mathbb{R}^n \to \mathbb{R}$ and the $s_j : \mathbb{R}^n \to \mathbb{R}$ can be arbitrary real polynomials maps.

Any finite union of basic semi-algebraic sets is called a ***semi-algebraic set****.*

Easy Exercise: If S_1 and S_2 are semi-algebraic, then both $S_1 \cup S_2$ and $S_1 \cap S_2$ are semi-algebraic.

Furthermore $\mathbb{R}^n \setminus S_1$ is semi-algebraic.

Non-Trivial Properties

- A semi-algebraic set has finitely many connected components, and each of them is semi-algebraic.

- The topological closure of a semi-algebraic set is semi-algebraic.

- **Tarski-Seidenberg Theorem:** The image of a semi-algebraic set under projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-k}$ is semi-algebraic.

- Every semi-algebraic set can be triangulated (and hence is locally connected).

Reference: Bochnak, Coste, and Roy,
"Real Algebraic Geometry", Springer 1998.

Recall the Theorem:

If each $f \in H$ has exactly $n-1$ attracting cycles (one for each critical point), then the boundary ∂H and the closure $\overline{H}$ are semi-algebraic sets.

To prove this we will first mark $n-1$ periodic points.

Let $p_1, p_2, \ldots, p_{n-1}$ be the periods of these points, and let $\mathcal{P}_n(p_1, p_2, \ldots, p_{n-1})$ be the set of all

$$(f, z_1, z_2, \ldots, z_{n-1}) \in \mathcal{P}_n \times \mathbb{C}^{n-1}$$

satisfying two conditions:

- Each z_j should have period exactly p_j under the map f ;
- and the orbits of the z_j must be disjoint.

Lemma. *This set* $\mathcal{P}_n(p_1, p_2, \ldots, p_{n-1}) \subset \mathbb{R}^{4n-4}$ *is semi-algebraic.*

The proof is an easy exercise. □

Proof of the Theorem

Let U be the open set consisting of all

$$(f, z_1, \ldots, z_{n-1}) \in \mathcal{P}_n(p_1, p_2, \ldots, p_{n-1})$$

such that the multiplier of the orbit for each z_j satisfies

$$|\mu_j|^2 < 1 .$$

This set U is semi-algebraic.

Hence each component $\widetilde{H} \subset U$ is semi-algebraic.

Hence the image of $\widetilde{H}$ under the projection
$\mathcal{P}_n(p_1, p_2, \ldots, p_{n-1}) \to \mathcal{P}_n$ is a semi-algebraic set H,
which is clearly a hyperbolic component in $\mathcal{P}_n$.

In fact any hyperbolic component $H \subset \mathcal{P}_n$ having attracting cycles with periods $p_1, p_2, \ldots, p_{n-1}$ can be obtained in this way.

This proves that H, its closure $\overline{H}$, and its boundary
$\partial H = \overline{H} \cap (\overline{\mathcal{P}_n \smallsetminus H})$ are all semi-algebraic sets. $\square$

Postcritical Parabolic Orbits

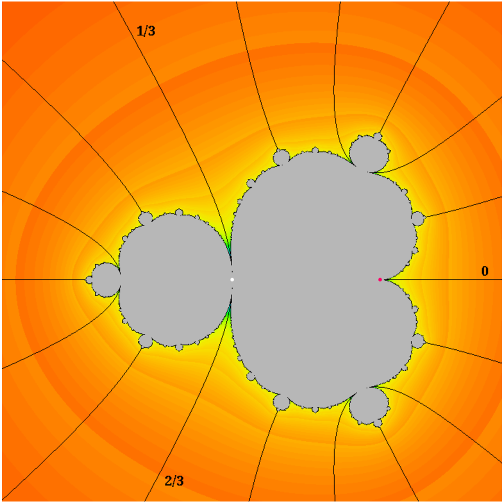
Definition. A parabolic orbit with a primitive q-th root of unity as multiplier will be called ***simple*** if each orbit point has just q attracting petals.

My strategy for trying to prove the Non Local Connectivity Conjecture is to split it into two parts (preliminary version):

Conjecture A. If maps in the hyperbolic component H have an attracting cycle which attracts two or more critical points, then some map $f \in \partial H$ has a postcritical simple parabolic orbit.

