Perspectives on Elliptic PDE's

joint work with Blaine Lawson

Non Linear, Degeneracies welcomed, uniformly elliptic, convex.

Our initial interest (2009 paper) was to try and emulate, for other calibrations, some of the theory of plurisubharmonic functions in several complex variables.

\( \phi \) calibration, \( G(\phi) \) Grassmannian of \( \phi \)-planes

(Old) Geometric Side: \( \phi \)-submanifolds

(New) Analytic Side: Functions with the property that restrictions to \( \phi \)-planes are \( \Delta \)-subharmonic.
We were successful—maybe too successful.

There was something disturbing, that became clearer over time.

Nothing much was used about calibrations.

An operator \( d^\phi u = \nabla u \perp \phi \) was used to define \( \phi \)-plurisubharmonic by requiring

\[
(dd^\phi u)(W) \geq 0 \quad \forall W \in \mathcal{G}(\phi).
\]

Theorem \( u \phi \)-plurisubharmonic, and
\[
\begin{cases}
(u \text{ smooth}) & M \phi \text{-submanifold} \\
(\phi \text{-parallel}) & u|_M \text{ is } \Delta_M \text{-subharmonic}.
\end{cases}
\]

Proof: 1) \( (dd^\phi u)(W) = \text{trace}_W \text{ Hess}_u \quad \forall W \in \mathcal{G}(\phi) \)

2) \( M \phi \text{-submanifold} \Rightarrow M \text{-minimal} \Rightarrow \)

\[
\text{trace}_{TM} \text{ Hess}_u = \Delta_M u \quad \text{on } M.
\]
An approach bypassing the calibration $\phi$

Suppose $G$ is a closed subset of $G(p,\mathbb{R}^n)$, the Grassmannia of $p$-planes in $\mathbb{R}^n$ (Take $G = G(\phi)$ if $\phi$ is calibration).

Define $u$ to be $G$-plurisubharmonic if

$$tr_w Du \geq 0 \quad \forall W \in G.$$ 

This non-linear inequality is now the focus.

**Theorem** If $u$ is $G$-plurisubharmonic, and $M$ is a $G$-submanifold which is minimal (for example any affine $G$-plane), then $u|_M$ is $\Delta M$-subharmonic.

√ Converse.

Proof: 2nd half of step 2.

New Objective: Understand
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An introduction to potential theory in calibrated geometry, Amer. J. Math. 131 no. 4 (2009), 893-944.
Duality of positive currents and plurisubharmonic functions in calibrated geometry, Amer. J. Math. 131 (2009), 1211-1239.
Plurisubharmonicity in a general geometric context, Geometry and Analysis 1 (2010), 363-401.

To Appear:

The foundations of p-convexity and p-plurisubharmonicity in riemannian geometry, Indiana Univ. Math. J.
Gårding’s theory of hyperbolic polynomials, Communications in Pure and Applied Math.
Existence, uniqueness and removable singularities for nonlinear partial differential equations in geometry, to appear in “Surveys in Geometry”.

Preprints:

Lagrangian plurisubharmonicity and convexity in symplectic geometry.
Removable singularities for nonlinear subequations.
Characterizing the strong maximum principle.
Convex subequations - the strong Bellman Principle.
Aspects of nonlinear potential theory.
Pure Second Order Constant Coefficients

I. Definitions "subequation", "subharmonic". (subsolution)

II. Some Elementary Properties of Subharmonics.

III. When does the Maximum Principle hold?

IV. An Example: Subaffine Functions.

V. When does the Strong Maximum Principle hold?

VI. What is an "Equation"? The "Dual Subequation".

VII. Monotonicity

VIII. The appropriate notion of boundary convexity (omitted).

IX. The Dirichlet Problem

Two final topics where general subequations (rather than pure 2nd order constant coefficient) will be included.

