Some Recent Development on Spinor Fields and Dirac Operators

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Stony Brook University
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Dedicated to Blaine LAWSON for his 70th Birthday
Dedication

I had the privilege of meeting Blaine for the first time in 1973 here in Stony Brook where Jim SIMONS had invited me to spend a year after he heard me lecture in Paris in June 1972. At that time Blaine was in Berkeley.

Earlier, I had been working with Edmond MAZET on a joint paper he had written with Shing Tung YAU on compact manifolds of nonpositive curvature.

Our collaboration started in earnest while he and Marie-Louise spent the academic year 1978-1979 at IHÉS during a sabbatical leave just at the time he was considering moving to Stony Brook. It developed while I spent the Spring of 1980 at the Institute for Advanced Study.
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Marie-Louise MICHELSOHN and Blaine have been interested for an extended period of time by spinors and Dirac operators. They produced one of the undisputed reference document on the subject, the book *Spin Geometry*, that contains also numerous new results.

The title is not innocent as it hints to the following point of view: “the use of spinors by mathematicians should lead to a geometry of its own, in particular shed light on numerous questions traditionally considered in Riemannian Geometry”.

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- Basic Facts on Spinors and Dirac Operators
  - Harmonic Spinors
  - Special Spinor Fields
  - Conformally Covariant Operators
  - Perspectives
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Section 1
Basics on Spinors and Dirac Operators
Spinors

To any $n$-dimensional Euclidean vector space $(V, g)$ is attached its Clifford algebra

$$Cl(V, g) = \bigotimes V / \langle x \otimes x + g(x, x)1 \rangle .$$

Clifford algebras over $\mathbb{C}$ have a 2-fold periodicity:

- if $n = 2m$, $Cl(V, g)$ is a simple algebra, hence $Cl(V, g) = \text{End}(\Sigma V)$;
- if $n = 2m + 1$, the $Cl(V, g) = \text{End}(\Sigma V) \oplus \text{End}(\tilde{\Sigma} V)$.

$\Sigma V$ and $\tilde{\Sigma} V$ are the spaces of spinors. They are $2^m$-dimensional. These definitions make the dependence of spinors upon the metric quite implicit.
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Spinors (continued)

Spinors are \textit{vectors in a fundamental representation space of the group} Spin\(_n\), the universal cover of the group SO\(_n\) for \(n \geq 3\).

A key ingredient in the whole theory is the exact sequence of groups

\[0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Spin}_n \longrightarrow \text{SO}_n \longrightarrow 1,\]

which holds true for \(n \geq 3\).

The link with Clifford algebras goes as follows:

- the group \(\text{Spin}(V, g)\) can be realized as the multiplicative subgroup of the Clifford algebra stabilizing the image of \(V\) inside \(Cl(V, g)\) and satisfying a certain normalization condition.
- therefore, through the adjoint representation, \(\text{Spin}(V, g)\) acts on \(V\).
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Of great importance in the study of spinors in even dimensions is \textit{chirality}:

- under the action of $i^m v_g$, the volume element determined by the metric $g$, the space $\Sigma V$ splits into

$$\Sigma V = \Sigma^+ V \oplus \Sigma^- V,$$

the spaces of \textit{half-spinors}, both of dimension $2^{m-1}$;

- Clifford multiplication by $v \in V$ exchanges $\Sigma^+ V$ and $\Sigma^- V$. 
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Spinors (continued)

Some general comments on the relation of spinors with other geometric objects:

- When differential \( k \)-forms have a specific “dimension”, to specify the “dimension” of a spinor is much trickier;

- The formal comparison of spinors for different metrics was a controversial question in the physics literature for some time; an explicit solution has been given by Paul GAUDUCHON and myself in 1992, but it is a little heavy;

- For conformally related metrics, the comparison can be made explicit as the change of metric is scalar;

- By using appropriate weights, it is possible to construct spinors attached to a conformal class of metrics, a construction due to Nigel HITCHIN.
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Spinor fields

The purpose is of course to carry these constructions over to an oriented Riemannian manifold \((M, g)\).

Lifting usual constructions over the bundle of oriented orthonormal bases to a bundle of spinorial bases can be done if \(w_2(M) = 0\).

