Minimal surfaces as extremals of eigenvalue problems

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Cycles, Calibrations and Nonlinear PDE

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Plan of Lecture

The general lecture plan:

Part 1: Introduction to the problem; compact surfaces.

Part 2: Surfaces with boundary.

Part 3: Main theorems.

Joint project with A. Fraser, arXiv:1209.3789

Part 1: Introduction

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Basic Question:

Assuming we fix the area to be 1, what is the metric which maximizes the first eigenvalue?

Does such a metric exist?

If so what can we say about its geometry?

The Euler-Lagrange equation

If we assume that we have a smooth metric g which realizes the maximum, then it turns out that the multiplicity of the eigenvalue is always at least 3, and the maximizing condition implies that there are independent eigenfunctions $u_1, \ldots, u_{n+1}$ with the property that

$$\sum |u_i|^2 = 1 \text{ on } M$$

and the map

$$u = (u_1, \ldots, u_{n+1}) \text{ defines a conformal map to } \mathbb{S}^n \ (n \geq 2).$$

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This implies that the image surface

$$\Sigma = u(M) \text{ is a minimal surface in } \mathbb{S}^n$$

that is, the mean curvature of Σ is zero. Furthermore, the optimal metric g is a positive constant times the induced metric on Σ from $\mathbb{S}^n$.

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• For T^2 the flat metric on the 60^0 rhombic torus is the unique maximum by a result of N. Nadirashvili from 1996. It can be minimally embedded into $\mathbb{S}^5$ by first eigenfunctions.
(The Clifford torus in $\mathbb{S}^3$ is a critical point, but not a maximum.)

Known results II

- For the Klein bottle the extremal metric is smooth and unique but not flat. This follows from work of Nadirashvili (1996 existence of maximizer), D. Jacobson, Nadirashvili, and I. Polterovich (2006 constructed the metric), and El Soufi, H. Giacomini, and M. Jazar (2006 proved it is unique). The metric arises on a minimal immersion of the Klein bottle into $\mathbb{S}^4$.

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The case of the torus and the Klein bottle rely on a difficult existence theorem which was posed along with an outlined proof by Nadirashvili.

Asymptotics in the genus

For large genus one might hope to understand the asymptotic behavior. If we fix a surface M of genus γ, then we can define

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• P. Buser has shown that for $\gamma > 1$ there is a hyperbolic metric metric with $\lambda_1 \geq \frac{3}{16}$. This implies the lower bound

$$\lambda^*(\gamma) \geq \frac{3}{4}\pi(\gamma - 1).$$

Part 2: Surfaces with boundary

A minimal submanifold Σ^k in $\mathbb{S}^n$ is naturally the boundary of a minimal submanifold of the ball, the cone $C(\Sigma)$ over Σ.

The coordinate functions of $\mathbb{R}^{n+1}$ restricted to $C(\Sigma)$ are harmonic functions which are homogeneous of degree 1, so on the boundary they satisfy $\nabla_\eta x_i = x_i$ where η is the outward unit normal vector to $\partial C(\Sigma)$.

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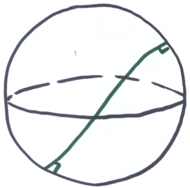
More generally a proper minimal submanifold Σ of the unit ball B^n which is orthogonal to the sphere at the boundary is called a *free boundary submanifold*. These are characterized by the condition that the coordinate functions are Steklov eigenfunctions with eigenvalue 1 (explanation to follow); that is, $\Delta x_i = 0$ in $\nabla_\eta x_i = x_i$.

It turns out that surfaces of this type arise as eigenvalue extremals.

Free boundary submanifolds

$$(M^k, \partial M^k) \longrightarrow (B^n, \partial B^n)$$



M minimal, meeting ∂B^n orthogonally along ∂M

$\uparrow$ $\qquad\qquad\qquad\qquad$ $\uparrow$

$H = 0$ $\qquad\qquad\qquad$ $\eta = \vec{x}$

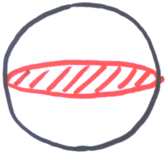
$M \subset \mathbb{R}^n$ minimal $\iff \Delta_M x_i = 0 \quad i = 1, \ldots, n$

$(x_1, \ldots x_n$ are harmonic$)$

M meets ∂B^n orthogonally $\iff \frac{\partial x_i}{\partial \eta} = x_i,\ i = 1, \ldots, n.$

