Mass in Kähler Geometry

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Stony Brook University

Colloquium, October 6, 2016

Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

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Hans-Joachim Hein
Fordham University

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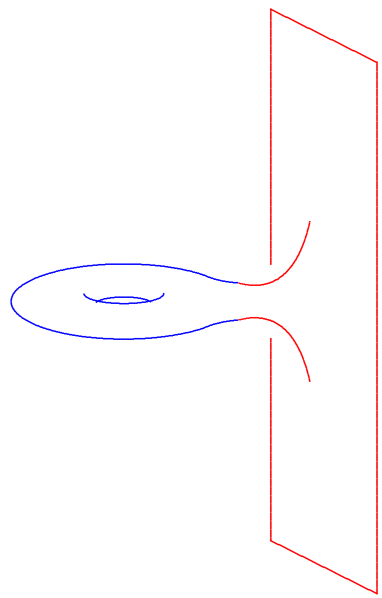
Definition. *A complete, non-compact Riemannian n-manifold* (M^n, g)



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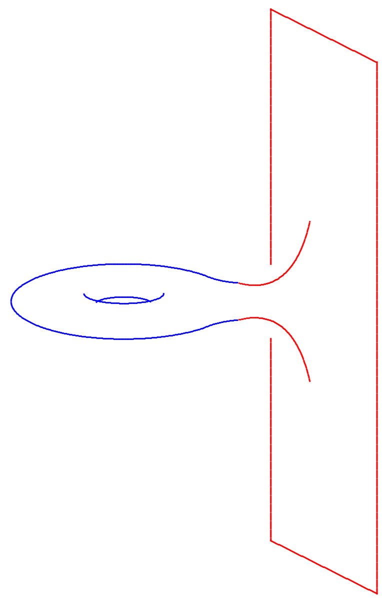

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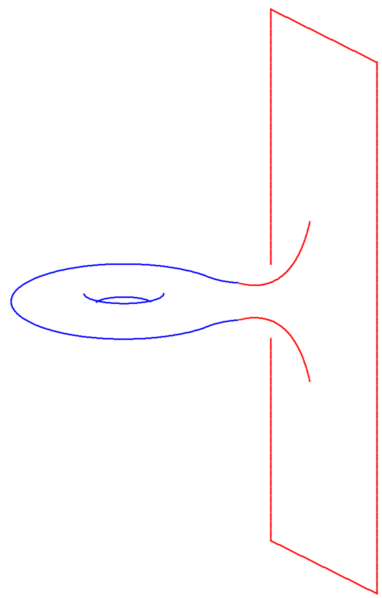

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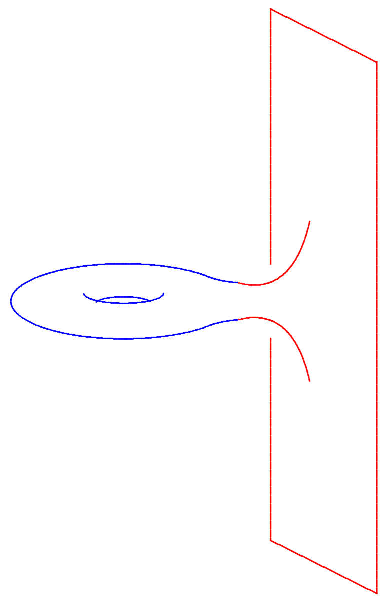

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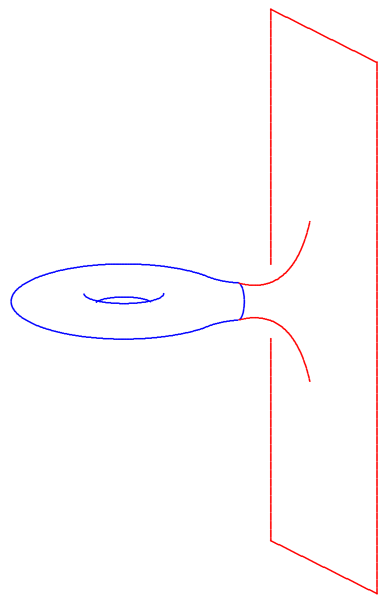

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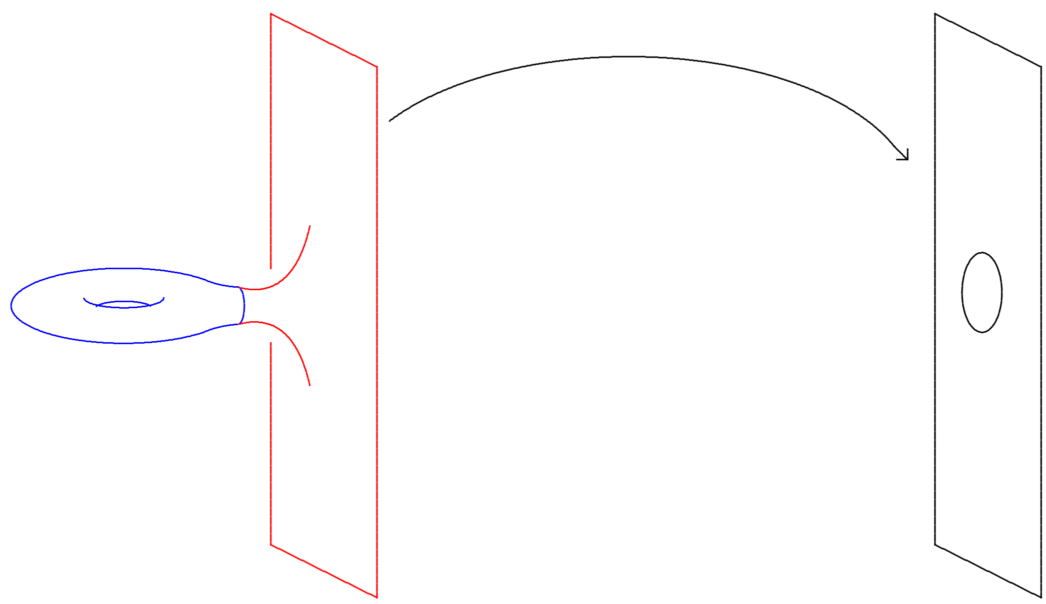

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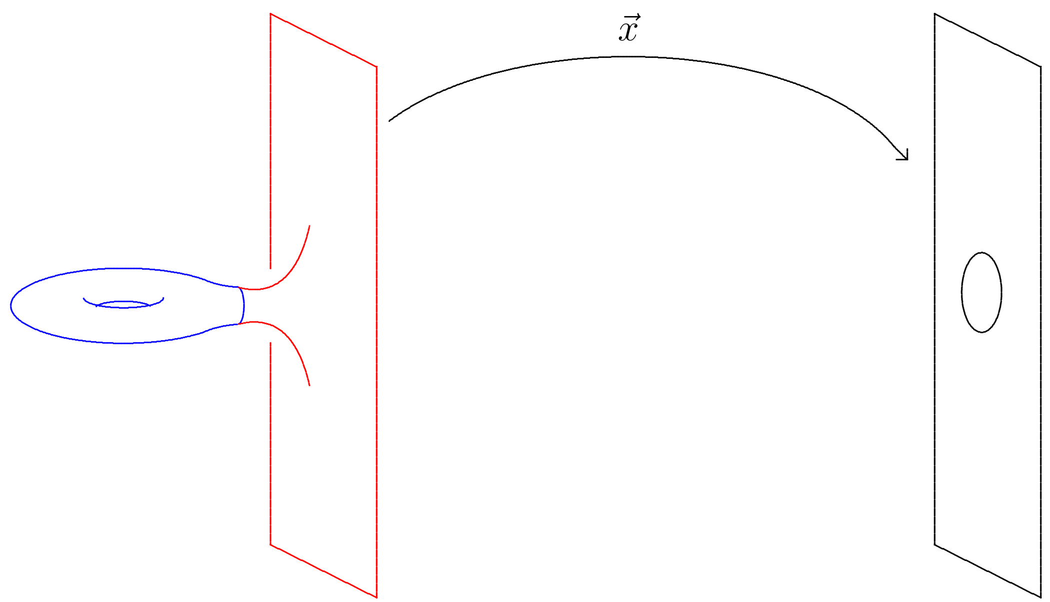

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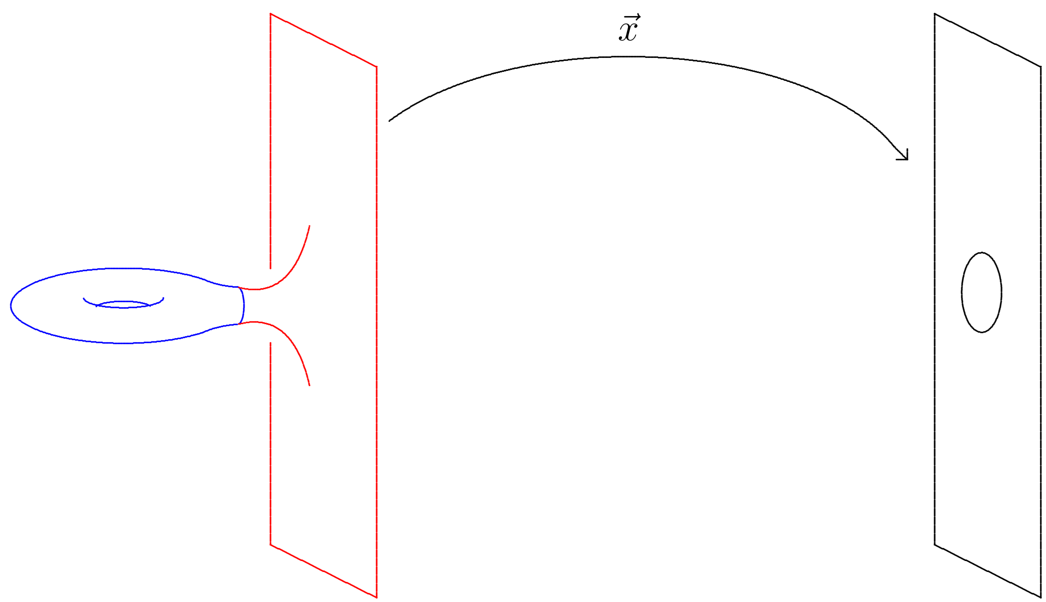