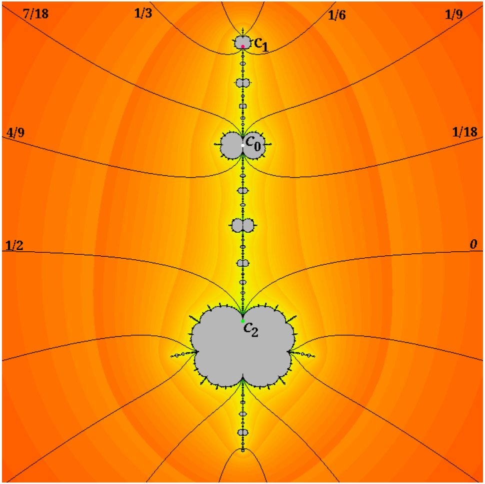
Conjecture B. If some $f \in \partial H$ has a postcritical simple parabolic orbit, then $\overline{H}$ and ∂H are not locally connected.

Example: $f(z) = z^3 + 2z^2 + z$



Here $f(-1) = 0$, where -1 is critical, and 0 is a parabolic fixed point of multiplier $f'(0) = 1$. Furthermore $f \in \partial H_0$.

Example: $f(z) = z^3 + 2.5319\, i\, z^2 + .8249\, i$

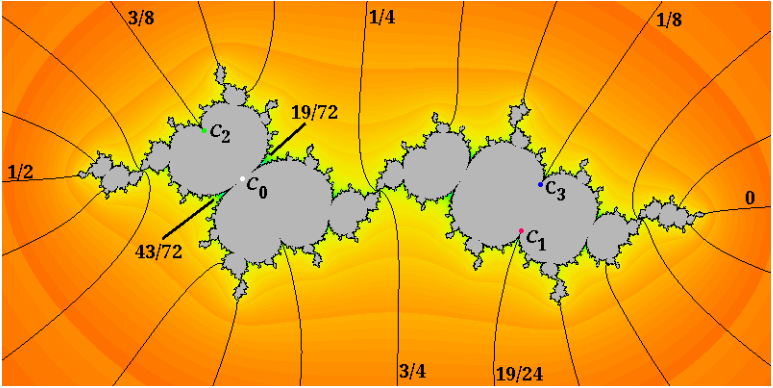


Here f is on the boundary of a capture component, with

$$c_0 = 0 \mapsto c_1 = .8249\, i \mapsto c_2 = -1.4596\, i \,,$$

where $f(c_2) = c_2\,, \quad \mu = f'(c_2) = 1\,.$

Example: $f(z) = z^3 + (-2.2443 + .2184\, i)z^2 + (1.4485 - .2665\, i)$



Here: $c_0 \mapsto c_1 \mapsto c_2 \leftrightarrow c_3$

with $\mu = f'(c_2)\, f'(c_3) = 1$.

The corresponding ray angles are

$$\left\{\frac{19}{72}, \frac{43}{72}\right\} \mapsto \underline{\frac{19}{24}} \mapsto \frac{3}{8} \leftrightarrow \frac{1}{8}\,.$$

Simplified Example: A dynamical system on $\mathbb{C} \sqcup \mathbb{C}$

$$g_\mu \circlearrowleft \mathbb{C} \xleftarrow{\;\;f_{\widehat{z}}\;\;} \mathbb{C}$$

z−plane $\qquad$ w−plane

Here g_μ maps the z-plane to itself by

$$z \;\mapsto\; z^2 + \mu\, z\,,$$

and $f_{\widehat{z}}$ maps the w-plane to the z-plane by

$$w \;\mapsto\; z = w^2 + \widehat{z}\,.$$

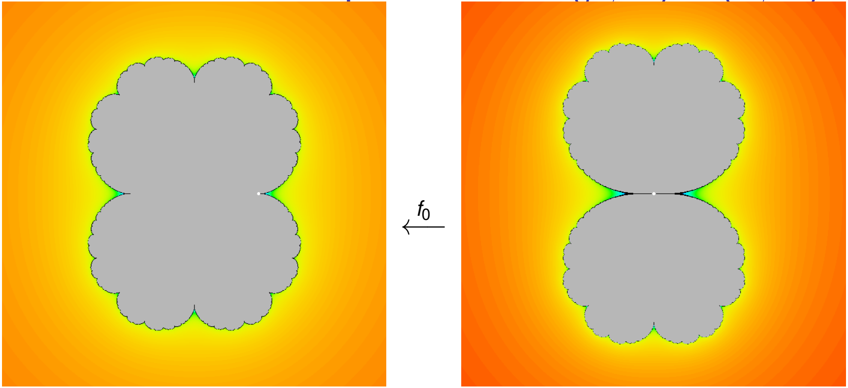
Thus the parameter space consists of all $(\mu\,,\ \widehat{z}) \in \mathbb{C}^2$.

