VOLUME ESTIMATES ON SETS OF POINTS AT WHICH THE REGULARITY SCALE IS SMALL

Jeff Cheeger

This talk is on joint work with Aaron Naber.

We describe a general method for improving estimates on Hausdorff dimension or Hausdorff measure of singular sets of elliptic or parabolic equations to estimates on the volumes of sets on which the "regularity scale" is small.

Applications include Einstein metrics, minimal hypersurfaces, harmonic maps, mean curvature flow and critical sets of solutions to linear homogeneous elliptic equations.

The key point in the proofs is the introduction of a quantitative stratification $\mathcal{S}^k_{\eta,r}$ and an estimate for the volume of $\mathcal{S}^k_{\eta,r}$.

This provides an quantitative replacement for iterated blow up arguments which enables one to work on a fixed scale.

The regularity scale.

Riemannian manifolds and their limits.

For riemannian manifolds, or more generally for for noncollapsed Gromov-Hausdorff limit spaces we define the *regularity scale* to be the curvature radius

$$r_{|Rm|}(x) := \max\{0 < r < 1 \,|\, \sup_{B_r(x)} r^2 \cdot |Rm| \leq 1\}\,,$$

where, we put $r_{|Rm|}(x) = 0$ if the space is not smooth in any neighborhood of x.

Harmonic maps.

For $f : M \to N$ a measurable map, we define the *regularity scale* $r_f(x)$ by

$$r_f(x)$$
$$:= \max\{0 < r < 1 \,|\, \sup_{B_r(x)} \leq r \cdot |\nabla f| + r^2 |\nabla^2 f| \leq 1\}\,,$$

where we put $r_f(x) = 0$ if f is not C^2 in a neighborhood of x.

Minimal hypersurfaces.

Let I be a minimizing hypersurface taken to be either an integral rectifiable current or a varifold in a smooth manifold M.

Let A denote the second fundamental form and fix a positive integer $N > 0$.

Put $r_I^N(x) = 0$ if I is not the union of at most N connected graphical C^2 submanifolds in a neighborhood of x.

Otherwise, let $r_{0,I}^N$ denote the sup of those r such that $I \cap B_r(x)$ is such a union.

Define the regularity scale $r_I^N(x)$ by

$$r_I^N(x) := \max\{0 < r \le r_{0,I}^N(x) \mid \sup_{B_r(x)} r \cdot |A| \le 1\}\,.$$

Uniform control on rescaled balls.

In each instance, the regularity scale at x is the sup of $r \leq 1$ such that when the data on $B_r(x)$ is rescaled to unit size, then the data on the rescaled ball is uniformly bounded in norm by 1.

Thus, the regularity scale at x provides much more information than a bound on the quantities,

$$|Rm(x)|,\ |\nabla f(x)|,\ |A(x)|\,.$$

In particular,

$$|Rm(x)| \leq r^{-2}_{|Rm|}(x)\,,$$

etc.

Riemannian manifolds.

Theorem A. Let Y^n denote the Gromov-Hausdorff limit of a sequence of Einstein manifolds satisfying

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,, \tag{1}$$

$$\mathrm{Vol}(B_1(p_i)) \geq v > 0\,. \tag{2}$$

Then:

1. For all $0 < q < 1$, there exists $C = C(n, v, q)$ such that

$$⨍_{B_1(p)} |Rm|^q \leq ⨍_{B_1(p)} r_{|Rm|}^{-2q} \leq C\,.$$

2. If M^n is Kähler, then for all $0 < q < 2$, there exists $C = C(n, v, q)$ such that

$$⨍_{B_1(p)} |Rm|^q \leq ⨍_{B_1(p)} r_{|Rm|}^{-2q} \leq C\,.$$

Equivalently, for q as in 1., 2., above:

$$\mathrm{Vol}(\{x \in B_1(p) \,|\, r_{|Rm|}(x) \leq r\}) \leq C \cdot r^{2q}\,.$$

3. If M^n is Kähler and satisfies

$$\int_{B_2(p)} |Rm|^q \leq \Lambda\,,$$

then for $C = C(n, v, q, \Lambda)$, we have

$$\mathrm{Vol}(\{x \in B_1(p)\,|\,r_{|Rm|}(x) \leq r\}) \leq C \cdot r^{2q}\,.$$

Remark. Theorem A improves the following earlier results (of Cheeger, Colding, Tian) for the Hausdorff dimension of $\mathcal{S}$ the singular set of the Gromov-Hausdorff limit of a sequence of Einstein manifolds satisfying the above bounds:

1. $\dim \mathcal{S} \leq n - 2$.