X. Restriction

XI. An Almost Everywhere Theorem and Additions.
I. Definitions

It is clear what we want for \( u \) smooth. Given a closed subset \( F \) of \( \text{Sym}^2(\mathbb{R}^n) \) define \( F \)-subharmonic by requiring:

\[ D_x^2 u \in F \quad \forall x \in X \quad (X \text{open } \mathbb{R}^n). \]

More generally consider \( u \in \text{USC}(X) \)

A test function \( \phi \) for \( u \) at \( x \) is a smooth function defined near \( x \) with

\[ u \leq \phi \text{ near } x \]

\[ = \text{ at } x \]

Viscosity Defn of \( F \)-subharmonic

Require \( D_x^2 \phi \in F \) \( \forall \) test functions \( \phi \) for \( u \) at \( x \), \( \forall x \in X \).

Let \( F(X) \) denote the space of all \( F \)-subharmonic functions on \( X \).
II. Some Elementary Properties of $F$-subharmonic Functions

(A) (Maximum Property) If $u, v \in F(X)$, then $w = \max\{u, v\} \in F(X)$.

(B) (Coherence Property) If $u \in F(X)$ is twice differentiable at $x \in X$, then $J^2_uu \in F_x$.

(C) (Decreasing Sequence Property) If $\{u_j\}$ is a decreasing ($u_j \geq u_{j+1}$) sequence of functions with all $u_j \in F(X)$, then the limit $u = \lim_{j \to \infty} u_j \in F(X)$.

(D) (Uniform Limit Property) Suppose $\{u_j\} \subset F(X)$ is a sequence which converges to $u$ uniformly on compact subsets to $X$; then $u \in F(X)$.

(E) (Families Locally Bounded Above) Suppose $\mathcal{F} \subset F(X)$ is a family of functions which are locally uniformly bounded above. Then the upper semicontinuous regularization $v^*$ of the upper envelope

$$v(x) = \sup_{f \in \mathcal{F}} f(x)$$

belongs to $F(X)$. 
To good to be true! But no mistakes in the elementary proofs can be found.

The explanation:

If $u$ is smooth then:

$\varphi(y) = u(y)$ is a test function for $u$ at $x$, and

more generally $\varphi(y) = u(y) + \frac{1}{2} \langle A(y-x), y-x \rangle$ is a test function for $u$ at $x$

for any $A \geq 0$.

Thus $D^2_x \varphi = D^2_x u + A$ for all $A \geq 0$

must belong to $F$. Said differently,

(P) $D^2_x u + \delta F$ “positivity”,

where $\delta F \equiv \{ A : A \geq 0 \}$, must hold.

Otherwise, the quadratic $4w = \frac{1}{2} \langle Bx, x \rangle$ will

not be $F$-subh. even though

$D^2_x u = B \in F$.

Note all the “elementary properties” are true when $F(X) = \emptyset$. 
Definition: A closed subset $F \subset \text{Sym}^2(\mathbb{R}^n)$ satisfying positivity (very weak form) of ellipticity

$$F + P \subset F$$

will be called a subequation.

For a subequation $F$ and a $C^2$-function $u$ on $X$

$$u \in F(X) \iff \mathcal{D}_x u \in F \quad \forall x \in F.$$

Examples could take up the remainder of the lecture.

$P$ is a subequation, but requires proof.

$$u \in P(X) \iff u \text{ is (locally) convex}.$$  

$P$ is arguably the most basic subequation.
Side Remark:

The elementary results listed above illustrate the power of the viscosity approach to non-linear problems. However, it is difficult to prove results involving addition. But, they can be formulated easily from our set point of view:

If $F$, $G$ subequations and $H = F + G$ is closed then $H$ is a subequation. Hence one can ask:

$$u \in F(x), v \in G(x) \Rightarrow u + v \in H(x) ?$$

This is an important question and non-trivial even for $\Delta = \{ A : \text{tr} A \geq 0 \}$.

Note: If $F = \{ A : f(A) \geq 0 \}$, $G = \{ B : g(B) \geq 0 \}$ finding $h$ with $F + G = \{ C : h(C) \geq 0 \}$ is not natural or easy in examples.
Contrast with Distributions

Distribution $u$

1) Can take derivatives (the space is “big enough”)
2) $u \geq 0 \implies u$ is non-negative measure (“not too big”)

Because of 2), linear inequalities $Lu \geq 0$ are meaningful, for subacts $F \in \text{Sym}^2(\mathbb{R}^n)$ which are convex distributionally $F$-subh makes sense.