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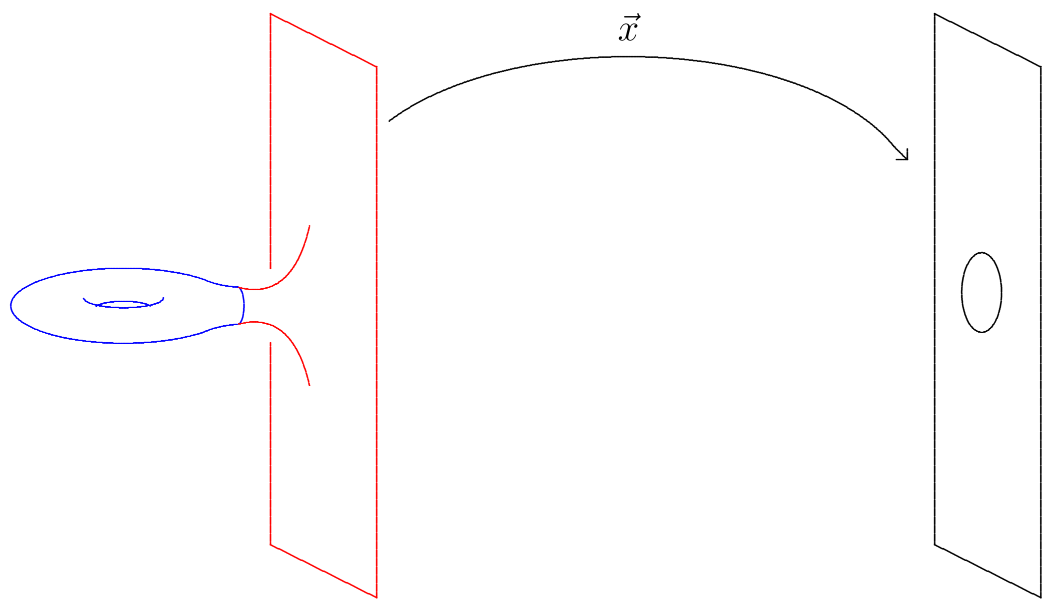

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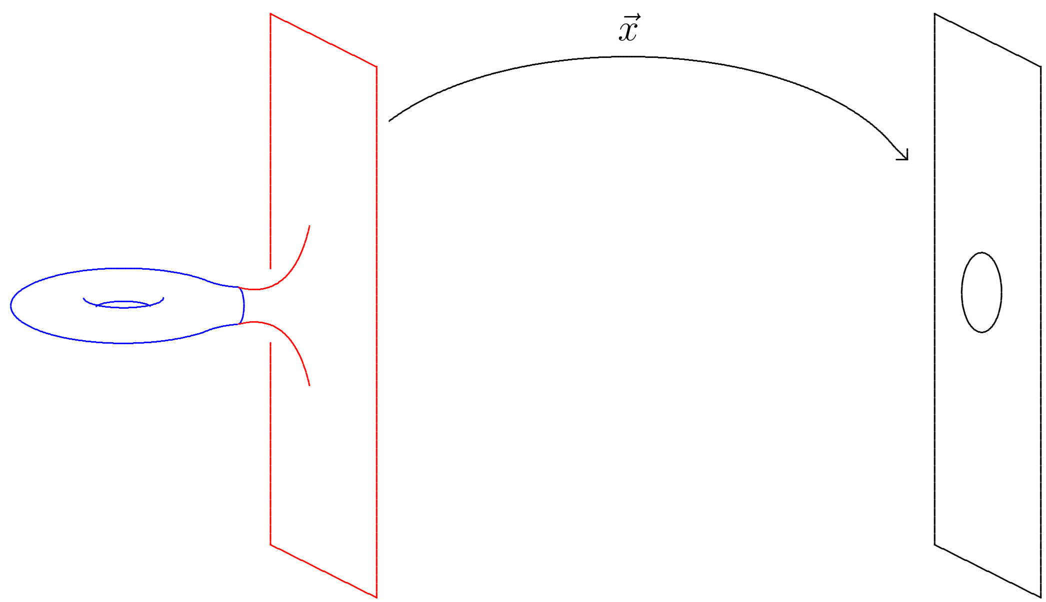

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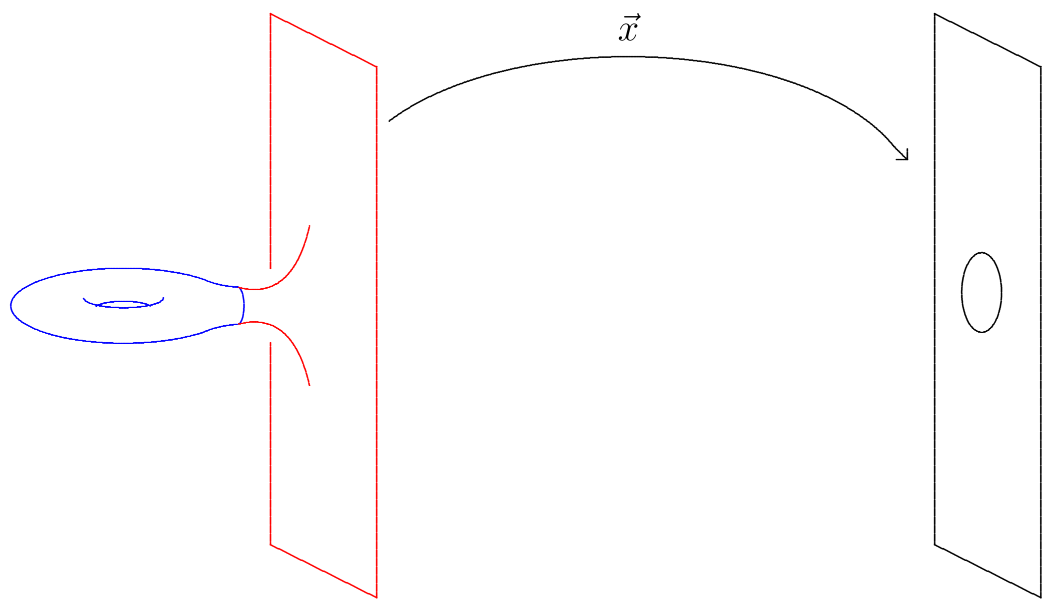

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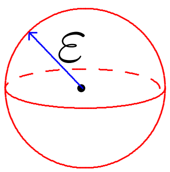

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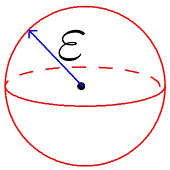

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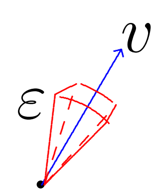

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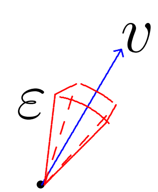

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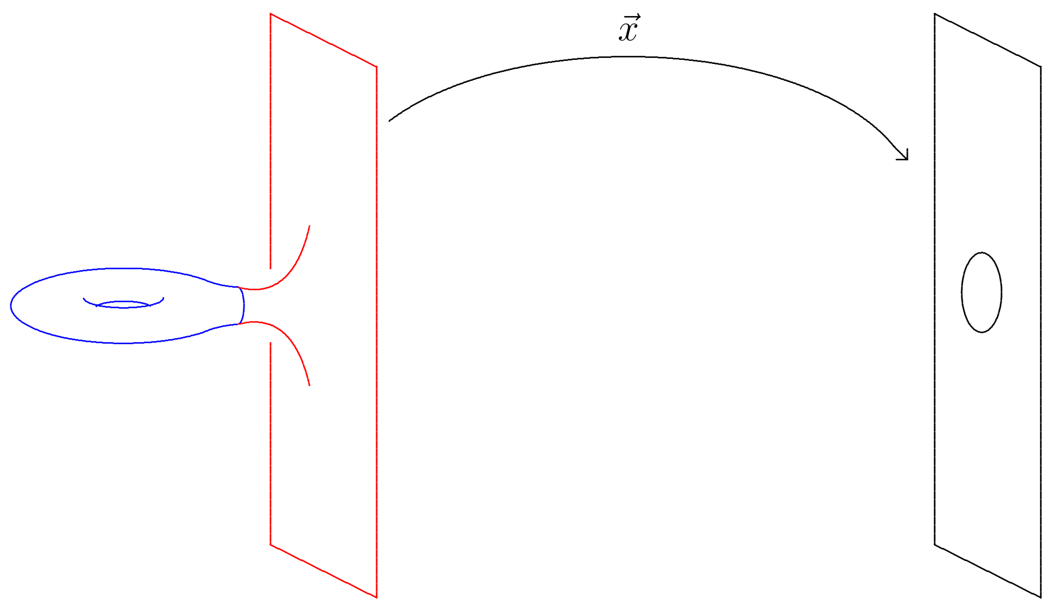

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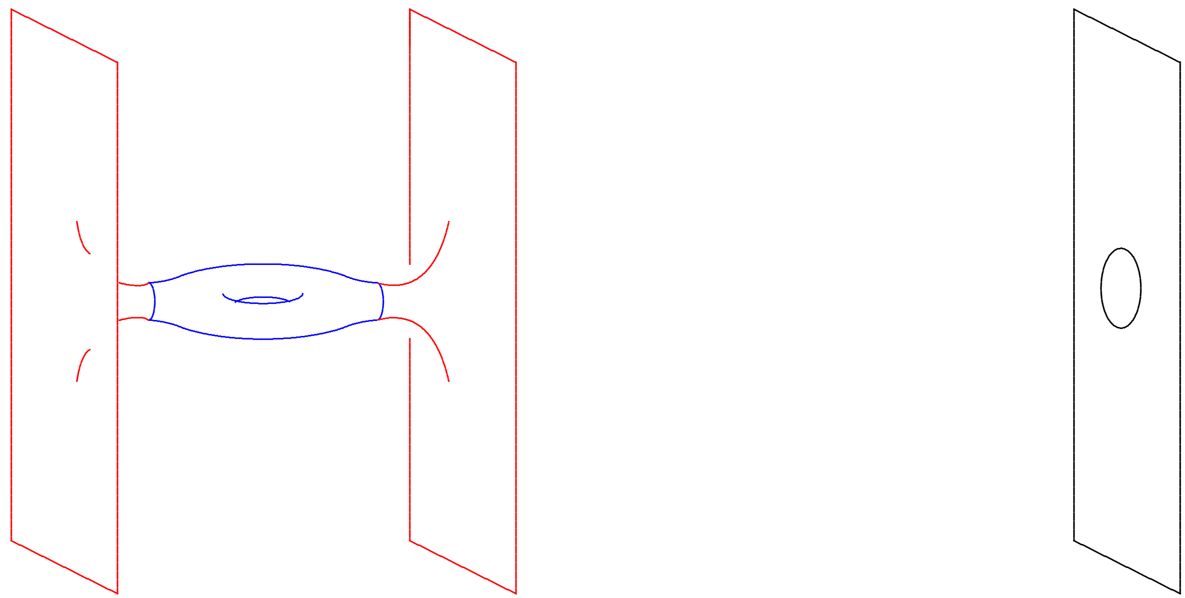