2. $\dim \mathcal{S} \leq n - 4$.

3. $\mathcal{H}^{n-2q}(\mathcal{S} \cap B_1(p)) \leq C\,$.

Remark. Corresponding improvements occur in the applications to harmonic maps, minimal hyper-surfaces and mean curvature flow.

Harmonic maps.

Theorem B. Let $B_2(0^n) \subset \mathbf{R}^n$ and let $f : B_2(0^n) \to N$ denote a minimizing harmonic map with

$$\int_{B_1(p)} |\nabla f|^2 \leq \Lambda\,.$$

Then for all $0 < q < 3$, there exists $C = C(N, \Lambda, q)$ such that

$$\fint_{B_1(p)} |\nabla f|^q + |\nabla^2 f|^{q/2} \leq \fint r_f^{-q} \leq C\,.$$

Equivalently, for all $q < 3$,

$$\mathrm{Vol}(\{x \in B_1(p) \,|\, r_f(x) \leq r\}) \leq C \cdot r^q\,.$$

Remark. With suitable modification, $\mathbf{R}^n$ can be replaced by M^n.

Minimal hypersurfaces.

Let $I = I^{n-1}$ denote a minimal hypersurface in $\mathbf{R}^n$ and A its second fundamental form.

Theorem C. Assume the mass bound

$$|I| \leq \Lambda\,.$$

Then for every $0 < q < 7$ there exists $C(n, \Lambda, q) > 0$ and a positive integer $N = N(n, \Lambda)$, such that

$$⨍_{B_1(p)\cap I} |A|^q \, d|I| \leq ⨍_{B_1(p)\cap I} (r_I^N)^{-q} \, d|I| \leq C\,.$$

Equivalently,

$$\mathrm{Vol}(\{x \in B_1(p) \,|\, r_f^N(x) \leq r\}) \leq C \cdot r^q\,.$$

Remark. With suitable modifications, $\mathbf{R}^n$ can be replaced by M^n.

General methodology.

We will illustrate the general methodology by indicating the proofs of 1. and 2. of Theorem A in the Einstein case.

In the harmonic map and minimal hypersurface cases, the titles of the transparencies which follow would remain unchanged.

Only the precise notions of "energy density", "cone", "splitting off a Euclidean factor" and "close" change appropriately.

Recall: 1. of Theorem A states that for all $0 < q < 1$ there exists $C = C(n, v, q)$ such that

$$\mathrm{Vol}(\{x \in B_1(p) \,|\, r_{|Rm|}(x) \leq r\}) \leq C \cdot r^{2q}\,. \tag{3}$$

2. States that in the Kähler-Einstein case, (3) holds for all $0 < q < 2$.

Remark. Conjecturally, (3) also holds for all $q < 2$ in the real case.

Setting and notation.

Let d_{GH} denote the Gromov-Hausdorff distance between metric compact spaces.

So in the riemannian case, “close” means *close in the Gromov-Hausdorff sense*.

From now on, our underlying space is either a riemannian manifold satisfying (1) and (2), or the Gromov-Hausdorff limit Y^n, of a sequence of such manifolds.

When we come to the ϵ-regularity theorem, we will also assume Einstein (respectively Kähler-Einstein).

Let denote the metric cone with cross-section $C(Z)$ and vertex z^*.

So in this case, “cone” means *metric cone* and “splitting off a Euclidean factor” means an *isometric splitting*

$$\mathbf{R}^{k+1} \times C(Z)\,.$$

Quantitative stratification.

Let $\underline{0}$ denote the origin in $\mathbf{R}^{k+1}$.