The set up goes as follows:

- choosing a \(\text{Spin}_n\)-principal bundle \(\Gamma\) covering the \(\text{SO}_n\)-bundle of oriented orthonormal frames determines the \textit{spin structure};
- the \textit{bundle of spinors} associated to the spin structure is the associated bundle \(\Sigma_M = \Gamma \times_{\text{Spin}_n} \Sigma_n\);
- a \textit{spinor field} is of course a section of the bundle \(\Sigma_M \rightarrow M\);
- the bundle in Clifford algebras \(\mathbb{C}l_g(M) \rightarrow M\) acts on the spinor bundle via pointwise Clifford multiplication.
- \(\Sigma_M \rightarrow M\) inherits a natural connection form from the Riemannian connection on the Riemannian frame bundle. Hence one can speak of covariant derivatives of spinor fields.
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3. a spinor field is of course a section of the bundle $\Sigma_\gamma M \to M$;
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On the space of spinor fields only two first order differential operators are universally defined, and combinations thereof:

- $T^* M \otimes \Sigma_\gamma M$ decomposes into exactly two invariant subspaces, a copy of $\Sigma_\gamma M$ and another space $\Sigma_\gamma^{3/2} M$;
- the Dirac operator $D$ maps spinor fields to spinor fields, and is defined, for a spinor field $\psi$, by
  $$D\psi = \sum_{i=1}^{n} e_i.D_{e_i}\psi ,$$
  where $(e_i)$ denotes an orthonormal basis of the tangent space;
- the twistor operator $P$, which maps spinor fields to sections of the bundle $T^* M \otimes \Sigma_\gamma M$, is defined as follows
  $$P\psi(X) = D_X\psi + \frac{1}{n} X.D\psi ,$$
  where $X \in TM$ and $\psi$ is a spinor field.
Natural Operators on Spinor Fields

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  where $(e_i)$ denotes an orthonormal basis of the tangent space;
- the twistor operator $P$, which maps spinor fields to sections of the bundle $\mathcal{T}^* M \otimes \Sigma \gamma M$, is defined as follows
  \[
  P \psi(X) = D_X \psi + \frac{1}{n} X \cdot \mathcal{D} \psi ,
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  where $X \in TM$ and $\psi$ is a spinor field.
Natural Operators on Spinor Fields

On the space of spinor fields only two first order differential operators are universally defined, and combinations thereof:

- $T^* M \otimes \Sigma \gamma M$ decomposes into exactly two invariant subspaces, a copy of $\Sigma \gamma M$ and another space $\Sigma^{3/2} M$;
- the Dirac operator $D$ maps spinor fields to spinor fields, and is defined, for a spinor field $\psi$, by

$$D \psi = \sum_{i=1}^{n} e_i \cdot De_i \psi,$$

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Dirac Operators

Here are the important properties of the Dirac operator:

- it is a square root of the Laplace-Beltrami operator, hence an elliptic operator in a Riemannian setting;
- its principal symbol is given by Clifford multiplication;
- it is self-adjoint;
- in even dimensions, it exchanges the chirality of spinors, hence non-trivial eigenspinors for the Dirac operator have necessarily components of both chiralities, unless they are harmonic.
- for a spinor field $\psi$, the Schrödinger-Lichnerowicz formula reads

$$D^2 \psi = D^* D \psi + \frac{1}{4} \text{Scal}_g \psi,$$

where $D^*$ denotes the adjoint of the covariant derivative $D$ and $\text{Scal}_g$ the scalar curvature of $g$. 
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Section 2

Harmonic Spinors
Harmonic Spinors

**Definition**

A harmonic spinor $\psi$ is a spinor field lying in the kernel of the Dirac operator, i.e. $D\psi = 0$.

On an $n$-dimensional spinorial manifold $(M, \gamma)$, we denote the space of harmonic spinors by $\mathcal{H}(M, \gamma)$. If $n = 2m$, we distinguish the positive (resp. negative) harmonic spinors $\mathcal{H}^+(M, \gamma) = \ker D^+$ (resp. $\mathcal{H}^-(M, \gamma) = \ker D^-$) where $D^+$ (resp. $D^-$) is the restriction of the Dirac operator to positive (resp. negative) spinor fields.