Theorem (2010) If $F$ convex subequation involving all the variables in $\mathbb{R}^n$ then:

$$ F_{\text{dist}}(X) \iff F(X) \text{ isomorphic}.$$

More generally, variable coeff. etc (in preprint 2012).
Some Examples of Subequations
(Tip of the Iceberg!)

\[ n = 2 \]

\[ \lambda_2(Du) \quad E \quad F \text{ On - invariant} \]
\[ \lambda_1(Du) \quad F \text{ satisfies positivity} \quad \iff E \text{ is } \mathbb{R}_+ \times \mathbb{R}_+ \text{ monotone} \]

\[ F_c : \text{ trace } \arctan(Du) \geq c \text{ is a subequation} \]
(The subequation for functions \( u \) governing special Lagrangian graphs.)

\[ F_c \text{ need not be convex} \]
\[ F_c \text{ is not uniformly elliptic} \]

Horizontal/Vertical Convexity \( F : \frac{\partial^2 u}{\partial x_1^2} \geq 0 \text{ or } \frac{\partial^2 u}{\partial x_2^2} \geq 0 \)
III. For Which Subequations \( F \) does the Maximum Principle hold?

The \( (\text{MP}) \) holds for \( F \) if

for all \( K \) compact \( C \times \text{open } \mathbb{C}^n \)

\( u \in F(X) \)

\( (\text{MP}) \)

\[ \sup_{K} u \leq \sup_{J K} u \]

Consider the subset \( \text{MP}(X) \subset \text{CUSC}(X) \) of functions \( u \) for which \( (\text{MP}) \) holds for all \( K \subset X \).

\( \text{MP}(X) \) is not determined by local properties and hence not by a subequation.

Example:

\[ \begin{array}{c}
\text{Example: } \begin{cases}
\text{u} \\
\end{cases}
\end{array} \]

\[ \text{However, } u \in \text{MP}(X) \iff u \text{ is "sub" the constants } C. \]

Before answering the question we turn to an important subequation.
Subaffine Functions

The definition is: subaffine $\equiv$ "sub" the affines.

Defn. $u \in \text{USC}(X)$ is subaffine if $\forall K \text{opt } CX$
$u \preceq a \text{ on } 2K \Rightarrow u \preceq a \text{ on } K \ \forall a(x) = p \cdot x + c$
affine.

Note: $u$ subaffine $\Rightarrow$ $u$ "sub" the constants $\iff$ (MP)

Theorem (2008) Let $\tilde{\mathcal{P}} \equiv \{ A : \lambda_{\text{max}}(A) \geq 0 \}$
(which is a subequation).

$u$ is subaffine $\iff u \in \tilde{\mathcal{P}}(X)$.

Cor. Being subaffine is a local condition.

Theorem (2008) The (MP) holds for a subequation $F$

$\iff F \subset \tilde{\mathcal{P}}$ (the subequation $\tilde{\mathcal{P}}$ is "universal")
for (MP)

$\iff 0 \not\in \text{Int}F$ (provides an easy test)

The (MP) fails for $F$

$\iff -\varepsilon 1 \times 1^2 \text{ is } F\text{-subh. for some } \varepsilon > 0$
(the function $-\varepsilon 1 \times 1^2$ is a "universal" example)
V. When Does the Strong Maximum Principle Hold?

\[ u \in \text{USCC}(X) \quad \forall K \text{cpt. } X \text{ connected} \]

(SMP) \[ u(x) = \sup_{K} u \quad \text{some } x \in \text{Int } K \implies u \text{ constant} \]

First note:

\[ 0 \in \text{Int } F \implies (MP) \text{ fails } \implies (SMP) \text{ fails} \]

Relatively easy to show

\[ 0 \notin F \implies (SMP) \text{ holds} \]

Left with the borderline case \( 0 \in \partial F \).