For all $\eta > 0$, $0 < r \leq 1$, we define the k-th **effective singular stratum**:

$$\mathcal{S}^k_{\eta,r} := \{y \,|\, d_{GH}(B_s(y), B_s((\underline{0}, z^*)) \geq \eta s,$$

$$\text{for all } \mathbf{R}^{k+1} \times C(Z) \text{ and all } r \leq s \leq 1\}\,.$$

Clearly,

$$\mathcal{S}^k_{\eta,r} \subset \mathcal{S}^{k'}_{\eta',r'} \qquad (\text{if } k' \leq k,\ \eta' \leq \eta,\ r \leq r')\,.$$

Also, for Gromov-Hausdoff limit spaces, if $\mathcal{S}^k$ denotes the kth stratum of the standard stratification of the singular set $\mathcal{S}$, then

$$\bigcup_{\eta} \bigcap_{r} \mathcal{S}^k_{\eta,r} = \mathcal{S}^k\,.$$

In the smooth case, $\mathcal{S} = \emptyset$, but the $\mathcal{S}^k_{\eta,r}$ need not be empty.

Volume estimate for $\mathcal{S}^k_{\eta,r}$.

Theorem D. Let Y^n denote the Gromov-Hausdorff limit of a sequence of manifolds satisfying

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,,$$

and for all $x \in M_i^n$,

$$\mathrm{Vol}(B_1(x)) \geq v > 0\,.$$

Then for all $\eta > 0$, there exists $c(n,v,\eta) > 0$ such that for all $y \in Y^n$,

$$\mathrm{Vol}(\mathcal{S}^k_{\eta,r} \cap B_{\frac{1}{2}}(y)) \leq c(n,v,\eta) r^{n-k-\eta}.$$

Equivalently, $\mathcal{S}^k_{\eta,r} \cap B_{\frac{1}{2}}(y)$ can be covered by at most

$$c(n,v,\eta) r^{-(k+\eta)}$$

balls of radius r.

ϵ-regularity theorem.

Suppose $y \notin \mathcal{S}^k_{\eta,r}$.

Then for some s with $r \leq s \leq 1$, there exists $C(Z)$ such that for z^* the vertex of $C(Z)$ and $\underline{0}$ the origin in $\mathbf{R}^{k+1}$,

$$d_{GH}(B_s(y), B_s((\underline{0}, z^*)) < \eta s \,. \qquad (4)$$

Theorem E. (Cheeger, Colding, Tian) There exists $\underline{\eta} = \underline{\eta}(n, v) > 0$ such that in the Einstein case, if (4) holds with $k + 1 = n - 1$ and $\eta \leq \underline{\eta}$, then

$$r_{|Rm|}(y) \geq \frac{1}{2}s \geq \frac{1}{2}r \,. \qquad (5)$$

In the Kähler-Einstein case, the same is true for $k + 1 = n - 3$.

Proof (of Theorem A): 1. and 2. of Theorem A follow immediately by combining Theorem D and Theorem E.

Proof of Theorem D; top down.

Fix $\eta > 0$.

To obtain the general case, it clearly suffices to restrict attention to r of the form γ^j, where $\gamma = \gamma(\eta) < 1$, and an appropriate choice turns out to be

$$\gamma = c_0^{-\frac{2}{\eta}}, \qquad (6)$$

where in particular, $c_0 = c_0(n)$ is such that $\gamma < \eta$.

We will define a collection of sets $\{\mathcal{C}^k_{\eta,\gamma^j}\}$, such that:

Lemma. There exists $c_1 = c_1(n) \geq c_0$, $K = K(n, v, \eta)$, such that for every $j \in \mathbf{Z}_+$,

1. The set $\mathcal{S}^k_{\eta,\gamma^j} \cap B_1(\underline{x})$ is contained in the union of $\leq j^K$ nonempty sets $\mathcal{C}^k_{\eta,\gamma^j}$.

2. Each set $\mathcal{C}^k_{\eta,\gamma^j}$ is the union of at most $(c_1\gamma^{-n})^K \cdot (c_0\gamma^{-k})^{j-K}$ balls of radius γ^j.

By volume comparision,

$$\mathrm{Vol}(B_{\gamma^j}(x)) \le c_2(n)(\gamma^j)^n\,. \qquad (7)$$

From (7) and, 1., 2., we get the volume bound

$$j^K \cdot \Big[(c_1\gamma^{-n})^K \cdot (c_0\gamma^{-k})^{(j-K)}\Big] \cdot c_2(\gamma^j)^n\,. \quad (8)$$

By (6) we have $c_0^j = (\gamma^j)^{\frac{\eta}{2}}$.

Also $j^K \le c(n,v,\gamma)(\gamma^j)^{-\frac{\eta}{2}}$.