Let $\mathcal{H} \subset \mathbb{C}^2$ be the "hyperbolic component" consisting of all pairs $(\mu\,,\ \widehat{z})$ such that $|\mu| < 1$ (so that $z = 0$ is an attracting fixed point), and such that $\widehat{z}$ belongs to its basin of attraction.

Thus a map belongs to $\mathcal{H}$

$\iff$ **both** critical orbits converge to $z = 0$.

Julia set in $\mathbb{C} \sqcup \mathbb{C}$ for parameters $(\mu, \hat{z}) = (1, 0)$



z-plane: $g_1(z) = z^2 + z$ w-plane: $f_0(w) = w^2$

Here f_0 maps the critical point $w = 0$ to the fixed point $z = 0$, which is parabolic with multiplier $g_1'(0) = 1$.

Thus for $(\mu, \hat{z}) = (1, 0)$ we have a map in $\partial\mathcal{H}$ with a postcritical parabolic point.

Empirical "Proof" that $\overline{\mathcal{H}}$ is not locally connected.

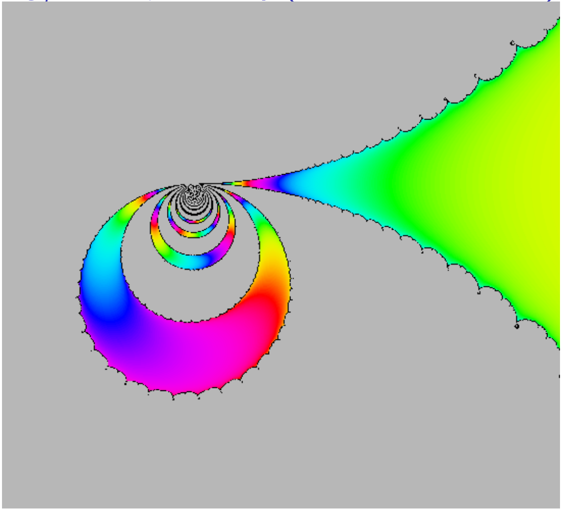
Non Local Connectivity Assertion. *There exists a convergent sequence in* $\overline{\mathcal{H}}$ *,*

$$\lim_{j\to\infty}(\mu_j,\ z_j)\ =\ (1,\ z_*)\ ,$$

and an $\epsilon > 0$ *, such that no* (μ_j, z_j) *can be joined to* $(1,\ z_*\ )$ *by a path of diameter* $< \epsilon$ *.*

This will imply that the set $\overline{\mathcal{H}} \subset \mathbb{C}^2$ is not locally connected.

Julia set of g_μ for $\mu = \exp(-.0001 + .01\,i)$.



Showing a neighborhood of zero in the z-plane.

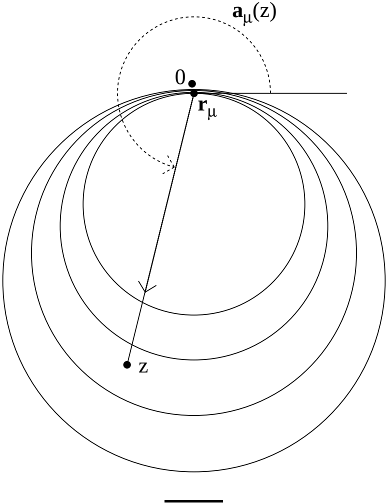
All orbits in the "Hawaiian earring" spiral away from the repelling fixed point $\mathbf{r}_\mu = 1 - \mu$.

The **a**rgument function $\mathbf{a}_\mu : K(g_\mu) \setminus \{\mathbf{r}_\mu\} \to \mathbb{R}$

For any $\mu \in \overline{\mathbb{D}}$, let $\mathbf{r}_\mu$ be the fixed point $1 - \mu$.

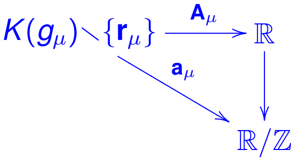
Thus $\mathbf{r}_\mu$ is repelling whenever $\mu \neq 1$.

For any $z \neq \mathbf{r}_\mu$, let $\mathbf{a}_\mu(z) = arg(z - \mathbf{r}_\mu) \in \mathbb{R}/\mathbb{Z}$ be the angle of the vector from $\mathbf{r}_\mu$ to z.



Now lift $\mathbf{a}_\mu$ to a real valued function

Since each set $K(g_\mu) \setminus \{\mathbf{r}_\mu\}$ is simply connected, this function $\mathbf{a}_\mu$ lifts to a real valued function $\mathbf{A}_\mu$.