Recall the formula for the second derivative of a radial function:

\[ D_x^2 \psi(\lambda x) = \frac{\psi'(\lambda x)}{\lambda^2} P_{x^\perp} + \psi''(\lambda x) P_x \]

where \( P_{x^\perp} \) is orthogonal projection onto \( x^\perp \) and \( P_x \) is the line of \( x \).
Given \( e = \frac{x}{|x|} \) define

the \underline{radial slice of} \( F \) to be

\[
\{ xP_e + aP_e \} \cap F
\]

\[
\begin{array}{c}
\text{Positivity} 
\Rightarrow \text{this slice is } \mathbb{R}_+ \times \mathbb{R}_+ \text{ monotone} \\
\Rightarrow \exists ! f : \mathbb{R} \to [-\infty, \infty] \text{ non-dec. u.s.c.} \\
a + f(x) \geq 0 \iff xP_e + aP_e \in F.
\end{array}
\]

\underline{Radial Subequation}

\[
\psi'(t) + f\left(\frac{\psi(t)}{t}\right) \geq 0.
\]

For simplicity assume that

\( F \) is "invariant".

\underline{Theorem (2012) (SMP) holds for} \( F \iff \)

\[
\begin{align*}
\text{(Borderline } & \Leftrightarrow \\
0 & \in \partial F \Leftrightarrow \\
-f(0) & \leq 0 \leq f(0) \tag{Borel's Law}) \quad \int \frac{dx}{f(x)} = \infty.
\end{align*}
\]
VI. What is an Equation?

The equation associated to a subequation is the boundary \( E = \partial F \).
One can show \( F = E + P \) so that \( E \) uniquely determines \( F \). We want \( D_u \cap E \).

Note \( E = F \cap (\sim \text{Int}F) \)
but \( \sim \text{Int}F \) is not a subequation.
However, \( -(\sim \text{Int}F) \) is a subequation.

The Dual Subequation

Definition: The dual to \( F \) is the subequation \( \tilde{F} \equiv -(\sim \text{Int}F) \).
Example: The dual of $P \equiv \{ A : \lambda_{\min}(A) \geq 0 \}$ is the subaffine subequation $\overline{P} \equiv \{ A : \lambda_{\max}(A) \geq 0 \}$.

Properties

\[ \widehat{F} = F \]

\[ F + \mathcal{P} \mathcal{C} F \iff \widehat{F} + \mathcal{P} \mathcal{C} \widehat{F} \]

\[ F \mathcal{C} G \iff \widehat{G} \mathcal{C} \widehat{F} \]

Definition: \( u \) is \( \widehat{F} \)-harmonic (a solution) if \( u \) is \( F \)-sub and \( -u \) is \( \widehat{F} \)-sub.
VII  Monotonicity

Definition: For a subset $M \subseteq \text{Sym}^2(\mathbb{R}^n)$ we say a subequation $F$ is $M$-monotone if

$$F + M \subset F.$$

(Thus $F$ satisfies positivity $\iff F$ is $P$-monotone.)

Key algebraic fact for translates by $A \in \text{Sym}^2(\mathbb{R}^n)$

$$\tilde{F} + A = \tilde{F} - A.$$

Cor (2008) $F + M \subset F \iff \tilde{F} + M \subset \tilde{F}$

$$\iff F + \tilde{F} \subset \tilde{M}$$

In particular, $F + \tilde{F} \subset \tilde{M}$ is always true.

Proof: $F + A \subset F \iff \tilde{F} + \tilde{F} = \tilde{F} - A$ ($A \in M$)

$M + A \subset F \iff \tilde{F} + \tilde{M} + A = \tilde{M} - A$ ($A \in F$).
IX. The Dirichlet Problem

Given \( \Omega \) a domain in \( \mathbb{R}^n \) with smooth \( \partial \Omega \).

Existence

Given \( \varphi \in C(\partial \Omega) \) find \( u \in C(\overline{\Omega}) \)

\( u \mid_{\partial \Omega} \) \( F \)-harmonic

\( u \mid_{\partial \Omega} = \varphi \).

Uniqueness

At most one such \( u \).

Comparison:

\( u, v \in \text{USC}(K), u \in F(\text{Int}K), v \in \tilde{F}(\text{Int}K) \)

\( \Rightarrow \) (MP) on \( K \) for \( u + v \).

Comparison \( \Rightarrow \) Uniqueness (obvious).