Substituting these two in (8) and recalling $\gamma = \gamma(\eta)$ gives

$$\mathrm{Vol}(\mathcal{S}^k_{\eta,\gamma^j} \cap B_{\frac{1}{2}}(y)) \le \underline{c}(n,v,\eta) \cdot (\gamma^j)^{n-k-\eta}\,.$$

Modulo proving 1. and 2. of the lemma, this completes the proof of Theorem D.

Almost conicality apart from a definite number of scales.

Proposition 10 below, whose proof will be given later, is a quantitative expression of the fact that for noncollapsed limit spaces satisfying (1), (2), every tangent cone is a metric cone.

Fix $0 < \epsilon = \epsilon(n, \eta) << \eta$ sufficiently small, to be specified below in the cone-splitting lemma.

Associate to each x an ∞-tuple $T^\infty(x)$ whose i-th entry is 0 if there exists a metric cone $C(W)$ with vertex w^* such that

$$d_{GH}(B_{\gamma^{-n}.\gamma^i}(x), B_{\gamma^{-n}.\gamma^i}(w^*)) \le \epsilon \cdot \gamma^i \, . \quad (9)$$

Otherwise, the i-th entry of $T^\infty(x)$ is 1.

Let $|T^\infty(x)|$ denote number entries of $T^\infty(x)$ that are equal to 1.

Proposition 10. For all x

$$|T^\infty(x)| < K(n, v, \eta) < \infty \, .$$

The decomposition lemma and item 1.

Let $T^j(x)$ be the j-tuple consisting of the first j entries of $T^\infty(x)$.

For each j-tuple T^j whose entries consist of 0's and 1's define

$$E_{T^j} = \{x \in B_1(\underline{x}) \,|\, T^j(x) = T^j\}.$$

Proposition 10 immediately implies:

Lemma. (Decomposition) At most j^K of the sets E_{T^j} are nonempty.

Each set $\mathcal{C}^k_{\eta,\gamma^j} = \mathcal{C}^k_{\eta,\gamma^j}(E_{T^j})$ in item 1. is a covering by balls of radius γ^j of some nonempty set E_{T^j}.

Remark. The fact that we are able to consider each set E_{T^j} individually, vastly simplifies the geometry, leading to the cardinality bound on the number of balls as in item 2.

Coverings of the E_{T^j} and item. 2.

We fix T^j, and inductively construct coverings for $i \leq j$ of $E_{T^j} \cap B_1(\underline{x})$..

At a bad scale, where the i-th entry of $T^j(x)$ is 1, we take a standard recovering of $B_{\gamma^{i-1}}(x) \cap E_{T^j}$ by:

$$\sim \gamma^{-n} \textit{ balls of radius } \gamma^i.$$

Claim: At a good scale, where the i-th entry of $T^j(x)$ is 0, we can recover $B_{\gamma^{i-1}}(x) \cap E_{T^j}$ by:

$$\sim \gamma^{-k} \textit{ balls of radius } \gamma^i.$$

Note that from the bound

$$|T^j(x)| \leq K\,,$$

if E_{T^j} is nonempty, *there are at most K bad scales*.

This gives item 2.

The *Claim* is a consequence of the **Cone-splitting principle** which we now explain.

Cone-splitting.

Let X be a metric space and let $x_1, x_2 \in X$ be distict points, $x_1 \neq x_2$.

Suppose there exist metric cones, $C(W_i)$ with vertices w_i^* and isometries

$$F_i : (X, x_i) \to (C(W_i), w_i^*) \qquad (i = 1, 2)\,.$$

Then there is a metric cone $C(Z)$ with vertex z^* and an isometry

$$I : X \to \mathbf{R} \times C(Z)\,,$$

such that

$$I(x_i) \in \mathbf{R} \times \{z^*\} \qquad (i = 1, 2)\,.$$

Remark. In particular, if X can be given the structure of a cone in two different ways, it must split off a line isometrically.

Remark. The following lemma, which implies the *Claim*, relies on a quantitative version of the Cone-splitting principle.

Quantitative cone-splitting.

Suppose $x \in \mathcal{S}^k_{\eta,\gamma^j}$, that i-th entry of both $T^j(x)$ of and $T^j(x')$ is 0 and that $x' \in B_{\frac{1}{2}\gamma^{i-1}}(x)$.