Examples I

1) $M^k = D^k \subset B^n$ equatorial k-plane



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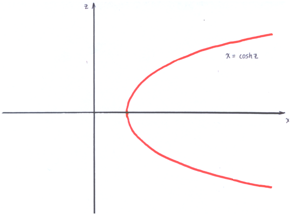

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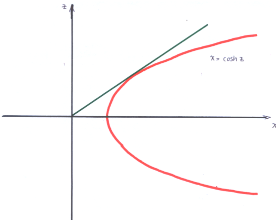

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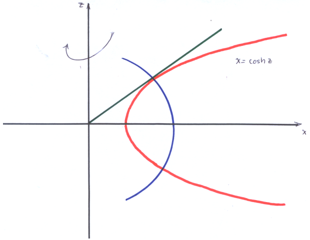

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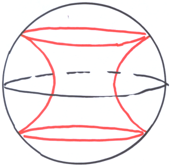

Examples II

3) We expect that there are arbitrarily high genus free boundary solutions with three boundary components in B^3 which converge to the union of the critical catenoid and a disk through the origin orthogonal to the axis.

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4) Critical Möbius Band

We think of the Möbius band M as $\mathbb{R} \times S^1$ with the identification $(t, \theta) \approx (-t, \theta + \pi)$. There is a minimal embedding of M into $\mathbb{R}^4$ given by

$$\varphi(t, \theta) = (2 \sinh t \cos \theta, 2 \sinh t \sin \theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta)$$

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For a unique choice of T_0 the restriction of φ to $[-T_0, T_0] \times S^1$ defines an embedding into a ball by first Steklov eigenfunctions.

We may rescale the radius of the ball to 1 to get the *critical Möbius band*.

Explicitly T_0 is the unique positive solution of $\coth t = 2 \tanh 2t$.

Steklov eigenvalues I

$(M, \partial M)$ Riemannian manifold

Given a function $u \in C^\infty(\partial M)$, let $\hat{u}$ be the harmonic extension of u:

$$\begin{cases} \Delta_g \hat{u} = 0 & \text{on } \ M, \\ \hat{u} = u & \text{on } \partial M. \end{cases}$$

The Dirichlet-to-Neumann map is the map

$$L : C^\infty(\partial M) \to C^\infty(\partial M)$$

given by

$$Lu = \frac{\partial \hat{u}}{\partial \nu}.$$

(non-negative, self-adjoint operator with discrete spectrum)

Eigenvalues $\sigma_0, \sigma_1, \sigma_2, , \sigma_3, \ldots$ (Steklov Eigenvalues)

Steklov eigenvalues II

Constant fcns are in the kernel of L
The lowest eigenvalue of L is zero, $\sigma_0 = 0$

The first nonzero eigenvalue σ_1 of L can be characterized variationally as:

$$\sigma_1 = \inf_{u \in C^1(\partial M),\ \int_{\partial M} u = 0} \frac{\int_M |\nabla \hat{u}|^2 \, dv_M}{\int_{\partial M} u^2 \, dv_{\partial M}}.$$

Example: B^m, $\quad \sigma_k = k, \quad k = 0, 1, 2, \ldots$
u homogeneous harmonic polynomial of degree k

$\sigma_1 = 1 \quad$ eigenspace $\quad x^1, \ldots, x^n$

An eigenvalue estimate I

Weinstock 1954 $\quad\Omega \subset \mathbb{R}^2$ simply connected domain

$$\sigma_1(\Omega)L(\partial\Omega) \leq 2\pi = \sigma_1(D)L(\partial D)$$
$$= \text{ only if } \Omega \text{ is a disk}$$

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Proof.

RMT $\Longrightarrow \exists$ proper conformal map of degree 1, $\varphi : \Omega \to D$

$\exists$ conformal $F : D \to D$ such that

$$\int_{\partial\Omega} (F \circ \varphi)\, ds = 0$$

An eigenvalue estimate II

i.e. WLOG, $\int_{\partial\Omega} \varphi \, ds = 0$. Then, for $i = 1, 2$,

$$\begin{aligned} \sigma_1 \int_{\partial\Omega} \varphi_i^2 \, ds &\leq \int_\Omega |\nabla \hat{\varphi}_i|^2 \, da \\ &\leq \int_\Omega |\nabla \varphi_i|^2 \, da \end{aligned}$$

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$$\sigma_1 L(\partial\Omega) \leq 2\, A(\varphi(\Omega)) = 2A(D) = 2\pi$$

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Payne, Hersch, Bandle, Schiffer, Escobar, Girouard-Polterovich . . .