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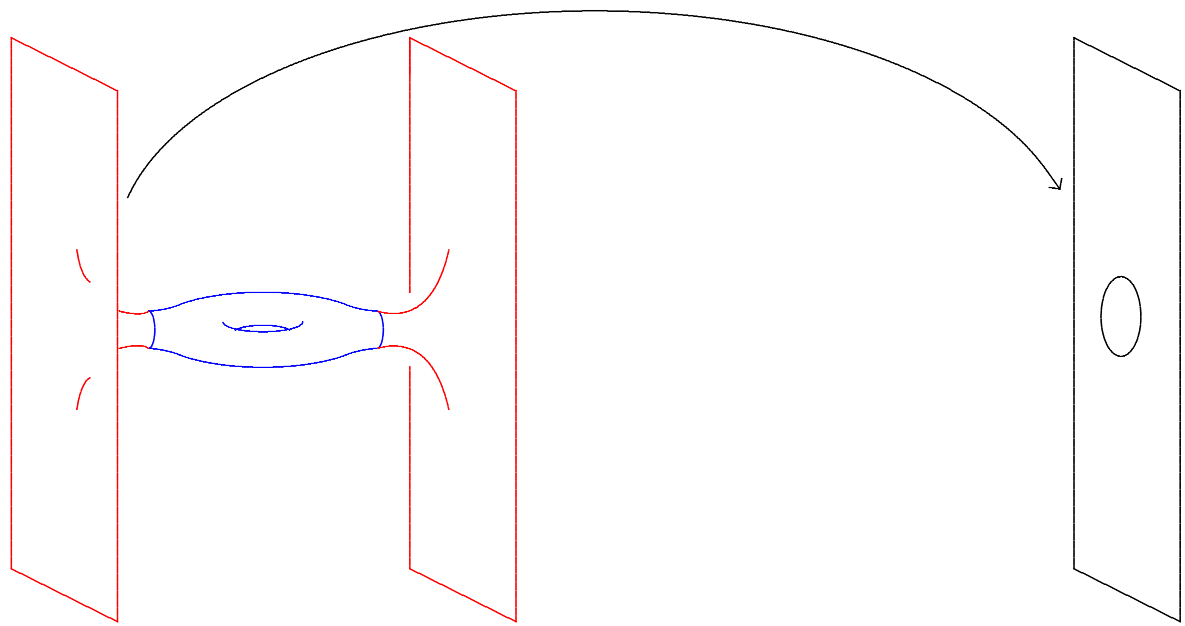

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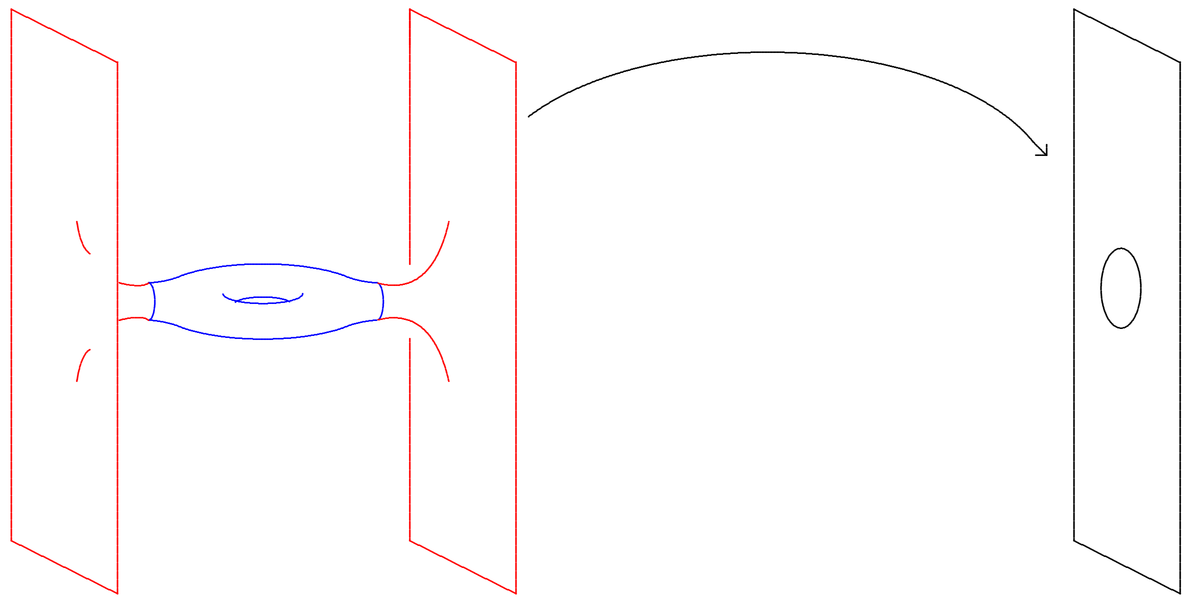

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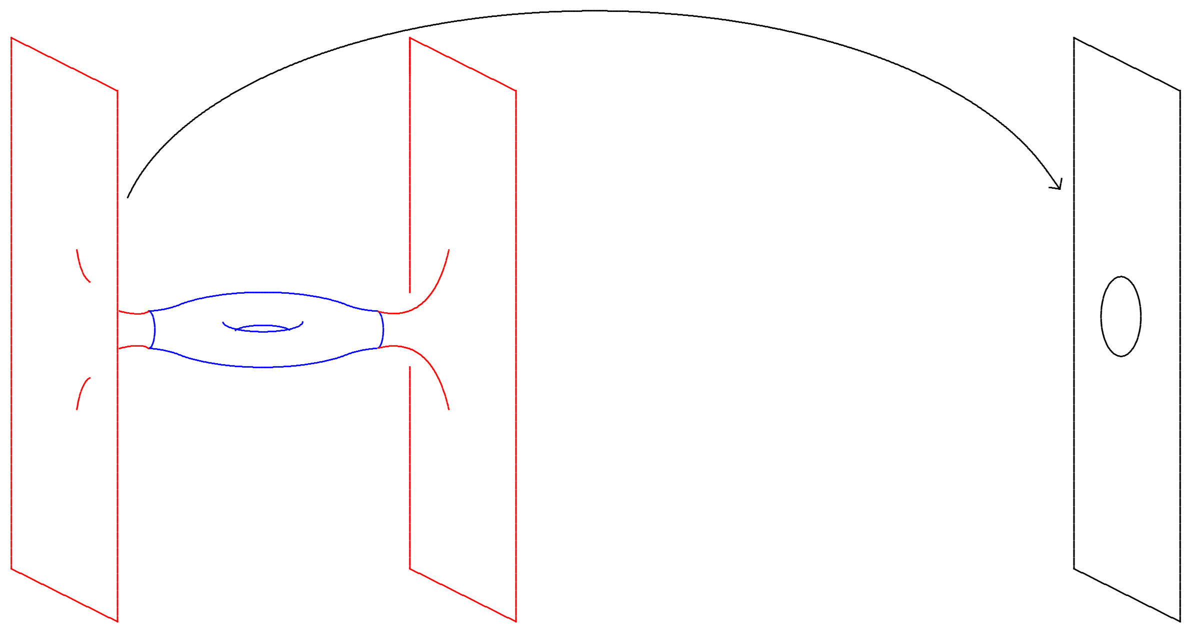

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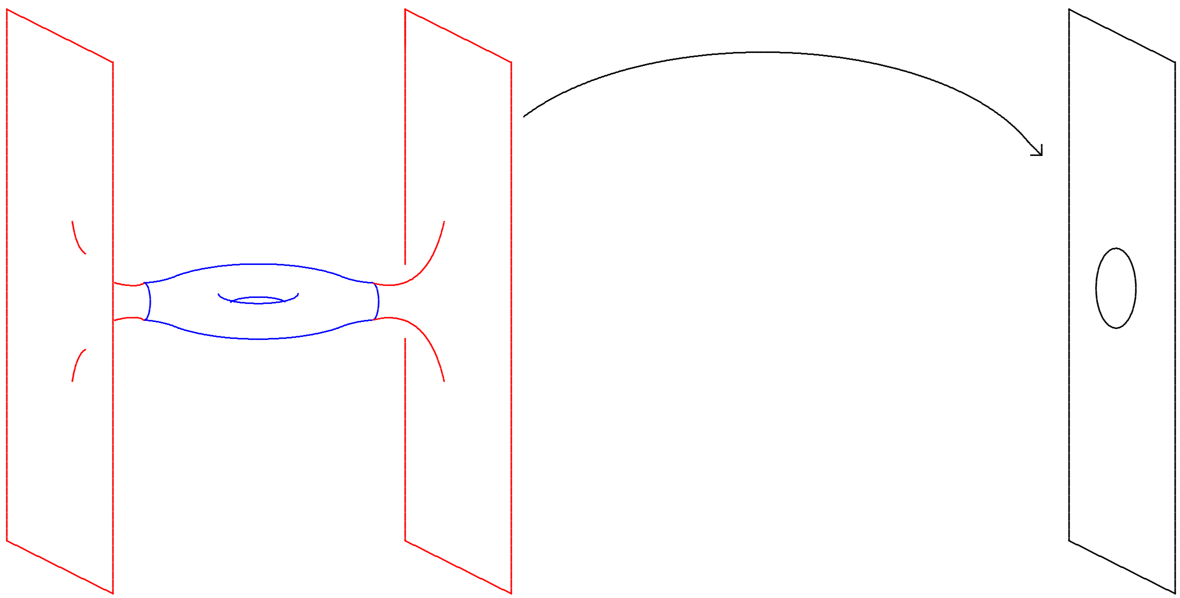

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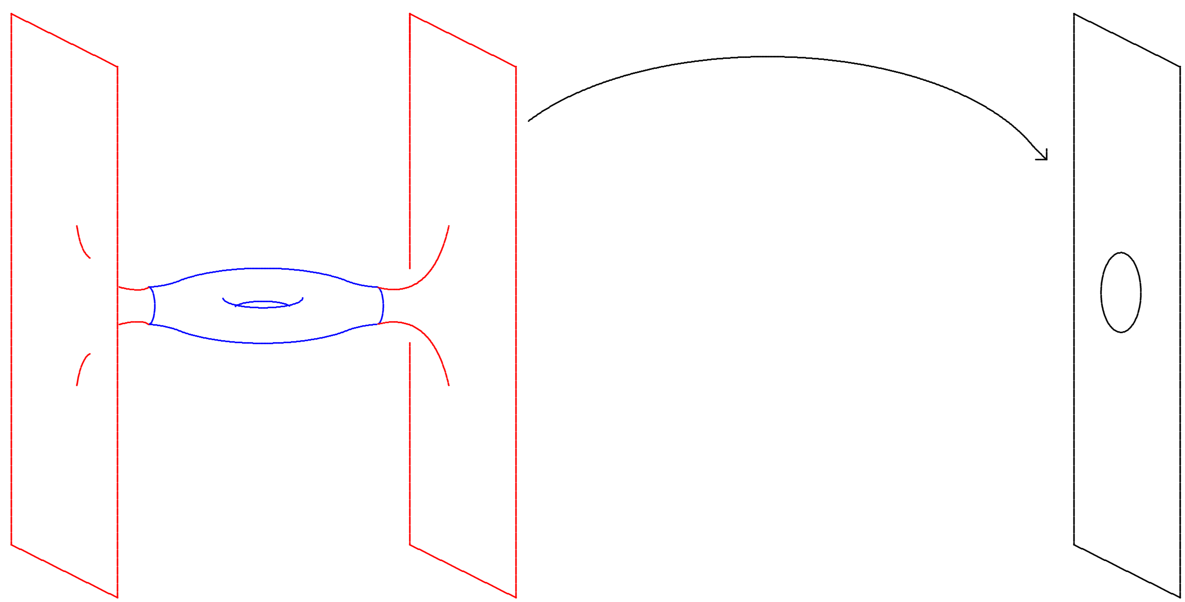