Lemma (Cone-splitting) If $\epsilon = \epsilon(n,\eta)$ in (9) is chosen sufficiently small, then there exists a cone

$$\mathbf{R}^\ell \times C(Z) \qquad (\ell \leq k)\,,$$

with vertex $(0^\ell, z^*)$, and an $\frac{1}{10} \cdot \gamma^i$ Gromov-Hausdorff equivalence,

$$I : B_{\gamma^{i-1}}(x) \to B_{\gamma^{i-1}}((0^\ell, z^*)))\,,$$

such that

$$I(x) = (0^\ell, z^*)\,,$$

and $I(x')$ is contained in the $\frac{1}{10}\gamma^i$ tubular neighborhood of the isometric factor

$$\mathbf{R}^\ell \times \{z^*\} \qquad (\ell \leq k)\,.$$

Indication of proof of cone-splitting.

By a contradiction argument and an induction argument, one reduces the assertion,

$$I(x') \in T_{\frac{1}{10}\gamma^i}((\mathbf{R}^\ell \times \{z^*\})\,,$$

to the case $\epsilon = 0$, $\ell = k$.

Thus, I is an isometry and in addition, there is a cone $C(W')$ and an isometry

$$F' : B_{\gamma^{i-1}}(x') \to B_{\gamma^{i-1}}(w'^*)\,.$$

By the *Cone-splitting principle*, unless

$$I(x') \in \mathbf{R}^k \times \{z^*\}\,,$$

the cone $\mathbf{R}^\ell \times C(Z)$ would split off an additional $\mathbf{R}$ factor.

This would contradict $x \in \mathcal{S}^k_{\eta,\gamma^i}$.

Monotone bounded energy density.

By (2) and the Bishop-Gromov inequality,

$$0 \leq E_r(x) = \log \frac{\mathrm{Vol}_{-1}(r)}{\mathrm{Vol}(B_r(x))} \uparrow \, .$$

where $\mathrm{Vol}_{-1}(r)$ denotes the volume of the unit ball in n-dimensional hyperbolic space.

The noncollapsing condition (2) implies that

$$E_1(x) \leq \Lambda(n, v)\, ,$$

where $\Lambda = \log \frac{\mathrm{Vol}_{-1}(r)}{v}$.

If the nondecreasing function $E_r(x)$ is constant on an interval $\frac{1}{2}r_1 \leq r \leq r_1$, then $B_{r_1}(x)$ is isometric to a $B_{r_1,}(\underline{x})$, in a warped product with warping function $\sinh r$.

Almost rigidity.

For $r \le \psi(n,\epsilon)$ sufficiently small, $B_r(\underline{x})$ is ϵr Gromov-Hausdorff close to a ball $B_r(w^*)$ in a metric cone $C(W)$, with vertex w^*.

Theorem (Cheeger-Colding) There exists $\delta = \delta(n,\epsilon)$ such that if $r \le \psi(n,\epsilon)$ and

$$E_{2r}(x) - E_{\frac{1}{2}r}(x) \le \delta\,,$$

then for some metric cone $C(W)$ with vertex w^*,

$$d_{GH}(B_r(x), B_r(w^*)) \le \epsilon r\,.$$

Proof of Proposition 10; $|T^\infty(x)| \le K$.

Let $\epsilon = \epsilon(n,\eta)$ be as in in (9).

By the theorem on the previous transparency, there exists $\delta = \delta(n,\epsilon)$, $\psi = \psi(n,\epsilon) \le \gamma^n$, such that if

$$2\gamma^i \le \psi\,, \tag{10}$$

$$E_{2\gamma^{-n}.\gamma^i}(x) - E_{\frac{1}{2}\gamma^{-n}.\gamma^i}(x) \le \delta\,, \tag{11}$$

then for some cone $C(W)$ with vertex w^*,

$$d_{GH}(B_{\gamma^{-n}.\gamma^i}(x), B_{\gamma^{-n}.\gamma^i}(w^*)) \le \epsilon \cdot \gamma^i\,. \tag{12}$$

Since

$$E_1(x) \le \Lambda(n,v)\,,$$

by Markov's inequality, there are at most $K(n,\eta)$ values of i for which (10), (11) fail to hold.

For the remaining values of i, (10), (11) hold, and hence, (12) holds as well.

This proves Proposition 10.