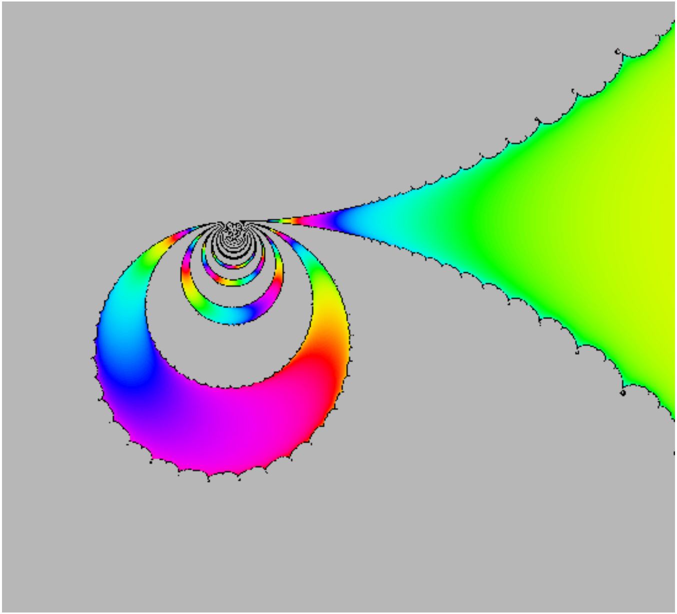


This lifting is only well defined up to an additive integer, but we can normalize (for $\mu \neq 1$) by requiring that

$$1/4 \; < \; \mathbf{A}_\mu(0) \; < \; 3/4 \, .$$

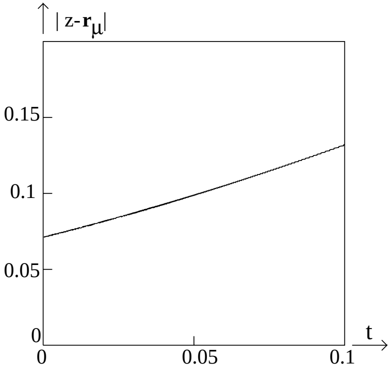
In fact $\mathbf{A}_\mu(z)$ is continuous as a function of both z and μ, subject only to the conditions that $z \in K(g_\mu)$ and $z \neq \mathbf{r}_\mu$.

Julia set of g_μ for $\mu = \exp(-.0001 + .01\, i)$.



A numerical calculation

Program: Given μ , start with the critical point $z = -\mu/2$ for g_μ and follow the backwards orbit of z within the half-plane $\Re(z) > \Re(-\mu/2)$, until it reaches a point with $A_\mu(z) > 1.75$. Then report the distance $|z - \mathbf{r}_\mu|$.



Graph of $|z - \mathbf{r}_\mu|$ as a function of $t \in [0, .1]$ for the family $\mu(t) = \exp(-t^2 + i\,t)$.

Note that $|z - \mathbf{r}_\mu| > .05$ for these t .

Construction of the points $(\mu_j,\ z_j)$

Choose points μ_j of the form $\exp(-t^2 + i\,t)$, with $t \searrow 0$, and choose corresponding points z_j with

$$\mathbf{A}_{\mu_j}(z_j) > 1.75 \quad \text{and with} \quad |z_j - \mathbf{r}_{\mu_j}| > .05\ .$$

Passing to a subsequence, we may assume that $\{z_j\}$ converges to some limit z_*.

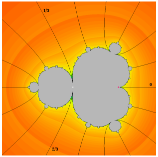
Now as we vary both μ_j and z_j along paths of diameter $< .02$ within $\overline{\mathcal{H}}$, the $\mathbf{A}_\mu(z)$ must still be > 1.5.

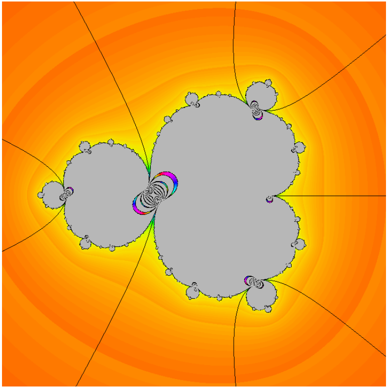
However, the limit point $(1,\ z_*)$, must satisfy $0 < \mathbf{A}_1(z_*) < 1$. Hence by following such small paths we can never reach this limit point.