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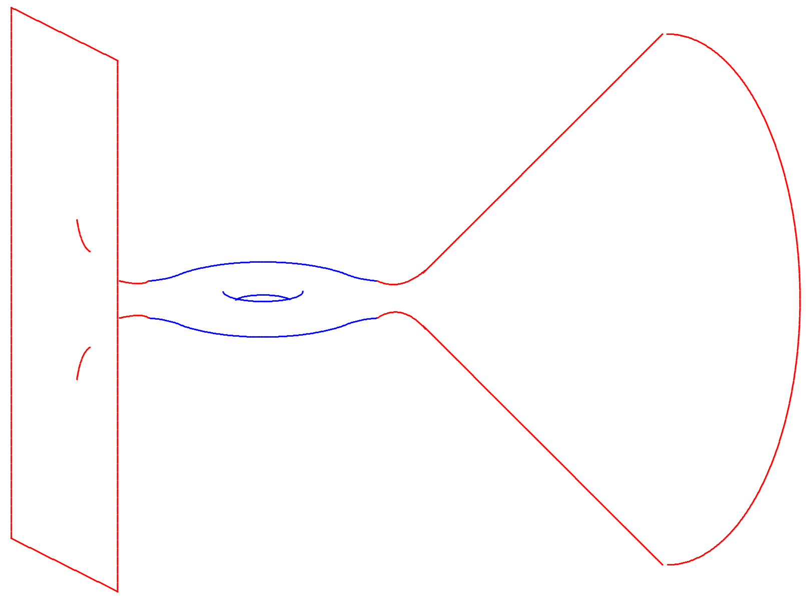

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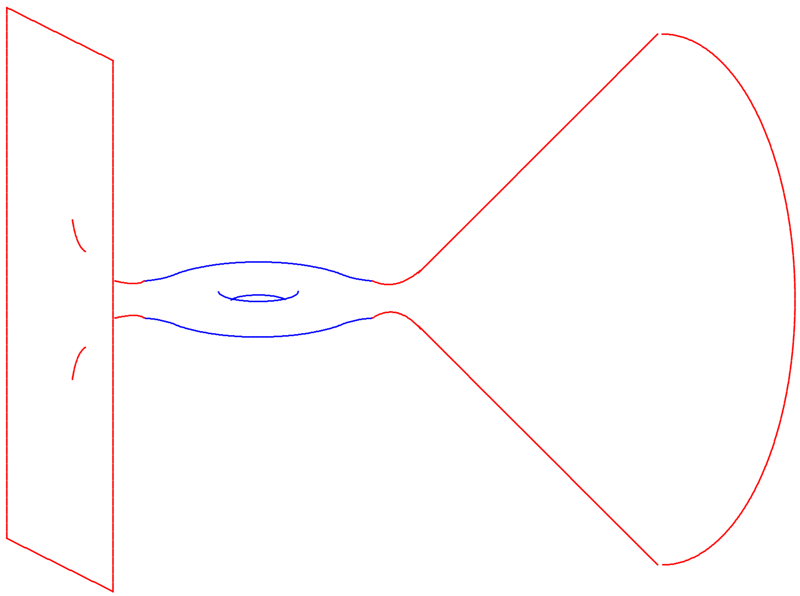

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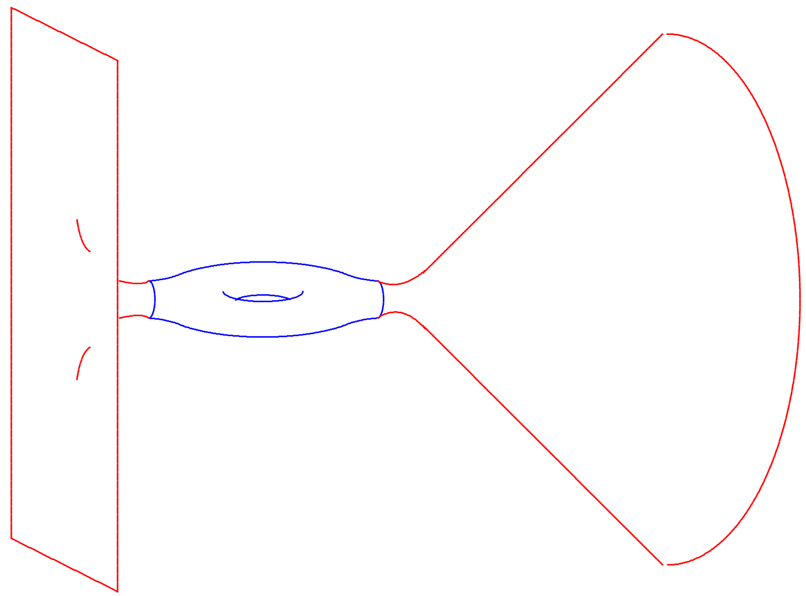

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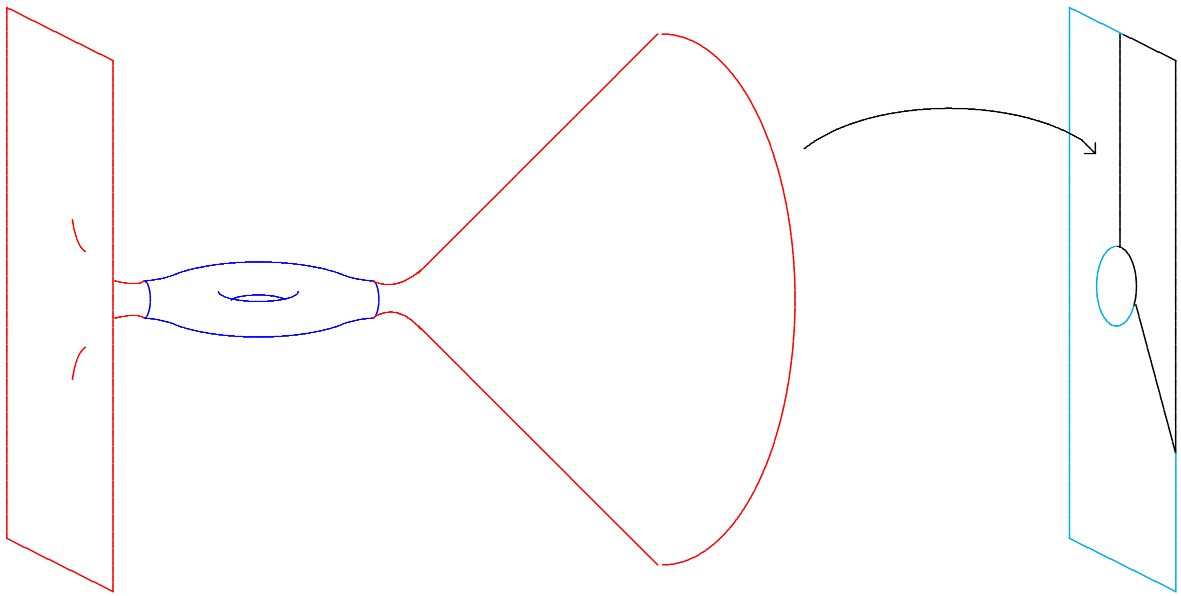

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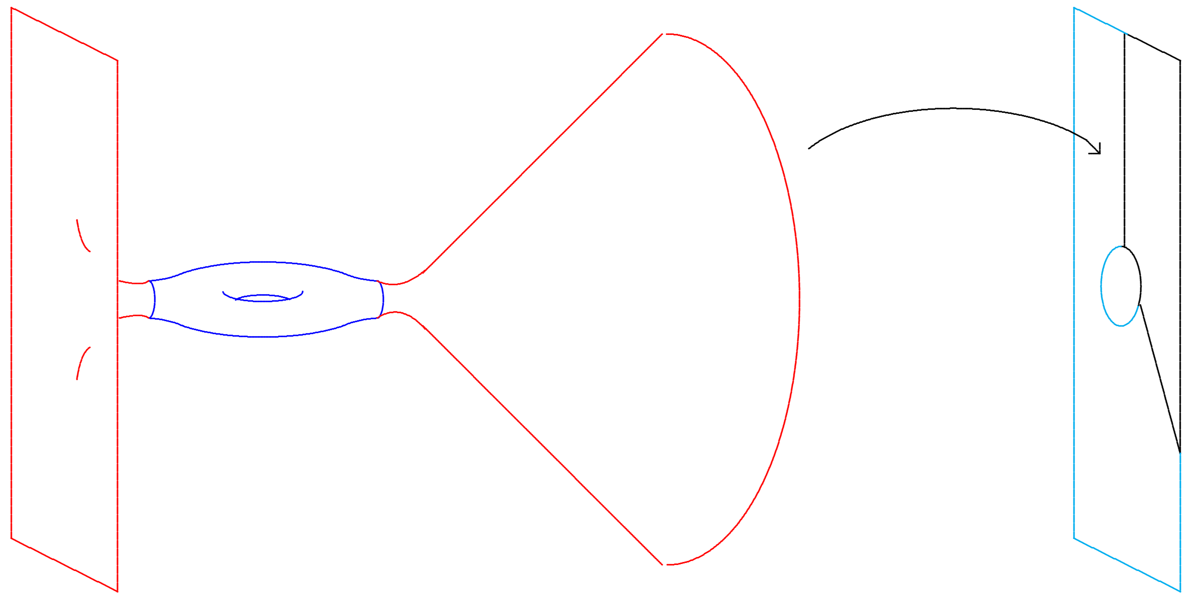

