A survey of results about G_2 conifolds

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Results include separate joint works with
Jason Lotay (University College London)
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G_2 manifolds

Manifolds with G_2 structure

Definition

Let M^7 be a smooth 7-manifold. A G_2 structure on M is a reduction of the structure group of the frame bundle from $\mathrm{GL}(7,\mathbb{R})$ to $G_2 \subset \mathrm{SO}(7)$.

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- A G_2 structure exists if and only if M is *orientable* and *spin*, which is equivalent to $w_1(M) = 0$ and $w_2(M) = 0$.
- A G_2 structure is encoded by a "non-degenerate" 3-form φ which nonlinearly determines a Riemannian metric g_φ and an orientation. We thus have a Hodge star operator $*_\varphi$ and dual 4-form $\psi = *_\varphi \varphi$.

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- On a manifold (M, φ) with G_2 structure, each tangent space $T_p M$ can be canonically identified with the *imaginary octonions* $\mathbb{O} \cong \mathbb{R}^7$.

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Properties of G_2 manifolds:

- The holonomy $\mathrm{Hol}(g_\varphi)$ is contained in G_2. If $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$, then (M, φ) is called an irreducible G_2 manifold. A *compact* G_2 manifold is irreducible if and only if $\pi_1(M)$ is finite.

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- G_2 manifolds admit a parallel spinor. They play the role in M-theory that Calabi-Yau 3-folds play in string theory.
- A G_2 structure is torsion-free if and only if $d\varphi = 0$ and $d *_\varphi \varphi = 0$. (Fernàndez–Gray, 1982.) Both φ and $*_\varphi \varphi$ are calibrations.

Comparison with Kähler and Calabi-Yau geometry

- G_2 manifolds are very similar to Kähler manifolds.
- Both admit calibrated *submanifolds* and *connections*.
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- Both admit calibrated *submanifolds* and *connections*.
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- However, unlike G_2 manifolds, *not all* Kähler manifolds are Ricci-flat. Those are the *Calabi-Yau* manifolds.
- By the Calabi-Yau theorem, we have a topological characterization of the Ricci-flat Kähler manifolds.
- We are still *very far* from knowing sufficient topological conditions for existence of a torsion-free G_2 structure.

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- Therefore, Kähler geometry can be thought of as 'decoupling' into complex geometry and symplectic geometry.

- However, if M admits a G_2 structure, the 3-form φ determines the metric g in a nonlinear way:

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = C\, g_\varphi(u, v)\, \mathrm{vol}_\varphi$$

- Thus, we cannot 'decouple' G_2 geometry in any way.

Examples of G_2 manifolds

Complete noncompact examples

- Bryant–Salamon (1989): these examples are total spaces of vector bundles $\Lambda^2_-(S^4)$, $\Lambda^2_-(\mathbb{CP}^2)$, $\not{S}(S^3)$; they are all asymptotically conical: far away from the base of the bundle, they "look like" *metric cones*.
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- These examples are all explicit cohomogeneity one G_2 manifolds — they have enough "symmetry" so that the nonlinear PDE reduces to a system of fully nonlinear ODEs, which can often be solved exactly.
- It can be shown (using the Bochner–Weitzenböck formula) that *compact* examples cannot have *any* symmetry. So the construction of compact examples is necessarily much more difficult.

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- Joyce–Karigiannis (2013?) — glueing a 3-dimensional family of Eguchi-Hanson spaces

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Let M be a compact manifold with a closed G_2 structure φ such that the torsion is sufficiently small. (One needs good control of the L^{14} norm of the torsion and some other estimates.) Then there exists a torsion-free G_2 structure $\widetilde{\varphi}$ close to φ in the C^0 norm, with $[\widetilde{\varphi}] = [\varphi]$ in $H^3(M, \mathbb{R})$.

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These constructions provide thousands of examples, but they are likely only a very small part of the "landscape."

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Ingredients [2] and [3] require compactness of M, and thus need to be modified in any noncompact setting.

