THE DIRAC OPERATOR FOR HIGGS BUNDLES

Nigel Hitchin (Oxford)

Blaine Lawson's 70th birthday

Stony Brook October 28th 2012

HIGGS BUNDLES

- Riemann surface Σ
- Vector bundle E + hermitian metric
- Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}\, E \otimes K)$
- equations $F_A + [\Phi, \Phi^*] = 0$

Volume 70B, number 3 PHYSICS LETTERS 10 October 1977

TWO- AND THREE-DIMENSIONAL INSTANTONS

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Received 6 June 1977

The four-dimensional Yang-Mills Lagrangian implies corresponding structures in lower dimensions. Instantons, characterized by a zero energy-momentum tensor as well as finite action, emerge as the solutions of coupled first order equations. For the Abelian case all such solutions are determined by the non-linear Poisson-Boltzmann equation.

term have acquired special values. One can obtain first order equations which imply the field equations, and these are deduced from eqs. (6):

$$F_{ij}^{a} = \pm e\, \epsilon_{ij} \epsilon^{abc} \psi^{b} \phi^{c} \ ,$$

$$D_i \psi^a \pm \epsilon_{ij} D_j \phi^a = 0 \ . \qquad (11)$$

Unfortunately, in this case the model is not interesting because the action is always zero.

LOCALLY...

- anti-self-dual connection on principal bundle over $\mathbf{R}^4$

- .... invariant under $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3 + t_1, x_4 + t_2)$

$$\nabla_3 = \frac{\partial}{\partial x_3} + \phi_1(x_1, x_2) \qquad \nabla_4 = \frac{\partial}{\partial x_4} + \phi_2(x_1, x_2)$$

$$[\nabla_1, \nabla_2] + [\phi_1, \phi_2] = 0 = [\nabla_1, \phi_2] + [\nabla_2, \phi_1] = 0 = [\nabla_1, \phi_1] + [\phi_2, \nabla_2]$$

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$$F + [\Phi, \Phi^*] = 0 \qquad \bar{\partial}_A \Phi = 0$$

CHALLENGE

Solve these equations like the ADHM construction of instantons

SPECIAL CASES

- $\Phi = 0$:

  find the flat unitary connection on a stable vector bundle

- $E = K^{1/2} \oplus K^{-1/2}, \Phi(u, v) = (0, u)$:

  find the hyperbolic metric on a Riemann surface

INSTANTONS

- ASD connection on E over S^4

- Dirac operator $D : C^\infty(S_+ \otimes E) \to C^\infty(S_- \otimes E)$

- $$D^*D = \nabla^*\nabla + \frac{1}{4}R \Rightarrow \ker D = 0$$

- $\dim \ker D^* = k$

- $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$, G-connection ∇

- $D = -\nabla_1 + i\nabla_2 + j\nabla_3 + k\nabla_4$ quaternions

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- $D = -\nabla_1 + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \nabla_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla_3 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \nabla_4$

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- D^*D real iff $[\nabla_1, \nabla_2] + [\nabla_3, \nabla_4] = 0$ etc.

- anti-self-dual Yang-Mills

- $D^*D = -\nabla_1^2 - \nabla_2^2 - \nabla_3^2 - \nabla_4^2$

- $\mathcal{L}^2$ solutions $\psi_1, \ldots, \psi_k$ of $D^*\psi = 0$

- matrices $T_1, \ldots, T_4$

$$\langle T_i \psi_\alpha, \psi_\beta \rangle = \int_{\mathbf{R}^4} (x_i \psi_\alpha, \psi_\beta)$$

- $\sim$ ADHM construction

DIRAC OPERATOR FOR HIGGS BUNDLES

- dimensional reduction

- ASD connection $A_1 dx_1 + A_2 dx_2 + \phi_1 dx_3 + \phi_2 dx_4$

- $D^* = \nabla_1 + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \nabla_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_1 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \phi_2$

- Dirac operator $\quad D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix}$

GLOBALLY...