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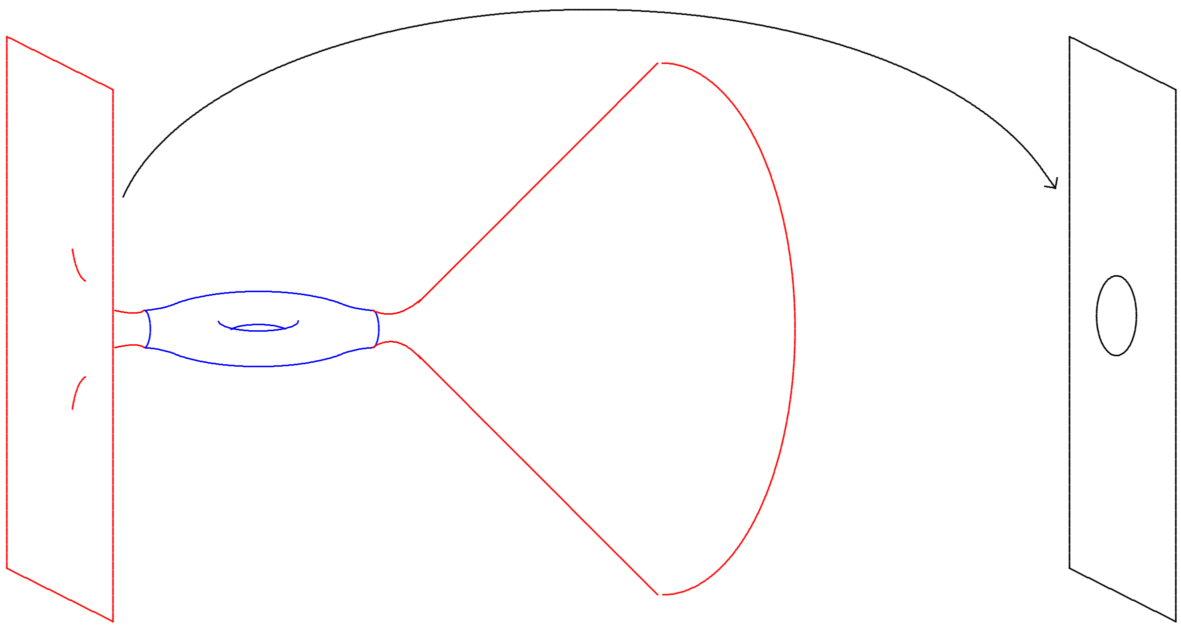

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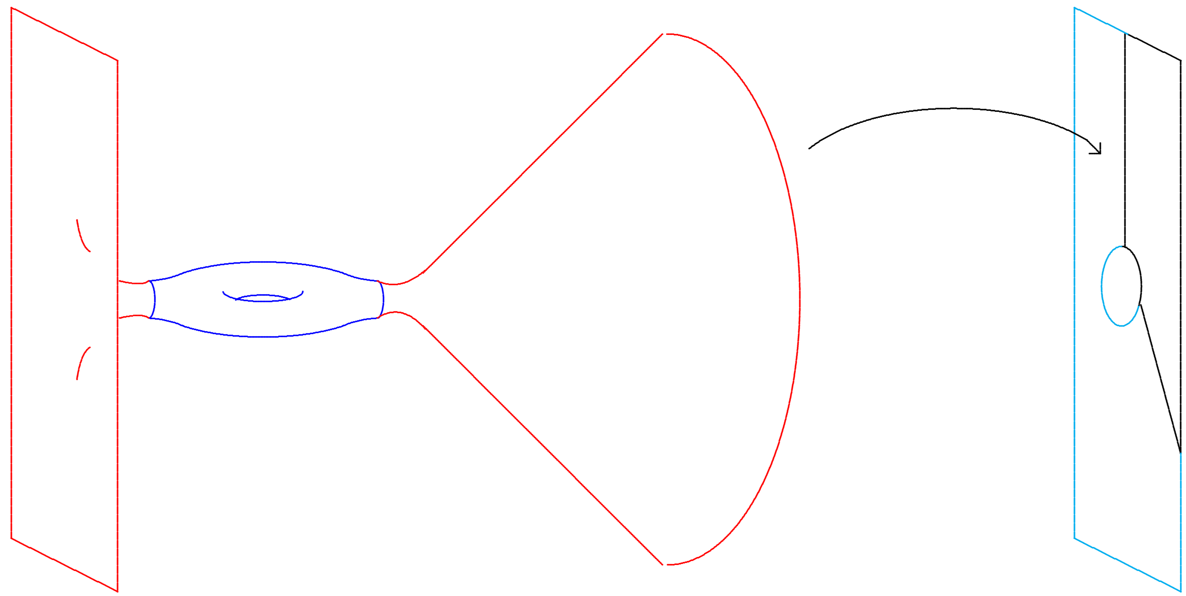

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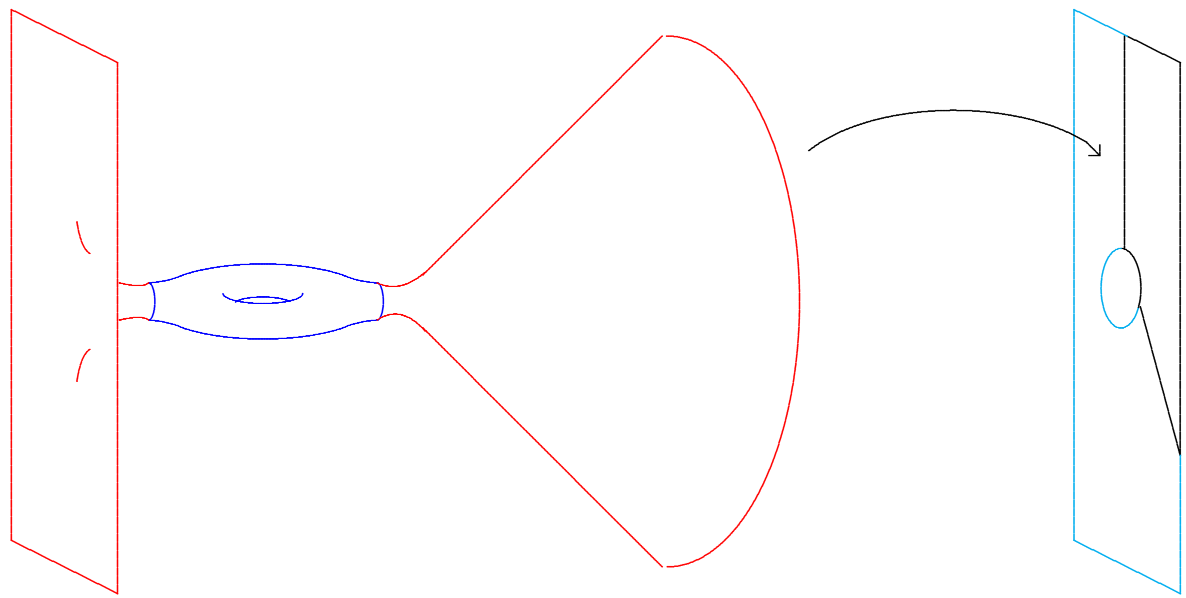

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Why consider ALE spaces?

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The G-H metrics are hyper-Kähler, and were soon independently rediscovered by Hitchin.

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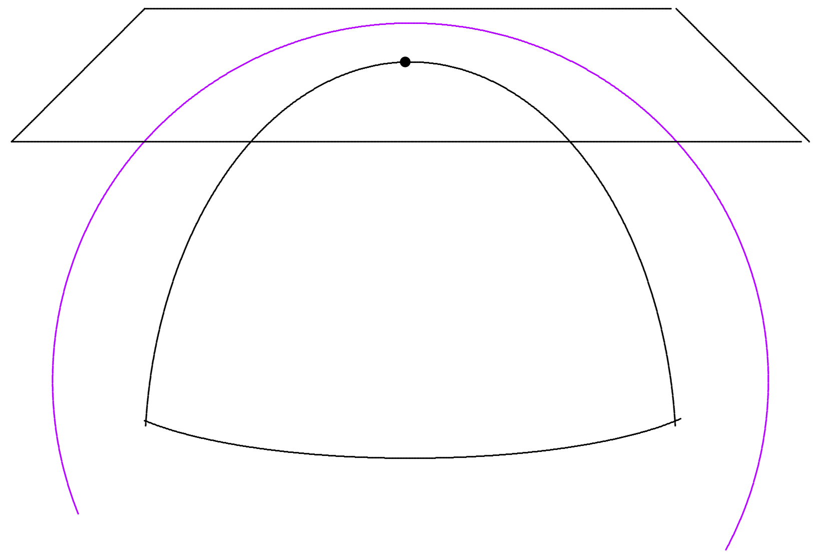

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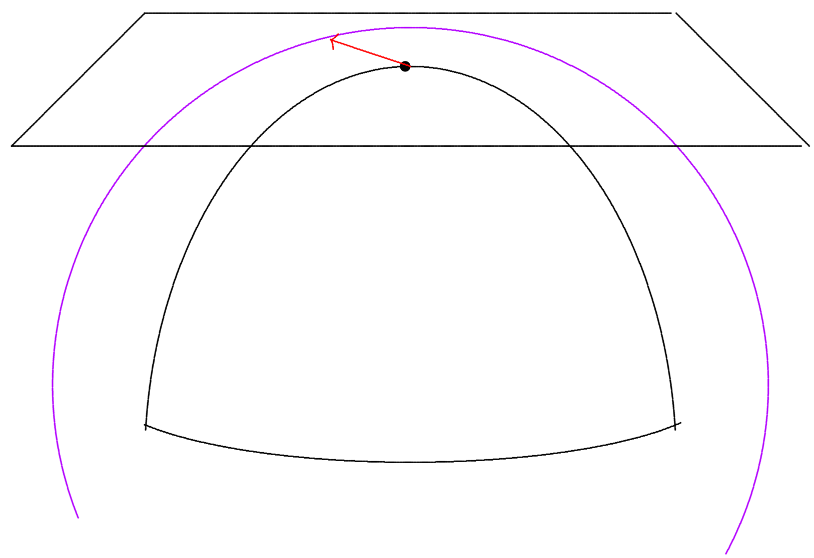

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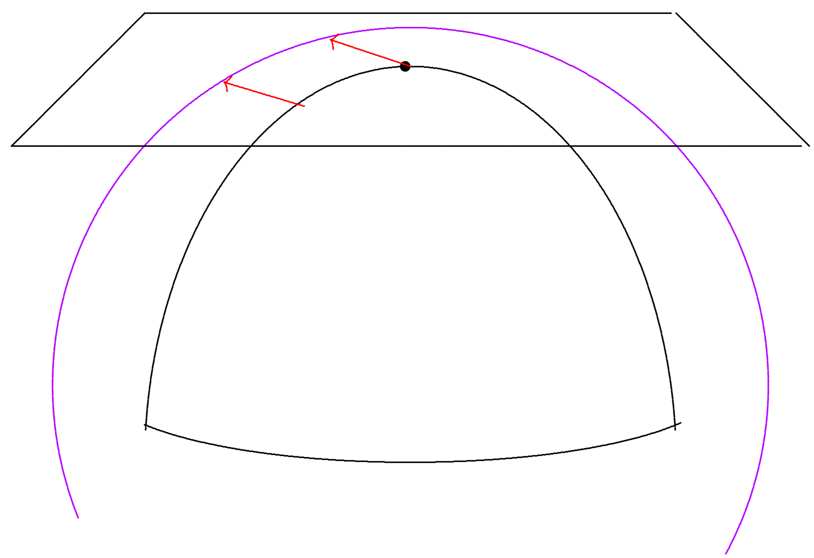