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G_2 cones

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A G_2 cone is a 7-manifold $C = (0, \infty) \times \Sigma$, with Σ compact, and a torsion-free G_2 structure φ_C with induced metric

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- The *link* Σ of a G_2 cone C is necessarily a compact strictly nearly Kähler 6-manifold (also called a *Gray manifold*.)
- These are almost Hermitian manifolds (Σ, J, g, ω) with $c_1(\Sigma) = 0$, such that

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- There are only three known compact examples, all homogeneous, but there are expected to exist *many examples*.

Asymptotically conical (AC) G_2 manifolds

Definition

We say (N, φ_N) is an AC G_2 manifold of rate $\nu < 0$, asymptotic to the G_2 cone (C, φ_C), if outside of a compact set $K \subseteq N$, we have $N \setminus K \cong (R, \infty) \times \Sigma$, and

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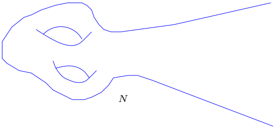

Conically singular (CS) G_2 manifolds

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Let $\overline{M}$ be a topological space with $M = \overline{M} \setminus \{x_1, \dots, x_n\}$ a noncompact smooth 7-manifold. We say (M, φ_M) is an CS G_2 manifold of rate $(\nu_1, \dots, \nu_n)$, where $\nu_i > 0$, asymptotic to the G_2 cones (C_i, φ_{C_i}), if outside of a compact set $K \subseteq M$, we have $M \setminus K \cong \bigsqcup_{i=1}^n (0, R) \times \Sigma_i$, and

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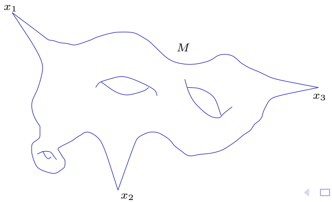

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- One way to show this, and thus to provide evidence for their likely existence, is to prove that they would *often* be desingularizable into families of compact smooth G_2 manifolds.
- A way to desingularize them is to cut out a neighbourhood of the singular points, and glue in *AC* G_2 *manifolds*, such as the Bryant–Salamon examples.

Desingularization of CS G_2 manifolds

Theorem (Karigiannis, Geometry & Topology, 2009)

Let M be a CS G_2 manifold with isolated conical singularities $x_1, \ldots, x_n$, modelled on G_2 cones $C_1, \ldots, C_n$. Suppose that $N_1, \ldots, N_n$ are AC G_2 manifolds modelled on the same G_2 cones, with all rates $\nu_i \leq -3$.

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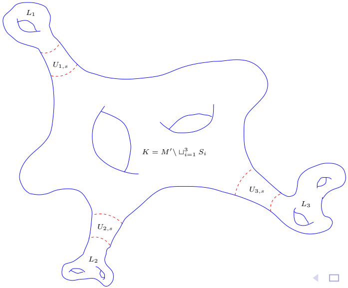

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Let M be a G_2 conifold of rate ν. Define $\mathcal{M}_\nu$ to be the *moduli space* of all torsion-free G_2 structures on M, asymptotic *to the same* G_2 *cones at the ends, with the same rates* ν_i, modulo the action of diffeomorphisms which preserve this condition.

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There are natural maps $\Upsilon^k : H^k(M) \to \oplus_{i=1}^n H^k(\Sigma_i)$. Let $K_i(\lambda)$ be the space of *homogeneous* closed and coclosed 3-forms on C_i of rate λ.

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- *In the AC case with $\nu < -4$, the moduli space may be obstructed, and its virtual dimension* $\text{v-}\dim \mathcal{M}_\nu$ *is*

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- The proof uses the Lockhart–McOwen machinery of weighted Sobolev spaces and its associated Fredholm theory, plus new Hodge-theoretic results in this context, and other G_2 specific ingredients (surjectivity of Dirac operator, L^2 harmonic 1-forms are parallel, more ...)

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[4] Statements [2] and [3] will be true *in general* if certain conjectures about the spectrum of the Laplacian on forms are true for *all* compact strictly nearly Kähler 6-manifolds.