Note that when $n = 8\ell + 4$, the spinor representation is naturally quaternionic, hence $\mathcal{H}^+(M, \gamma)$ and $\mathcal{H}^-(M, \gamma)$ are even-dimensional complex vector spaces.
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Dirac Operators and Index Theorems

When applied to the Dirac operator $\mathcal{D}^+$ acting in even dimensions on positive spinor fields, the Atiyah-Singer Index Theorem leads to the following crucial statement:

**Theorem (M.F. Atiyah, I.M. Singer)**

On a compact $4k$-dimensional spin manifold $M$ endowed with a spinorial metric $\gamma$,

$$\text{Index}(\mathcal{D}^+) = \dim \mathcal{H}^+(M, \gamma) - \dim \mathcal{H}^-(M, \gamma) = \hat{A}(M).$$

This statement has a number of consequences on the relation between metric and topology:
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Positive Scalar Curvature

The study of manifolds admitting metrics with positive scalar curvature has been immensely enhanced by this link. Several strategies have been developed to this effect:

- surgery was used to take advantage of the structure of the Spin cobordism ring (results by Misha GROMOV and Blaine in the simply connected case);
- the case of non-simply connected manifolds has also been addressed by Misha GROMOV and Blaine through the study of enlargeable manifolds, using extended notions of $\hat{A}$-genus;
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Dimension of the Space of Harmonic Spinors

Very early on, a lot of efforts has been devoted to the study of $\dim \mathcal{H}$:

- **In his thesis HITCHIN both sophisticates the topological information related to harmonic spinors and gives many examples of manifolds on which $\dim \mathcal{H}$ depends on the metric;**
- he also shows that all $n$-manifolds with $n \equiv 0, 1$ or $7$ mod $8$ admit a metric with $\dim \mathcal{H} \geq 1$;
- a powerful example was given by the *Berger metrics* on the sphere $S^3$, showing that, on a given manifold, $\dim \mathcal{H}$ can be unbounded;
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There has also been efforts going in the exact opposite direction, namely to prove that, on a given manifold, there are indeed metrics for which $\dim \mathcal{H}$ is the minimum possible, taken into account the topological constraint:

- **Stephan MAIER** proved it in dimension $\leq 4$;
- it was proved by BÄR and DAHL for simply connected $n$-manifolds with $n \geq 4$;
- then, in 2011, AMMANN, DAHL and Emmanuel HUMBERT established this fact in full generality even by a modification of the metric on an arbitrary small open set; such metrics are therefore generic in the space of Riemannian metrics;
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3. Special Spinor Fields
Killing Spinors

**Definition (R. PENROSE)**

A Killing spinor $\psi$ is a spinor field lying in the kernel of $P$ and an eigenspinor for $D$. Its characteristic equation is, for some $\lambda \in \mathbb{C}$,

$$\forall X \in TM, \ D_X \psi + \frac{1}{n} \lambda X.\psi = 0.$$

- The 1-form $\xi_\psi$ defined on $X \in TM$ by $\xi_\psi(X) = (X.\psi, \psi)$ is dual to a Killing vector field.
- Other components of $\psi \otimes \bar{\psi}$ also satisfy interesting conditions.
- A definition of a supersymmetric transformation can go as follows: one maps fermionic fields (such as spinor fields) $\Phi$ to bosonic fields (such as 1-forms) by $\Phi \mapsto \Re(X.\Phi, \psi)$.
- The curvature tensor acting on $\psi$ is very special, namely, for all $X, Y \in TM$, $R_{X,Y} \psi = \lambda^2/n^2 (X.Y - Y.X).\psi$.
- It follows that $Ric_g = 4 \lambda^2 (n - 1)/n^2 \ g$. 
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$$\forall X \in TM, \quad D_X \psi + \frac{1}{n} \lambda X.\psi = 0.$$ 

- The 1-form $\xi_\psi$ defined on $X \in TM$ by $\xi_\psi(X) = (X.\psi, \psi)$ is dual to a Killing vector field.
- Other components of $\psi \otimes \bar{\psi}$ also satisfy interesting conditions.
- A definition of a supersymmetric transformation can go as follows: one maps fermionic fields (such as spinor fields) $\Phi$ to bosonic fields (such as 1-forms) by $\Phi \mapsto \Re(X.\Phi, \psi)$.
- The curvature tensor acting on $\psi$ is very special, namely, for all $X, Y \in TM$, $R_{X,Y} \psi = \lambda^2/n^2 (X.Y - Y.X).\psi$.
- It follows that $\text{Ric}_g = 4 \frac{\lambda^2}{n} (n - 1)/n^2 g$. 
Killing Spinors

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Killing Spinors (continued)

It is useful to discuss cases according to the eigenvalue $\lambda$:

- if $\lambda = 0$, then the Killing spinor is parallel, and hence the metric has reduced holonomy;
- if $\lambda \in i\mathbb{R}^*$, then $M$ is non compact;
- if $\lambda \in \mathbb{R}^*$, then the Ricci curvature is uniformly positive, and by Myers’ Theorem, $M$ is compact.