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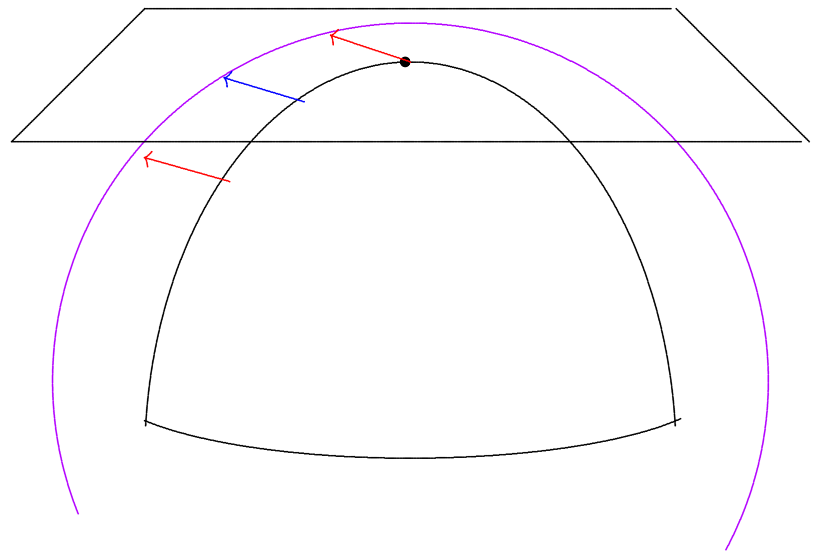

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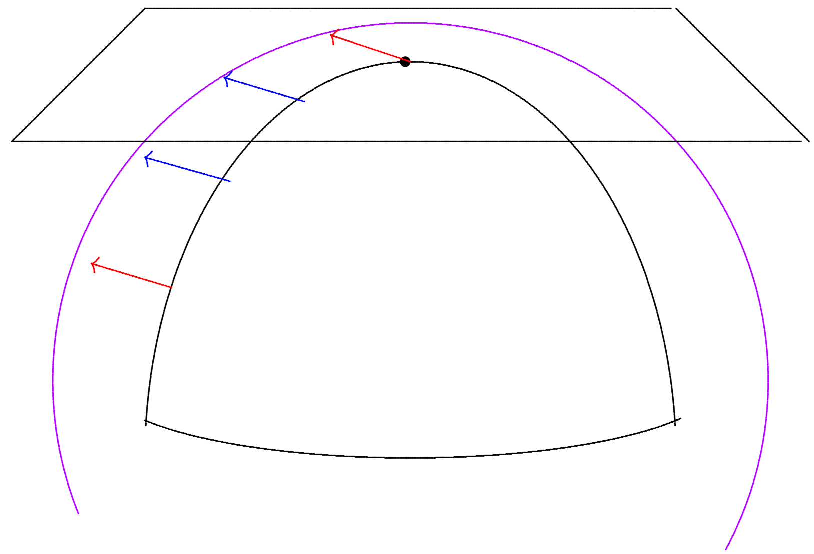

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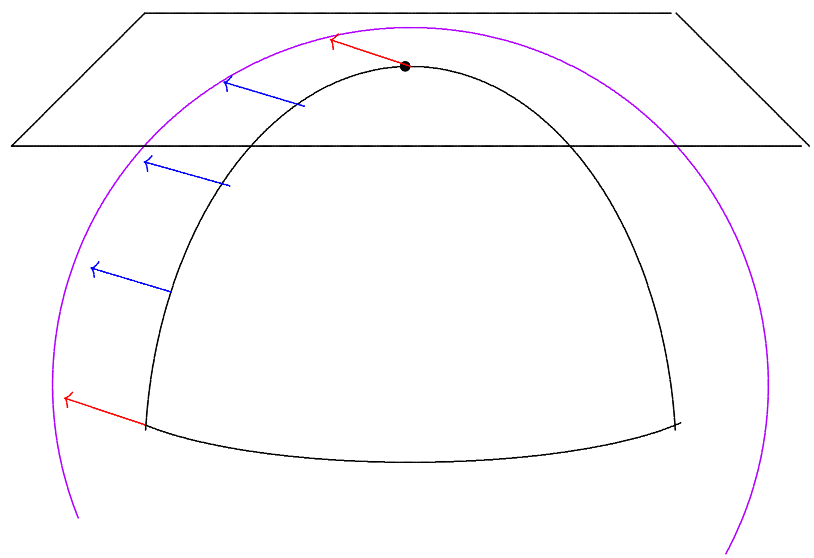

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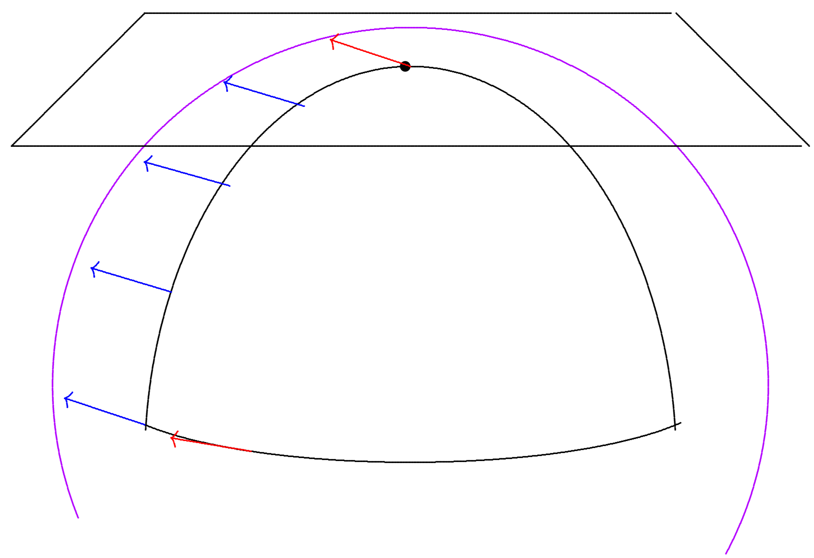

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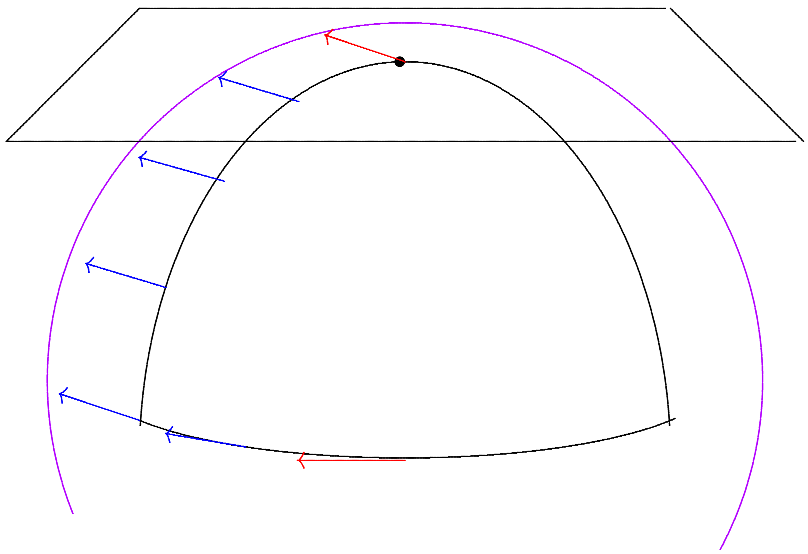

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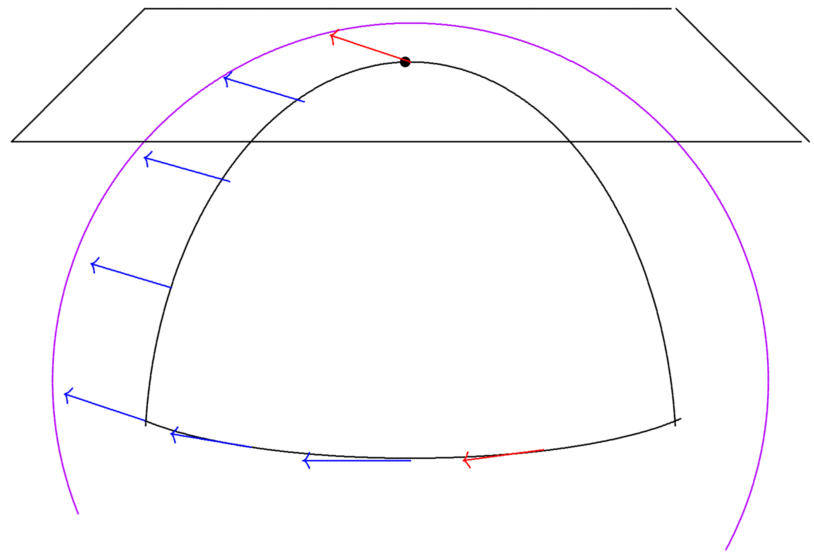

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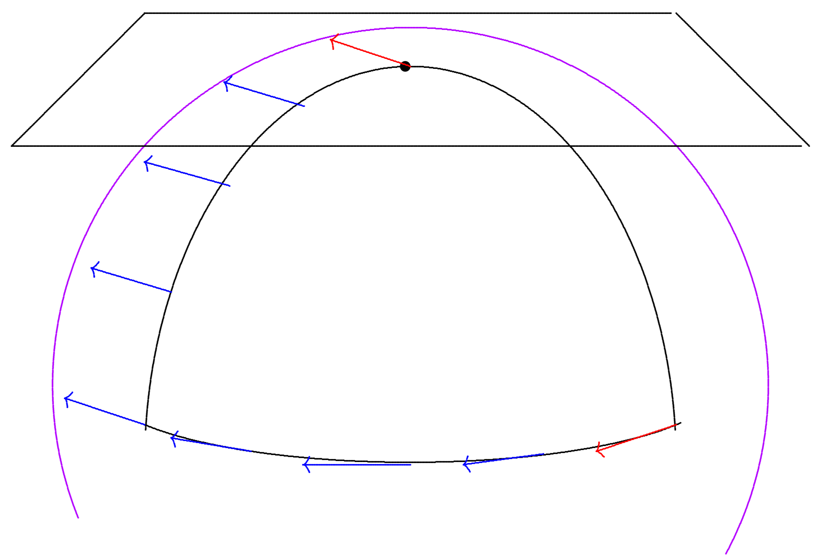

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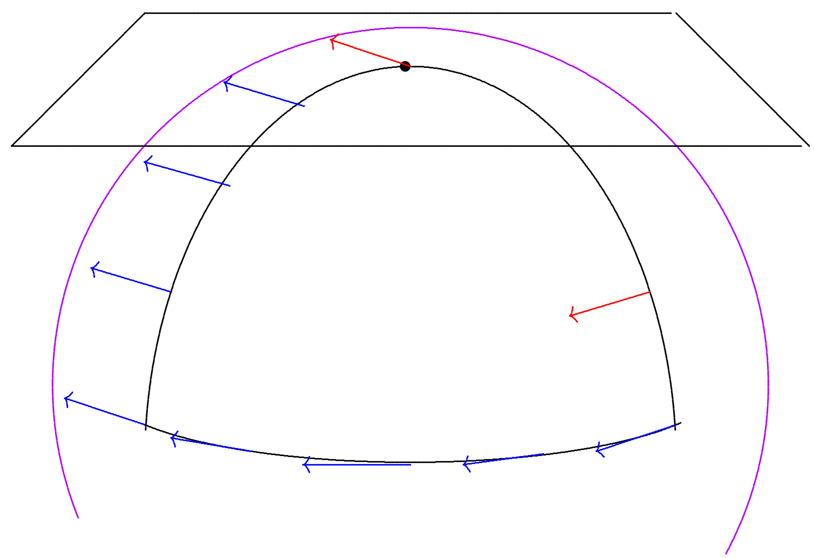