A new construction of compact G_2 manifolds

(which may possibly generalize to construct compact CS G_2 manifolds)

[Step 1] Construct an orbifold $\widehat{M}$

- Let $(N^6, g, \omega, \Omega, J)$ be a compact Calabi-Yau manifold admitting an *antiholomorpic isometric involution* τ:

$$\tau^*(g) = g, \qquad \tau^*(\omega) = -\omega, \qquad \tau^*(\Omega) = \overline{\Omega}, \qquad \tau^*(J) = -J.$$

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- Define $M^7 = N^6 \times S^1$. Then

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- Define $\sigma : M \to M$ by $\sigma(p, \theta) = (\tau(p), -\theta)$. Then σ is an involution of M such that $\sigma^*(\varphi) = \varphi$. The quotient space $\widehat{M} = M/\langle\sigma\rangle$ is a G_2 orbifold, with singularities locally of the form $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$.

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- The singular set $L^3 = A^3 \times \{\pm 1\}$, where $A^3 = \mathrm{Fix}(\tau)$ is a compact special Lagrangian submanifold of N^6, and L is totally geodesic in M.

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- We want to cut out a neighbourhood of the singular locus L in $\widehat{M}$ and glue in a noncompact smooth manifold to get a smooth compact 7-manifold $\widetilde{M}$, which hopefully will admit a *closed* G_2 *structure with small enough torsion*, to to apply Joyce's existence theorem.

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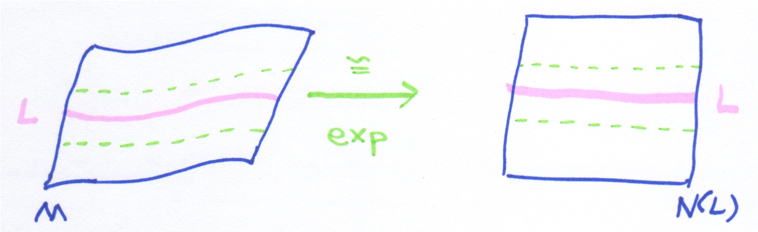

- The submanifold L is an associative submanifold. This implies that, *given a nonvanishing* 1-*form* α *on* L, the normal bundle $N(L)$ is actually a $\mathbb{C}^2$ bundle over L, and the above diffeomorphism descends to identify $\widehat{M}$ with $P = N(L)/\{\pm 1\}$ near L.

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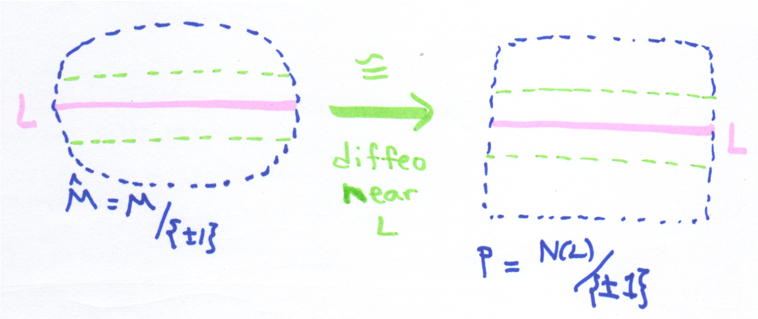

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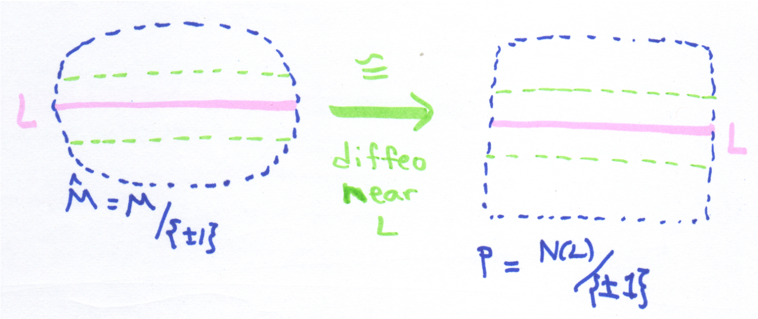

- The fibres of $P = N(L)/\{\pm 1\}$ are $\mathbb{C}^2/\{\pm 1\}$. We resolve P to $\widetilde{P}$ with a 'fibre-wise blow-up', replacing each fibre with $\widetilde{\mathbb{C}^2/\{\pm 1\}} \cong T^*S^2$.