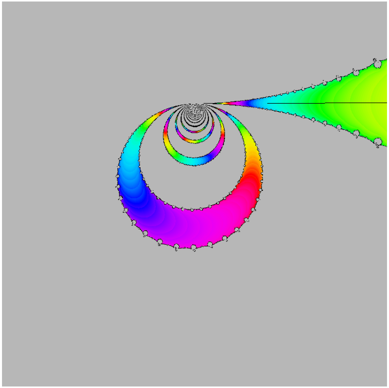
This "proves" the non local connectivity of $\overline{\mathcal{H}}$. $\square$

Example: Julia set for $f(z) = z^3 + 2z^2 + \mu z\,, \quad \mu \approx 1$

$\mu = 1$:



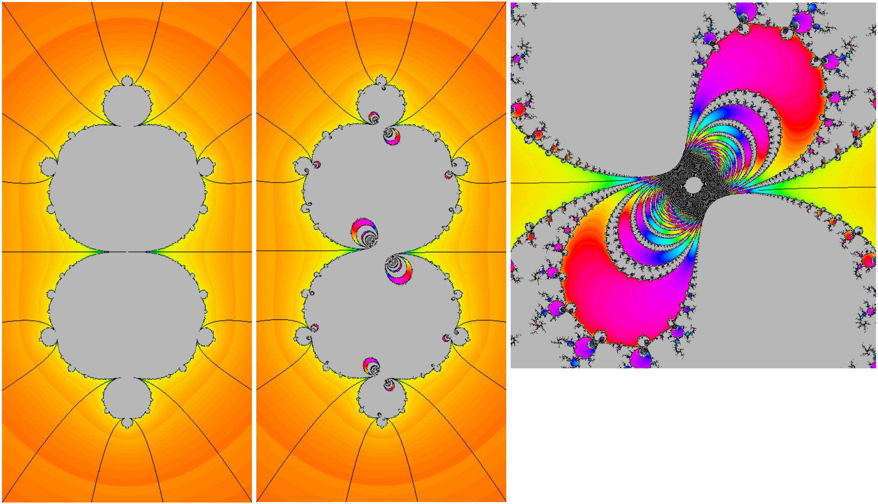




$\mu = \exp(-.0001 + .01\, i\,)$

Detail near $z = 0$.

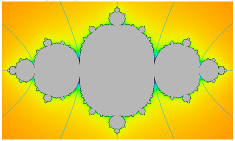
Example: Perturbing a non-simple parabolic point.

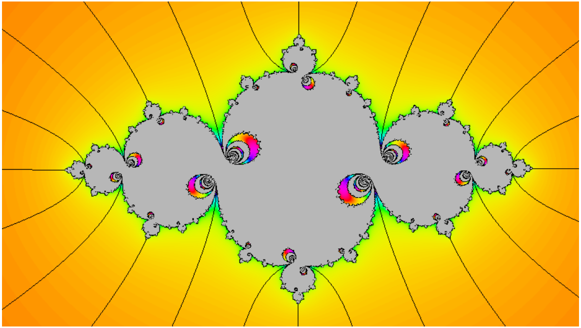


$f(z) = z^3 + z$

Example: Julia set for $f(z) = z^2 + \mu z, \quad \mu \approx -1$

$\mu = -1$:



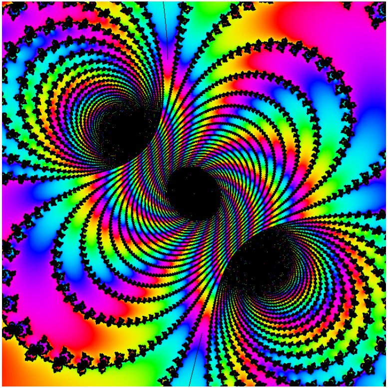


$$\mu = -\exp(-.0001 + .01\, i) \approx -1 .$$

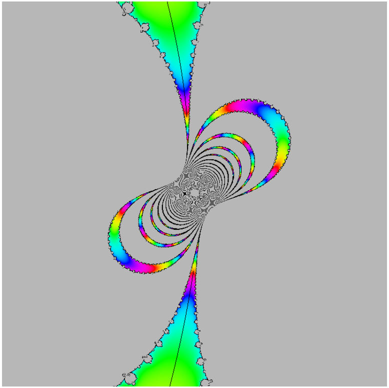
Thus we have moved from the "fat basilica" $z \mapsto z^2 - z$ to a map inside the main cardioid of the Mandelbrot set.

Example: $z \mapsto z^2 + \mu z\,,\ \ \mu \approx -1\,,$ again

Outside the Mandelbrot set.



Into the period two component



Conjectures A and B: Corrected Version

Consider the postcritical parabolic orbit $\mathcal{O}$ for $f \in \partial H$.

Suppose that the immediate basin for $\mathcal{O}$ corresponds to a cycle of Fatou components of period p for maps in H.

Then we must require that $\mathcal{O}$ be a simple parabolic orbit for the iterate $f^{\circ p}$.

THE END