Theorem (2008): Comparison holds for all subequations \( F \subset \text{Sym}^2(\mathbb{R}^n) \).

If \( \Omega \) is \( F \) and \( F \) strictly convex then the Perron function is a solution to (DP).

No assumptions of uniformly elliptic or convex.

Many such examples.
Comments on Proof:

Existence  Use classical barrier methods. Already completed in the S.C.V. case \( \Phi_c \equiv \Phi : A_c \geq 0 \).

by Bremermann 1959 and Walsh 1969.

The new part here is:

1) Adapt their methods to general case
2) Introduce the appropriate general notions of F-boundary convexity.

Comparison

Two proofs:

(2008) Use a Lemma of Sadowski 1984 concerned with convex functions, quasi convex approximations, and subaffine functions. (Recall \( F + FC \).

Restriction

Given a subequation $F$ and a submanifold $i: M \hookrightarrow X$, the restriction of $F$, denoted $H \overset{\text{def}}{=} i^*F$, is always defined via restriction of quadratic forms.

**Theorem (2012)** If $u \in F(X)$ and $M$ is minimal then $u|_M$ is $H$ subharmonic.

This is frequently not interesting, because $H$ is everything; but in many cases such as the geometric case, where

$$F(G) = \{ A : \tr_w A \geq 0 \quad \forall W \in G \}$$

this result is important. Here, with $M$ affine $H = \{ B : \tr_w B \geq 0 \quad \forall W \in G \text{ which are in } TM \}$ is also geometrically defined (by

$$\forall W \in G : W \subset TM$$

**Special Case:** $u \in F(X) \iff u$ convex.
Almost Complex Manifolds and the Pali Conjecture (2005).

On $X$ almost complex, one has three different definitions of plurisubharmonic (they agree when $u$ is smooth).

- Classical definition
- Distributional definition
- Viscosity Defn. with $F \in C^2_{\text{Fred}}(X)$ subharmonic

Theorem (2012) $u \ F$-sub. $\iff u \mid_M$ conformally sub

$\iff Nijenhuis-Waolf (1963)$

Theorem (2012) $u \ F$-sub. $\iff u$ is Dist. Sub.

Proof relies on fact that $F$ is convex and works for other convex subequations (the classical Bellman principle), generalizing the constant coefficient result mentioned on p. 11 above.
XI  An Almost Everywhere Theorem

Suppose $F$ is a "general" subequation on $X$. Assume that a function $w$ is $\frac{1}{r}$ quasi-convex. That is, the function $u(y) \equiv w(y) + \frac{1}{2r} |y|^2$ is convex.

Then Alexander's: $u$ (and hence $w$) 2nd diff. a.e.

A.E. Theorem. If $(w(x), D_x w, D_x^2 w) \in F$ a.e.
then $w \in F(X)$.

As a corollary one has:

Addition Theorem. Suppose $F + G$ $C H$ are three subequations (general), and $u, v$ are quasi-convex.

If $u \in F(X)$ and $v \in G(X)$ then $u + v \in H(X)$.

This is a non-linear result which is non-trivial and important (comparison).
The Proof of The A.E. Theorem

Follows in an elementary manner from either of two Lemmas (and Alexandrov).

As above, assume $u$ and $w$ are related by

$$u(y) = w(y) + \frac{1}{2r} |y|^2$$

$B_S(x_0)$ denotes the ball about $x_0$ of radius $S$.

_Slodkowski (1984)_ Assume that $p_0 = 0$, $A_0 = \frac{1}{r} I$ is a strict upper contact jet for $u$ at $x_0$.

For $S > 0$ small the set of points $x \in B_s(x_0)$, which have an upper contact jet on $B_s(x)$ of the form $(p, \frac{1}{r} I)$ for some $p \in \mathbb{R}^n$,

is a set of positive measure.

_Jensen (1988)_ Assume that $q_0 = 0$, $B_0 = 0$ is a strict upper contact jet for $w$ at $x_0$ (this means $w$ has a strict local maximum at $x_0$).

For $S > 0$ small the set of points $x \in B_s(x_0)$, which have an upper contact jet on $B_s(x)$

of the form $(q, 0)$ for some $q$

is a set of positive measure.