- Each fibre T^*S^2 admits an $S^2 \times (0, \infty)$ family of *Eguchi-Hanson metrics* (holonomy $\mathrm{SU}(2)$ metrics) that are parametrized by a choice of complex structure on $\mathbb{R}^4 = \mathbb{H}$ (a unit vector in $\mathbb{R}^3$) and a scaling.

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- We can use α to construct a closed G_2 structure $\varphi_{\widetilde{P}}$ on $\widetilde{P}$ with small torsion, but for the torsion to have any chance of being small enough, it is necessary that $d\alpha = 0$ and $d^*\alpha = 0$. For now, let us assume that we have such a nowhere vanishing harmonic 1-form α.

[Step 3] Construct a compact smooth manifold $\widetilde{M}$

- We construct a compact smooth manifold $\widetilde{M}$ as follows. Far from the zero section, identify P with $\widehat{M}$ using the *exponential map*. Close to the zero section, identify P with $\widetilde{P}$ using the *resolution map*.

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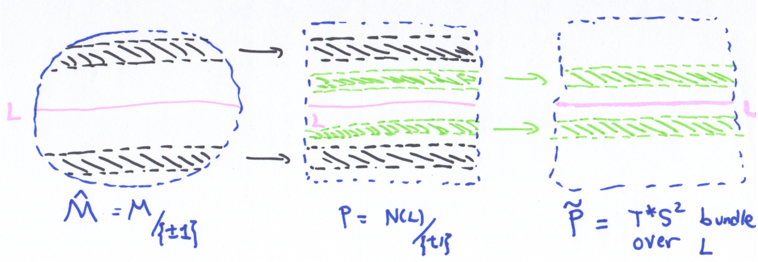

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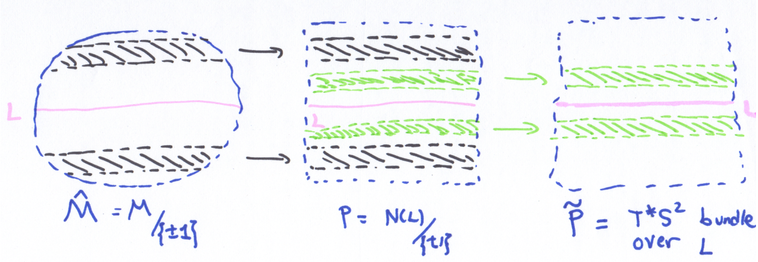

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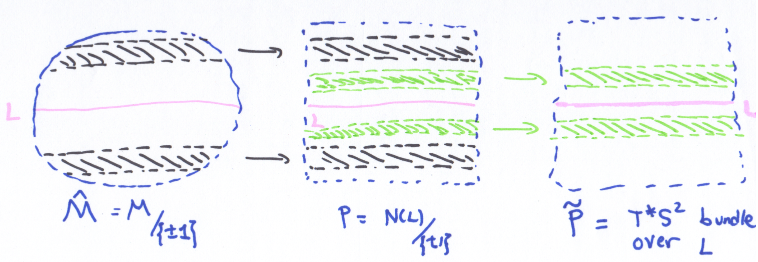

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- We want to construct a closed G_2 structure $\widetilde{\varphi}$ on $\widetilde{M}$ by interpolating between $\varphi_{\widehat{M}}$ and $\varphi_{\widetilde{P}}$ using $\overline{\varphi}$. We use the metric $\overline{g}$ of $\overline{\varphi}$ to measure the torsion of $\widetilde{\varphi}$, since we cannot compare $\widehat{M}$ and $\widetilde{P}$ directly.