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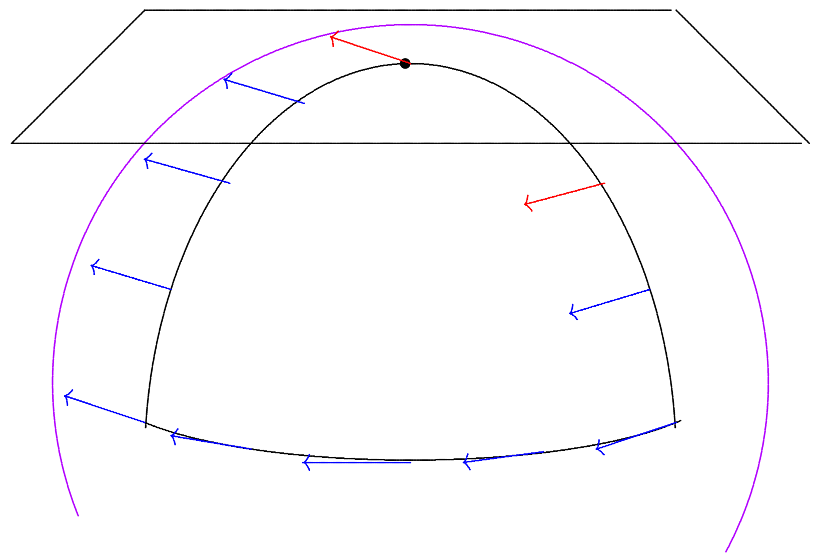

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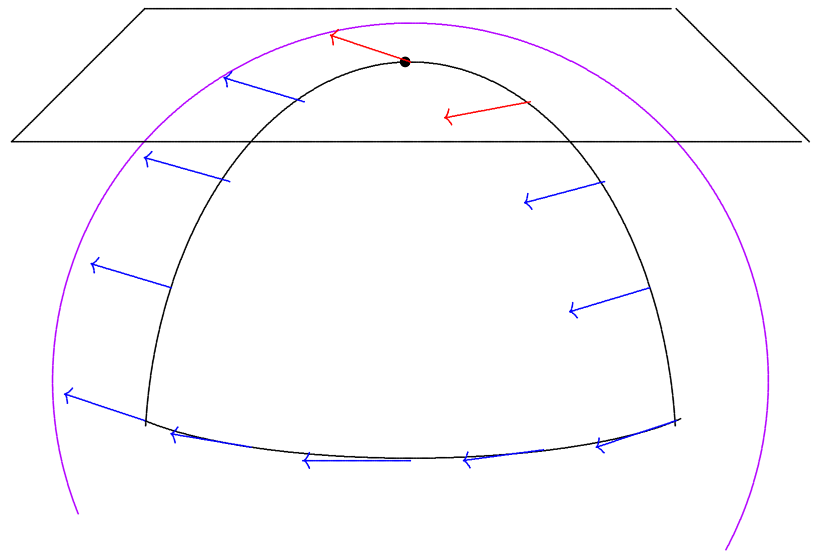

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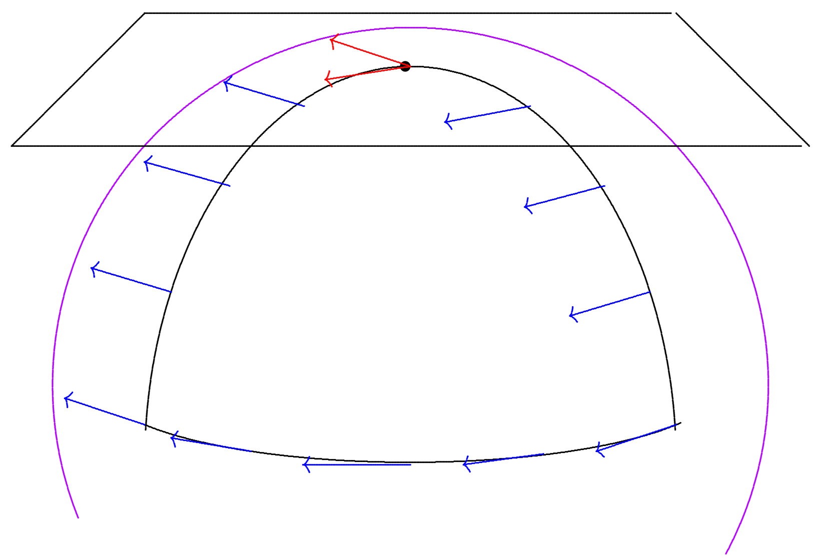

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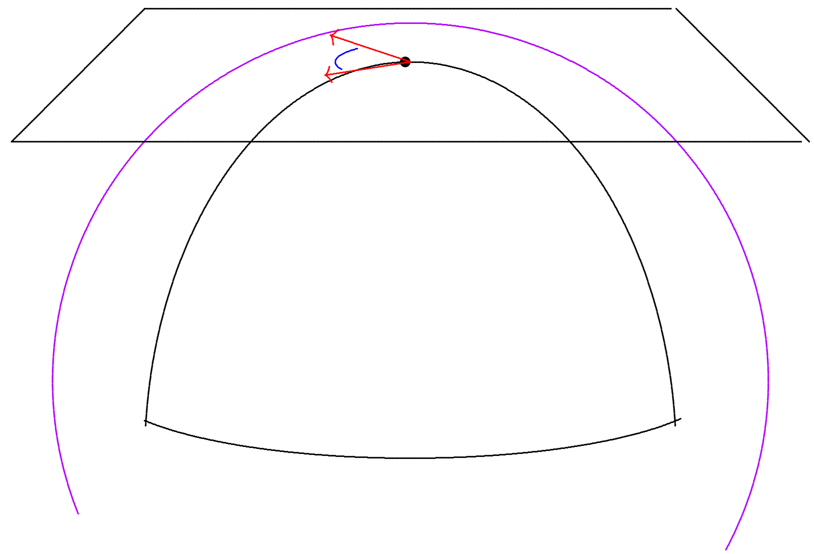
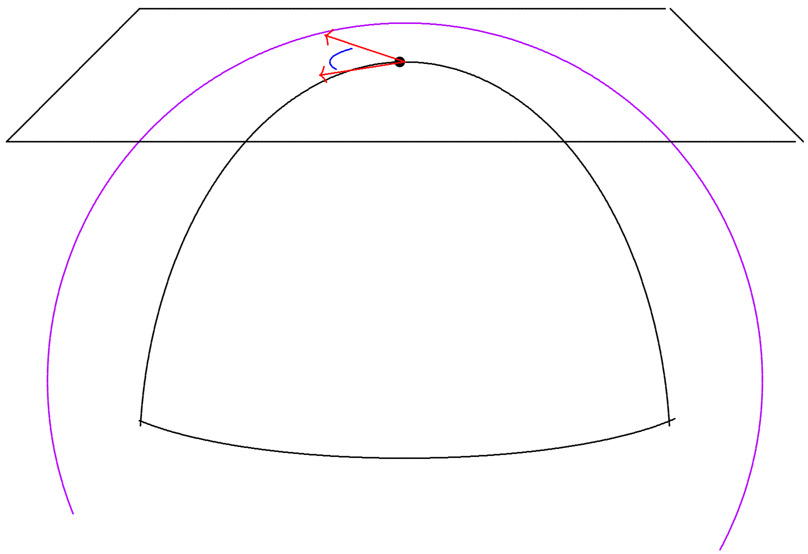
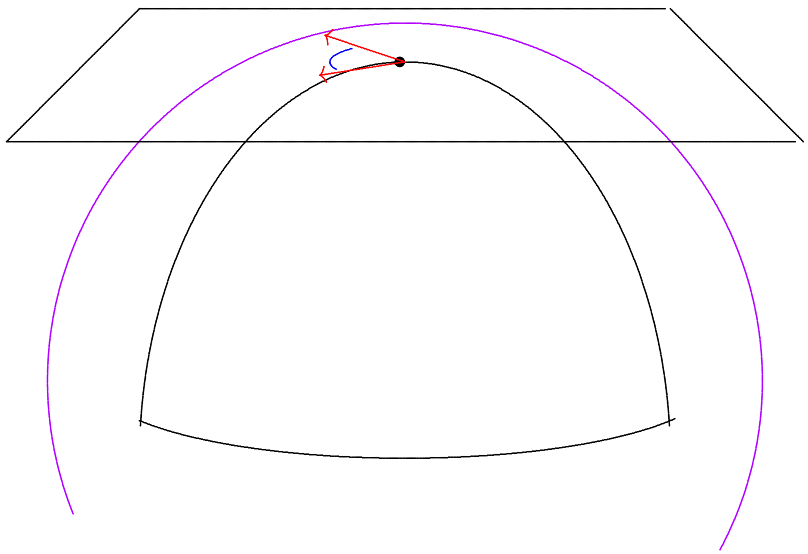


(M^n, g): holonomy $\subset \mathbf{O}(n)$



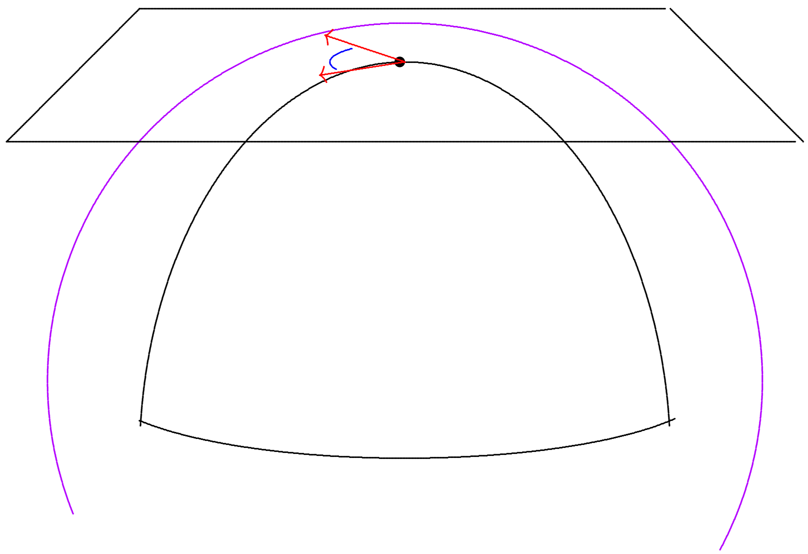
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\iff$ holonomy $\subset \mathbf{U}(m)$