Coarse upper bounds

The following result is a combination of bounds obtained with Fraser together with results of G. Kokarev.

Theorem: $(M^2, \partial M)$ *oriented Riemannian surface of genus* γ *with* k *boundary components. Then,*

$$\sigma_1 \, L(\partial M) \leq \min\{2\pi(\gamma + k), \; 8\pi[(\gamma + 3)/2]\}$$

The inequality is strict if $\gamma = 0$ *and* $k > 1$.

Note: $\gamma = 0$, $k = 1$ simply connected surface: Weinstock.

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<u>Question</u>: What is the sharp constant for annuli or other surfaces?

Part 3: Main theorems

In order to show that the critical catenoid achieves the maximum of $\sigma_1 L$ we must characterize it among the free boundary minimal annuli.

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Theorem A: *Assume that Σ is a free boundary minimal annulus in B^n such that the coordinate functions are first eigenfunctions. Then $n = 3$ and Σ is the critical catenoid.*

There is a corresponding result for the Möbius band.

Theorem B: *Assume that Σ is a free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions. Then $n = 4$ and Σ is the critical Möbius band.*

Main theorems on sharp bounds

Let

$$\sigma^*(\gamma, k) = \sup_g \sigma_1 L$$

where the supremum is over metrics on a surface of genus γ with k boundary components.

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We know $\sigma^*(0, 1) = 2\pi$. The next result identifies $\sigma^*(0, 2)$.

Theorem 1: *For any metric annulus M we have*

$$\sigma_1 L \leq (\sigma_1 L)_{cc}$$

with equality iff M is equivalent to the critical catenoid.
In particular,

$$\sigma^*(0, 2) = (\sigma_1 L)_{cc} \approx 4\pi/1.2.$$

Theorem 2: *The sequence $\sigma^*(0, k)$ is strictly increasing in k and converges to 4π as k tends to infinity. For each k a maximizing metric is achieved by a free boundary minimal surface Σ_k in B^3 of area less than 2π. The limit of these minimal surfaces as k tends to infinity is a double disk, and for large k, Σ_k is approximately a pair of nearby parallel plane disks joined by k boundary bridges.*

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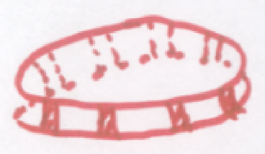
Here is a rough sketch of the surfaces for large k.



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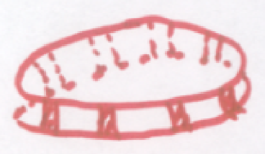

Corollary: *For every $k \geq 1$ there is an embedded minimal surface in B^3 of genus 0 with k boundary components satisfying the free boundary condition. Moreover these surfaces are embedded by first eigenfunctions.*

Maximizing metrics

The following result shows that a metric which maximizes $\sigma_1 L$ arises from a free boundary minimal surface in a ball.

Theorem: *Let M be a compact surface with nonempty boundary and assume that g is a metric on M for which $\sigma_1 L$ is maximized.*

Then the multiplicity of σ_1 is at least two, and there exists a proper conformal branched minimal immersion $\varphi : M \to B^n$ for some $n \geq 2$ by first eigenfunctions which is a homothety on the boundary.

The surface $\Sigma = \varphi(M)$ is a free boundary minimal surface.

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• Next we take a carefully chosen maximizing sequence and a weak* limit of the boundary measures. We then prove that first eigenfunctions give a branched conformal minimal immersion into the ball which is freely stationary. The regularity at the boundary then follows.

• Finally we observe that the metric on the boundary can be recovered from the map and it is smooth. Branch points do not occur on the boundary.

Proof of Theorem 1

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with equality iff M is equivalent to the critical catenoid.
In particular,

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• there exists a metric on the annulus with $\sigma_1 L = \sigma^*(0,2)$

• this maximizing metric arises from a free boundary minimal immersion of the annulus in the ball by first eigenfunctions

• by the uniqueness result this immersion is congruent to the critical catenoid

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• Σ_k does not contain the origin and is embedded and star shaped. This follows from the fact that the restrictions of the linear functions have no critical points on their zero set.

• The coarse upper bound implies that $A(\Sigma_k) \leq 4\pi$ and the star shaped property implies that each Σ_k is stable for variations which fix the boundary. Curvature estimates then imply uniform curvature bounds in the interior.