- Σ compact, genus g

- $$D^* = \begin{pmatrix} \bar\partial_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} : \begin{pmatrix} E \otimes K \\ E \otimes \bar K \end{pmatrix} \to \begin{pmatrix} E \otimes K\bar K \\ E \otimes K\bar K \end{pmatrix}$$

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- $D^* D = -\nabla_1^2 - \nabla_2^2 - \phi_1^2 - \phi_2^2 \Rightarrow \ker D = 0$

- index theorem $\Rightarrow \dim \ker D^* = (2g-2)\,\mathrm{rk}\, E$

- $(\psi_1, \psi_2) \in \mathbf{C}^\infty(EK \oplus E\bar{K})$

- conformally invariant inner product

$$\int_\Sigma \langle \psi_1, \psi_1 \rangle + \langle \psi_2, \psi_2 \rangle$$

- $\Rightarrow$ inner product on $\ker D^*$

EXAMPLE

- E trivial line bundle on $\mathbf{C}$; $\Phi = dz$
- $D^*(\psi_1, \psi_2) = 0$
- $\bar{\partial}\psi_1 + \psi_2 = 0, \psi_1 + \partial\psi_2 = 0$

$$\frac{\partial^2 \psi_1}{\partial z \partial \bar{z}} + \psi_1 = 0$$

- $\psi_2 = -\bar{\partial}\psi_1$

FOURIER TRANSFORM

M.Jardim, *Nahm transform for doubly-periodic instantons*, Commun.Math.Phys.. **225** (2002) 639–668

J.Bonsdorff *A Fourier transform for Higgs bundles*, Crelle **591** (2006) 21–48

J.Bonsdorff *Autodual Connection in the Fourier Transform of a Higgs Bundle*, Asian J. Math. **14** (2010), 153–174

S.Szabo, "Nahm transform for integrable connections on the Riemann sphere", Société Mathématique de France (2007)

- Higgs bundle (E, Φ)
- L flat line bundle, θ holomorphic 1-form
- family $(E \otimes L, \Phi + \theta 1)$ parameters $(L, \theta) \in \mathrm{J}(\Sigma) \times H^0(\Sigma, K)$
- tensoring with a $U(1)$ Higgs bundle

- $\ker D^*_{L,\theta}$ defines a vector bundle over $T^*\mathsf{J} = \mathsf{J} \times H^0(\Sigma, K)$

- Higgs bundle equations $\Rightarrow$ hermitian metric

- $\Rightarrow \mathcal{L}^2$ connection

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- the connection is *hyperholomorphic*

- Jacobian J flat Kähler torus
- T^*J flat hyperkähler manifold
- complex structures I, J, K
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hyperholomorphic = curvature type $(1,1)$ wrt I, J, K

- ADHM: $D = A + qB$, D^*D real

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- $a, \theta \in H^0(\Sigma, K)$

- $$\begin{pmatrix} \bar{\partial}_A + \bar{a} & \Phi + \theta \\ \Phi^* + \bar{\theta} & \partial_A - a \end{pmatrix} = D^* + q$$

- $q \in H^0(\Sigma, K) \oplus \bar{H}^0(\Sigma, K)$ quaternionic vector space

- complex structure I

$$\begin{array}{ccc} \Omega^{0,p}(E) & \stackrel{\Phi}{\to} & \Omega^{0,p}(EK) \\ \downarrow \bar\partial & & \downarrow \bar\partial \\ \Omega^{0,p+1}(E) & \stackrel{\Phi}{\to} & \Omega^{0,p+1}(EK) \end{array}$$

- total differential $\bar\partial \pm \Phi$

- Hodge theory: $\ker D^* \cong$ hypercohomology $\mathbf{H}^1$

- complex structure J

- $\nabla = \partial_A + \bar{\partial}_A + \Phi + \Phi^*$ flat $GL(n, \mathbf{C})$ connection

$$\Omega^0(E) \stackrel{\nabla}{\to} \Omega^1(E) \stackrel{d_\nabla}{\to} \Omega^2(E)$$

- Hodge theory: $\ker D^* \cong$ de Rham cohomology $H^1(\mathcal{E})$

FOURIER TRANSFORM

- complex structure I: (L, θ)

- hypercohomology $\mathbf{H}^1$ of

$$\mathcal{O}(EL) \stackrel{\Phi+\theta}{\to} \mathcal{O}(ELK)$$

- complex structure J: $\mathcal{L} \in H^1(\mathbf{\Sigma}, \mathbf{C}^*)$

- $H^1(\mathbf{\Sigma}, \mathcal{E}\mathcal{L})$

- hyperholomorphic $\Rightarrow$ holomorphic bundle on twistor space of $\mathsf{J} \times H^0(\Sigma, K)$

- $[\Theta] \in H^1(\mathsf{J}, T^*)$: extension $T^* \to V \to \mathcal{O}$

- $\mathsf{P}(V) \backslash \mathsf{P}(T^*)$ affine bundle over J

- $H^1(\Sigma, \mathbf{C}^*) \to H^1(\Sigma, \mathcal{O}^*) = \mathsf{J}$

- $\zeta \in \mathbf{C}, \zeta[\Theta] \Rightarrow$ bundle (V_ζ, ζ) over $\mathsf{J} \times \mathbf{C}$