- In fact, the G_2 structures $\overline{\varphi}$ on P and $\varphi_{\widetilde{P}}$ on $\widetilde{P}$ are *not closed*, so these have to be slightly modified, using smooth cut-off functions, to "closed versions" before we can construct $\widetilde{\varphi}$ on $\widetilde{M}$ by interpolation.

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- We need to perform two *corrections* to solve these problems.

[Step 4] 1^{st} correction: bend horizontal and vertical

- We begin with the easier correction: modifying the identification between M and $N(L)$ so that the canonical G_2 structure $\overline{\varphi}$ on $P = N(L)/\{\pm 1\}$ is close enough to φ_M on $\widehat{M}$.

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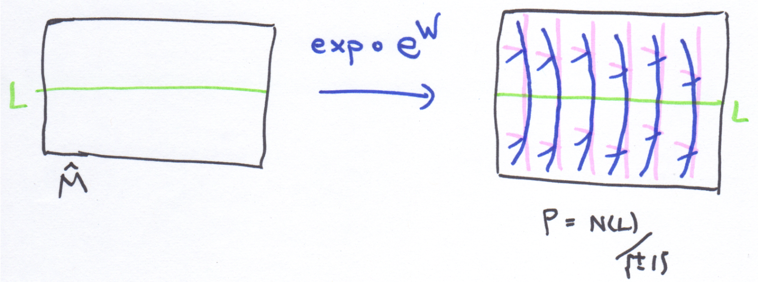

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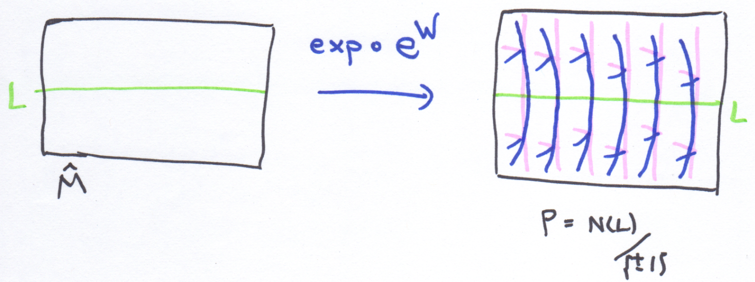

- These can in fact be chosen to make $\overline{\varphi}$ close enough to φ_M.

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- This is done using Lockhart–McOwen theory of Fredholm operators on noncompact manifolds with "well-behaved" geometry at infinity.

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- To do this, we need to be able to *solve an elliptic PDE on the noncompact Eguchi-Hanson space* T^*S^2 of the form

$$(d + d^*)\eta = \sigma$$

for some mixed-degree form σ given by the original G_2 structure $\varphi_{\widetilde{P}}$.
- This is done using Lockhart–McOwen theory of Fredholm operators on noncompact manifolds with "well-behaved" geometry at infinity.
- The theory says that such an equation can be solved if and only if σ has appropriate asymptotic behaviour at infinity, which it does.

Remarks on the construction

- Our construction is more general. We can take any G_2 manifold M admitting an involution σ such that $\sigma^*(\varphi) = \varphi$. Then $L = \mathrm{Fix}(\sigma)$ is an associative submanifold and everything proceeds as before.

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In general, we cannot guarantee that the submanifold L will admit a nowhere vanishing harmonic 1-form α. The metric on L is induced from the Calabi-Yau metric on N, which we do not know explicitly.

However, if N is near the "large complex structure limit" of the moduli space, from mirror symmetry arguments we expect it to contain a special Lagrangian torus that is *close to being flat*, so it will admit such 1-forms.

- Generically, a harmonic 1-form α on L has isolated zeroes. Then we can resolve M to $\widetilde{M}$ except for a finite number of singular points. In fact, near the singular points, $\widetilde{M}$ is *topologically* a cone over $\mathbb{CP}^3$.

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This is work in progress.

Happy birthday, Blaine!