Proof of Theorem 2

- There is a subsequence of the Σ_k which converges in a smooth topology to a smooth limiting minimal surface Σ_∞ possibly with multiplicity. This limit must have multiplicity since otherwise the limit would be a smooth free boundary solution, and the convergence would be smooth up to the boundary contradicting the fact that $k \to \infty$.

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- The limit of $A(\Sigma_k)$ is equal to $2A(\Sigma_\infty) = 2\pi$. It follows that $\sigma^*(0, k) = \sigma_1(\Sigma_k)L(\partial\Sigma_k) = 2A(\Sigma_k)$ converges to 4π as claimed.

The Möbius band

Finally we show that the critical Möbius band uniquely maximizes $\sigma_1 L$. After some calculation one can see that $(\sigma_1 L)_{cmb} = 6\sqrt{6}\pi$.

Theorem: *For any metric on the Möbius band M we have*

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The proof follows the same steps as for the critical catenoid.

• we show existence of a smooth maximizing metric on the Möbius band

• this then gives an immersion into B^n by first eigenfunctions

• by the uniqueness result, this immersion is congruent to the critical Möbius band

Statement of Theorem A

Theorem A: *Assume that Σ is a free boundary minimal annulus in B^n such that the coordinate functions are first eigenfunctions. Then $n = 3$ and Σ is the critical catenoid.*

Theorem A: Proof outline

A multiplicity bound implies that $n = 3$.

We may assume that Σ is parametrized by a conformal harmonic map φ from $M = [-T, T] \times S^1$ with coordinates (t, θ).

The vector field $X = \frac{\partial \varphi}{\partial \theta}$ is then a conformal Killing vector field along Σ.

Goal: Show that X coincides with a rotation vector field of $\mathbb{R}^3$.

The key step in doing this is to show that the three components of X are first eigenfunctions.

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For functions or vector fields Y defined along Σ we consider the quadratic form Q defined by

$$Q(Y, Y) = \int_\Sigma \|\nabla Y\|^2 \, da - \int_{\partial\Sigma} \|Y\|^2 \, ds.$$

Assumption that $\sigma_1 = 1$ implies: if $\int_{\partial\Sigma} Y\, ds = 0$ then

$$Q(Y, Y) \geq 0$$

with equality $\iff$ the components of Y are first eigenfunctions.

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The quadratic form Q is also the second variation of $\frac{1}{2}E$ provided that Y is tangent to S^2 along $\partial\Sigma$.

Find a vector field Y such that $Q(Y, Y) \leq 0$ and with $\int_{\partial\Sigma}(X - Y)\, ds = 0$.

It would then follow that $Q(X - Y, X - Y) \leq 0$

and also, since $\int_{\partial\Sigma}(X - Y)\, ds = 0$, $Q(X - Y, X - Y) \geq 0$,

It would follow that the components of $X - Y$ are first e.f.

We are not able to find such vector fields directly, so we consider the second variation of area for normal variations.

Note that for free boundary solutions there is a natural Jacobi field given by $x \cdot \nu$. It is in the nullspace of the second variation form S given by:

$$S(\psi, \psi) = \int_{\Sigma} (\|\nabla \psi\|^2 - \|A\|^2 \psi^2) \, da - \int_{\partial \Sigma} \psi^2 \, ds$$

where A denotes the second fundamental form of Σ and we are considering normal variations $\psi \nu$ where ν is the unit normal vector of Σ. Note that this variation is tangent to S^2 along the boundary.

We can show by a subtle argument that for any $v \in \mathbb{R}^3$

$$S(v \cdot \nu, v \cdot \nu) \leq 0$$

This is not sufficient for the eigenvalue problem because the normal deformation does not preserve the conformal structure of Σ in general.

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The way we get around this problem is to consider adding a tangential vector field Y^t to so that $Y = Y^t + \psi\nu$ preserves the conformal structure and is tangent to S^2 along $\partial\Sigma$. This involves solving a Cauchy-Riemann equation with boundary condition to determine Y^t.

This problem is generally not solvable, but has a 1 dimensional obstruction for its solvability (because Σ is an annulus). We then get existence for ψ in a three dimensional subspace of the span of $\nu_1, \nu_2, \nu_3, x \cdot \nu$. We can then arrange the resulting conformal vector field Y to satisfy the boundary integral condition and we have

$$Q(Y, Y) = S(\psi, \psi) = S(v \cdot \nu, v \cdot \nu) \leq 0.$$