The key construction goes as follows:

- construct the cone $CM = M \times \mathbb{R}^+$ over $M$ with the cone metric $\bar{g} = dr^2 + r^2 g$;
- then, through an identification of an action of the group $\text{Spin}_{n+1}$ within the Clifford algebra $Cl_g(M)$, map spinor fields on $M$ into spinor fields on $CM$;
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Generalized Killing Spinors

The notion of Killing spinors has been generalized in different ways. One has to do with the use of spinors in the study of submanifold theory.

Here is one of the constructions due to BÄR, GAUDUCHON and Andrei MOROIANU:

- on a Riemannian manifold \((M, g)\) consider a field of symmetric linear maps \(S\) satisfying the Codazzi equation on \(M\) and assume that there exists a generalized Killing Spinor \(\psi\) satisfying, for any \(X \in TM\), the equation

\[
D_X \psi - \frac{1}{2} S(X).\psi = 0 ;
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- let \(I \subset \mathbb{R}\) be an interval parametrized by \(t\), and endow \(I \times M\) with the metric \(dt^2 + g_t\), where \(g_t = g \circ (Id - tS)^2\);
- then there is a parallel spinor field on \(I \times M\) whose restriction to \(\{0\} \times M\) is precisely \(\psi\).
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4. Conformally Covariant Operators
Conformal Behaviour of the Dirac Operator

The simplicity of the behaviour of spinors for conformally related metrics $g$ and $\tilde{g} = e^{2u}g$ for some function $u$ on $M$ has been pointed out earlier.

This is reflected further at the level of the Dirac operator. Namely, if we denote the (natural) isomorphism between $\gamma$-spinors and $\tilde{\gamma}$-spinors by $b_{\tilde{\gamma}}$, then:

Conformal Covariance of the Dirac operator

The Dirac Operator is as conformally covariant as it can be: for a $\gamma$-spinor field $\psi$,

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e^{\frac{n+1}{2}u} D\tilde{\gamma}(b_{\tilde{\gamma}}(\psi)) = b_{\tilde{\gamma}}(D\gamma(e^{\frac{n-1}{2}u}\psi)) .$$
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$$e^{\frac{n+1}{2}u} \mathcal{D}^{\tilde{\gamma}}(b_{\tilde{\gamma}}(\psi)) = b_{\tilde{\gamma}} \left( \mathcal{D}^{\gamma}(e^{\frac{n-1}{2}u} \psi) \right).$$
Conformal Laplacian and Dirac Operator

The modified Laplacian $C_g = 4\frac{n-1}{n-2} \Delta_g + \text{Scal}_g$ is known to be also \textit{covariantly covariant}, hence called the Conformal Laplacian. Namely, for a function $u$ on $M$, one has

$$e^{n+2/2}(C_{\tilde{g}}u) = C_g(e^{n-2/2}u).$$

Using the conformal covariance allows to prove some inequalities between eigenvalues of these operators:

- already in 1986, Oussama HIJAZI proved that, for $n \geq 3$, if $\lambda_\gamma$ is an eigenvalue of $D_\gamma$,

$$\frac{n}{4(n-1)} c_g \leq (\lambda_\gamma)^2,$$

where $c_g$ is the lowest eigenvalue of the Conformal Laplacian;

- equality occurs only if there is a supersymmetry, i.e., if there exists a non-trivial Killing Spinor on $M$;

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5. Perspectives
Towards a Spinorial Geometry?

As you have found out, spinor fields proved to be powerful tools to deal with some specific Riemannian questions, such as the existence of metrics with positive scalar curvature.

More was expected:

- a link to the Ricci curvature, and in particular to the Einstein condition, in a vein similar to the one that appeared in the presence of a Killing spinor;
- so far mathematicians focused their attention on spinor fields that physicists call fields of spin $\frac{1}{2}$;
- attention should also probably be devoted to spinors with higher spins, such as $\frac{3}{2}$, that would lead to a more systematic study of the Rarita-Schwinger operator.
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I thank you for your attention.

Happy Birthday, Blaine!

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