- $\zeta = 0$ $T^*\mathsf{J}$: product $\mathsf{J} \times H^0(\Sigma, K)$

- $\mathsf{P}(V)\backslash\mathsf{P}(T^*)$ affine bundle $\cong H^1(\mathbf{\Sigma}, \mathbf{C}^*)$

$$H^1(\mathbf{\Sigma}, \mathbf{C}^*) \to \mathsf{J}(\mathbf{\Sigma})$$

flat connection ∇ holomorphic structure $\nabla^{0,1}$

- $\mathsf{P}(V) \backslash \mathsf{P}(T^*)$ affine bundle $\cong H^1(\boldsymbol{\Sigma}, \mathbf{C}^*)$

$$H^1(\boldsymbol{\Sigma}, \mathbf{C}^*) \to \mathsf{J}(\boldsymbol{\Sigma})$$

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$$H^1(\boldsymbol{\Sigma}, \mathbf{C}^*) \to \mathsf{J}(\bar{\boldsymbol{\Sigma}})$$

flat connection ∇ holomorphic structure on $\bar{\boldsymbol{\Sigma}}$ $\nabla^{1,0}$

- twistor space: identify over $\mathbf{C}^* \subset \mathbf{C}$

CONCLUSION

- Higgs bundle (E, Φ)

- Fourier transform = rank $\mathrm{rk}\, E(2g-2)$ bundle $\hat{E}$ on $\mathsf{J} \times H^0(\Sigma, K)$ with hyperholomorphic connection

- $\sim$ holomorphic $\mathrm{rk}\, E(2g-2)$ bundle on twistor space Z

QUESTIONS

1. Does this vector bundle encode the Higgs bundle and its metric?

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2. If so, what is the inverse transform?

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2. If so, what is the inverse transform?

3. Find the transform of the canonical Higgs bundle.

COMPLEX STRUCTURE I

HYPERCOHOMOLOGY

- $\mathcal{O}(E) \xrightarrow{\Phi} \mathcal{O}(EK)$

  two spectral sequences:

- cohomology of $\Phi : H^p(\Sigma, E) \to H^p(\Sigma, EK)$:

  $0 \to H^0(E) \to H^0(EK) \to \mathbf{H}^1(E) \to H^1(E) \to H^1(EK) \to 0$

- cohomology of $\mathcal{O}(U, E) \xrightarrow{\Phi} \mathcal{O}(U, EK)$

  $\mathbf{H}^1(E) \cong H^0(\Sigma, \mathrm{coker}\, \Phi)$

- hypercohomology $\Phi + \theta 1 : \mathcal{O}(E \otimes L) \to \mathcal{O}(E \otimes LK)$

- $x_0 \Phi + x_1 \theta 1 : \mathcal{O}(E \otimes L) \to \mathcal{O}(E \otimes LK)$

- bundle extends to $\mathrm{J} \times \mathrm{P}(H^0(\Sigma, K) \oplus 1) = \mathrm{J} \times \mathrm{P}^g$

J.Bonsdorff *A Fourier transform for Higgs bundles*, Crelle **591** (2006) 21–48

- $\mathsf{ch}(\hat{E}) = \mathsf{rk}\, E(g-1+(g-1)e^x + \theta(1-e^x))$

  $x \in H^2(\mathsf{P}^g, \mathbf{Z}), \theta \in H^2(\mathsf{J}, \mathbf{Z})$

- $\dim H^p(\mathsf{J} \times \mathsf{P}^g, \hat{E}) = \mathsf{rk}\, E \binom{g-1}{p-1}$

  $1 \le p \le g$ zero otherwise

EXAMPLE

- $\Phi = 0$: stable bundle E over Σ, $\deg E = 0$

- $H^0(\Sigma, E) = 0$, $\dim H^1(\Sigma, E) = \operatorname{rk} E(g-1)$

- $\mathcal{P} \to \Sigma \times \mathrm{J}$ Poincaré bundle

- $\Rightarrow$ Picard bundle $R^1(E) \sim H^1(\Sigma, EL)$ over J

- $0 \to H^0(\Sigma, ELK) \to \mathbf{H}^1(\Sigma, EL) \to H^1(\Sigma, EL) \to 0$

- $p : \mathsf{J} \times \mathsf{P}^g \to \mathsf{J}$
- $0 \to p^* R^0(E) \to \widehat{E} \to p^* R^1(E)(1) \to 0$
- $\Rightarrow \widehat{E} \cong p^* R^0(E) \oplus p^* R^1(E)(1)$

- hypercohomology $\Phi + \theta 1 : \mathcal{O}(E \otimes L) \to \mathcal{O}(E \otimes LK)$

- $x_0\Phi + x_1\theta 1 : \mathcal{O}(E \otimes L) \to \mathcal{O}(E \otimes LK)$

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- Fourier-Mukai: equivalence of derived categories

- .... recover (E, Φ) as a holomorphic object

THE HYPERHOLOMORPHIC CONNECTION

- connection on a bundle over $\mathsf{J} \times \mathbf{C}^g$
- What is its asymptotic behaviour?
- Why does the holomorphic structure extend to $\mathsf{J} \times \mathsf{P}^g$?

M.Jardim, *Construction of doubly-periodic instantons*, CMP **216** (2001) 1–15

M.Jardim, *Nahm transform for doubly-periodic instantons*, CMP **225** (2002) 639–668

- instanton on $T^2 \times \mathbf{R}^2$,

$$\int_{T^2 \times \mathbf{R}^2} |F|^2 < \infty$$

- $\Rightarrow$ Higgs bundle on $\hat{T}^2$ with two singularities

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- I- holomorphic structure extends from $T^2 \times \mathbf{C}$ to $T^2 \times \mathrm{P}^1$

- Dirac operator D^*

$$D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} + t \begin{pmatrix} 0 & \theta \\ \bar{\theta} & 0 \end{pmatrix}$$

- What is the asymptotic behaviour as $t \to \infty$ of solutions?

- Weitzenböck formula

  $$DD^* = \nabla^*\nabla + A + tB + t^2|\theta|^2$$

- $$D^*\psi = 0 \qquad \int_\Sigma |\psi|^2 = 1$$

- $$\int_\Sigma \langle A\psi, \psi\rangle + t\int_\Sigma \langle B\psi, \psi\rangle + t^2\int_\Sigma |\theta|^2|\psi|^2 \leq 0$$

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- $$t^2\int_\Sigma |\theta|^2|\psi|^2 \leq tM$$

- $U \subset \mathbf{\Sigma}$ $\qquad \displaystyle\int_U |\theta|^2 |\psi|^2 \leq t^{-1} M$

- choose U disjoint from the zeros of θ

- $\displaystyle m \int_U |\psi|^2 \leq \int_U |\theta|^2 |\psi|^2 \leq t^{-1} M$

- $\displaystyle \int_U |\psi|^2 \to 0$

- solutions localize around the $2g-2$ zeros of $\theta \in H^0(\Sigma, K)$

- $\mathcal{O}(E) \stackrel{\Phi}{\to} \mathcal{O}(EK)$

- hypercohomology $\mathbf{H}^1 \cong$ cokernel of Φ...

- supported on zeros of $\det \Phi \in H^0(\Sigma, K^n)$

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- $\det(\Phi + t\theta) = t^n\theta^n + \ldots + \det \Phi$

- zeros of $\theta \sim$ basis for $\hat{E}$ on $\mathrm{P}(T^*)$

SPECIAL CASES

- $\Phi = 0$:

  find the flat unitary connection on a stable vector bundle

- $E = K^{1/2} \oplus K^{-1/2}, \Phi(u, v) = (0, u)$:

  find the hyperbolic metric on a Riemann surface

- S^1-action $(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$

- S^1-action on $T^*\mathsf{J}$

- $\Phi = 0$ fixed $\Rightarrow \widehat{E}$ invariant

- $E = K^{1/2} \oplus K^{-1/2}, \Phi(u,v) = (0,u)$

- $(e^{i\theta/2}u, e^{-i\theta/2}v) : \Phi \mapsto e^{i\theta}\Phi$

- $\Rightarrow \hat{E}$ invariant

THM. (Feix, Kaledin). Let M be a real analytic Kähler manifold, then there is a unique S^1-invariant hyperkähler metric on a neighbourhood of the zero section extending the Kähler metric.

B. Feix, *Hyperkähler metrics on cotangent bundles*, J. Reine Angew. Math. **532** (2001), 33-46.

THM. (Feix). Let A be a connection in a vector bundle over M whose curvature is of type $(1,1)$, then this extends to a unique S^1-invariant hyperholomorphic connection on a neighbourhood of the zero section.

B. Feix, *Hypercomplex manifolds and hyperholomorphic bundles,* Math. Proc. Camb. Phil. Soc. **133** (2002), 443-457

CANONICAL HIGGS BUNDLE

- $u_1 \in H^0(\Sigma, K^{3/2}L) \quad v_2 \in H^1(\Sigma, K^{-1/2}L)$

- $\deg K^{3/2}L = 3g - 3$

- Picard bundle V, $\mathrm{P}(V) \cong S^{3g-3}\Sigma$

- $\mathcal{L}^2$ metric wrt hyperbolic metric.


Happy Birthday