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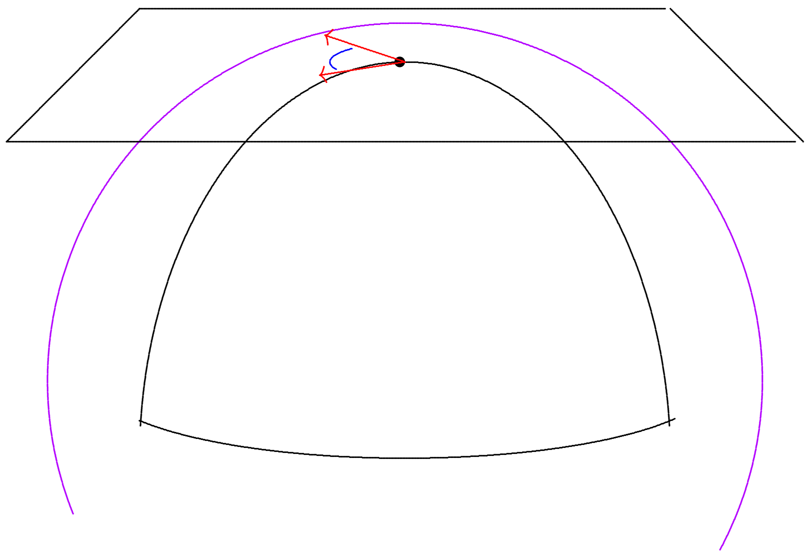

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

Kähler metrics:

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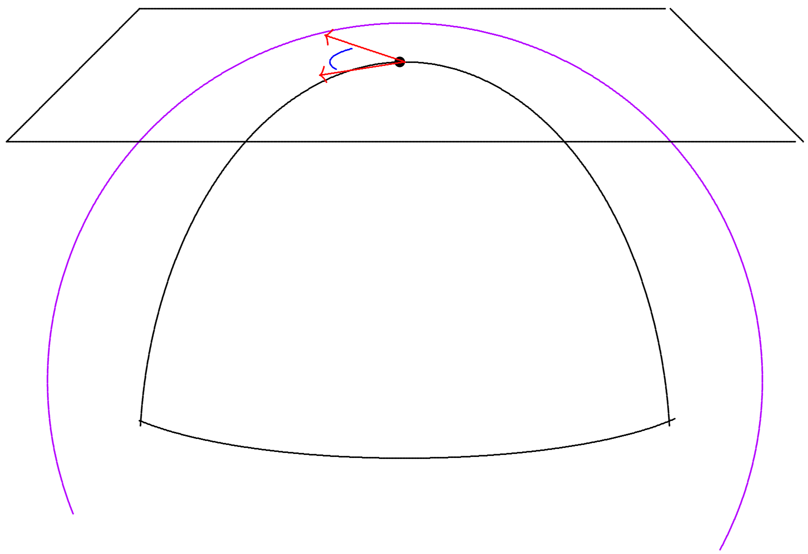

Makes tangent space a complex vector space!

Kähler metrics:

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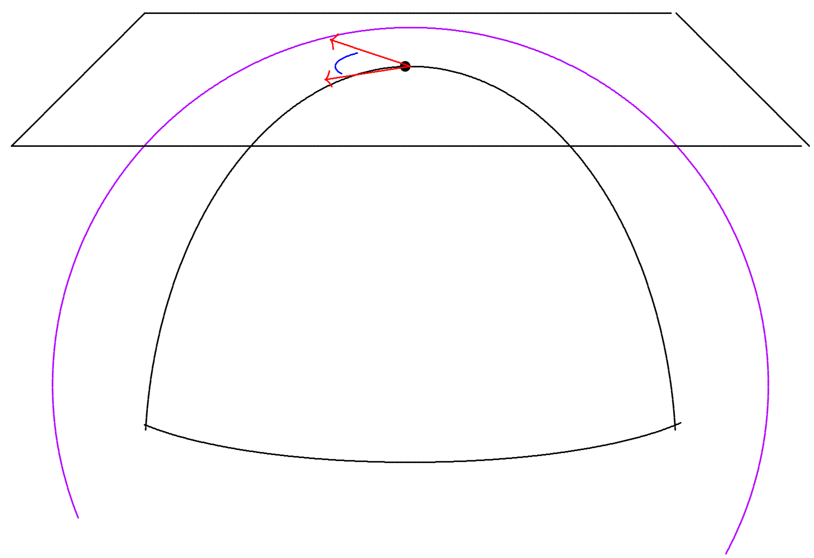

Makes tangent space a complex vector space!

$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

Kähler metrics:

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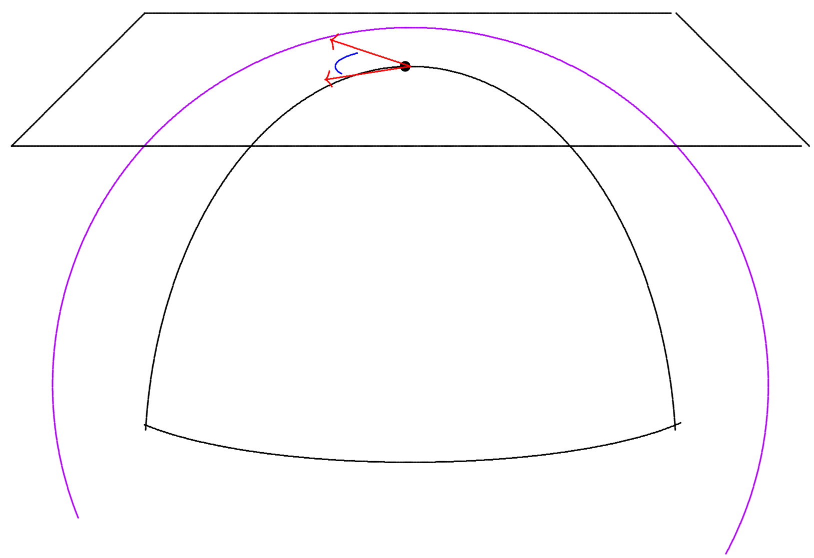

Makes tangent space a complex vector space!

Invariant under parallel transport!

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$$d\omega = 0$$

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$$[\omega] \in H^2(M)$$

"Kähler class"

Kähler metrics:

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$\Longleftrightarrow$ In local complex coordinates $(z^1, \ldots, z^m)$,

$$g = -\sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

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$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k$$

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

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Kähler magic:

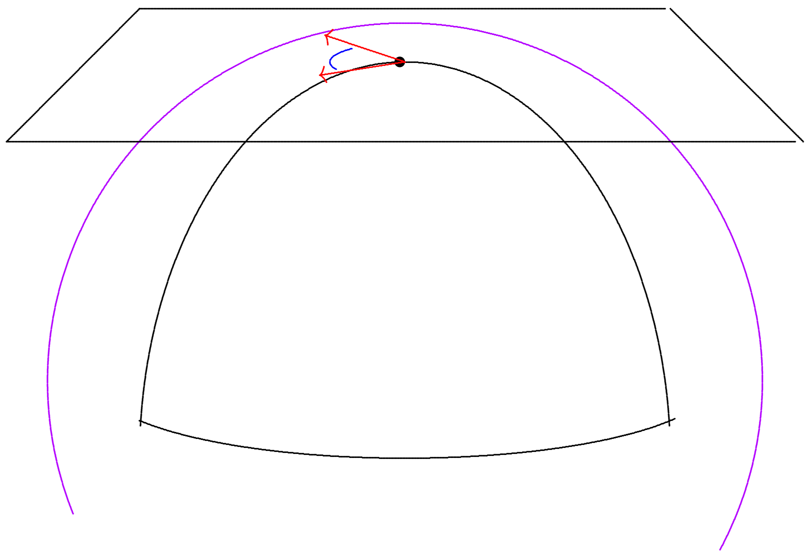
If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

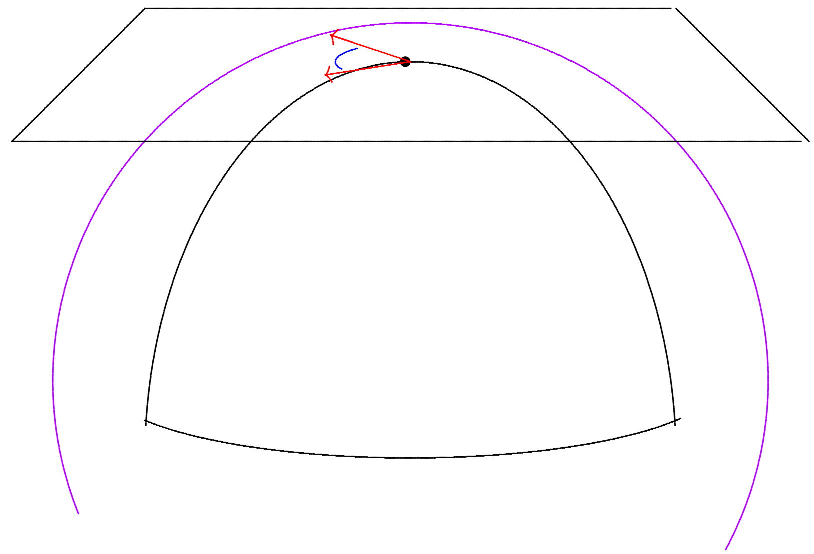
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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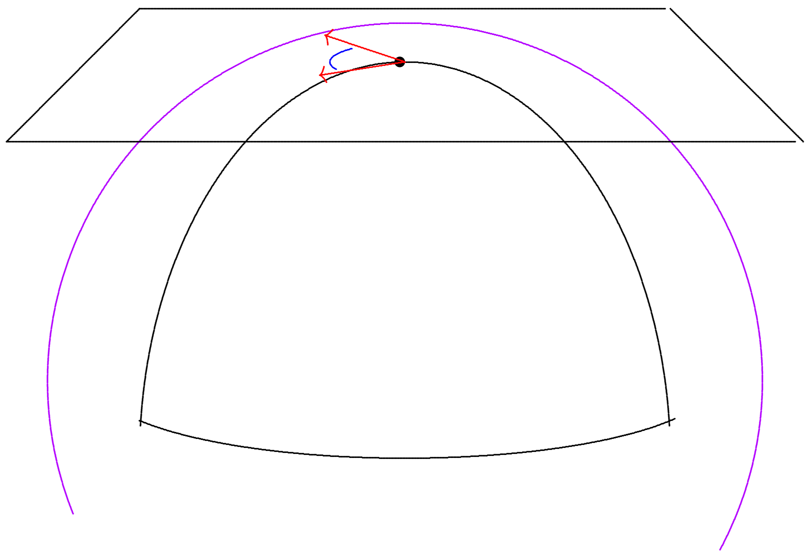

$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\{A \mid \det A = 1\}$

Kähler metrics:

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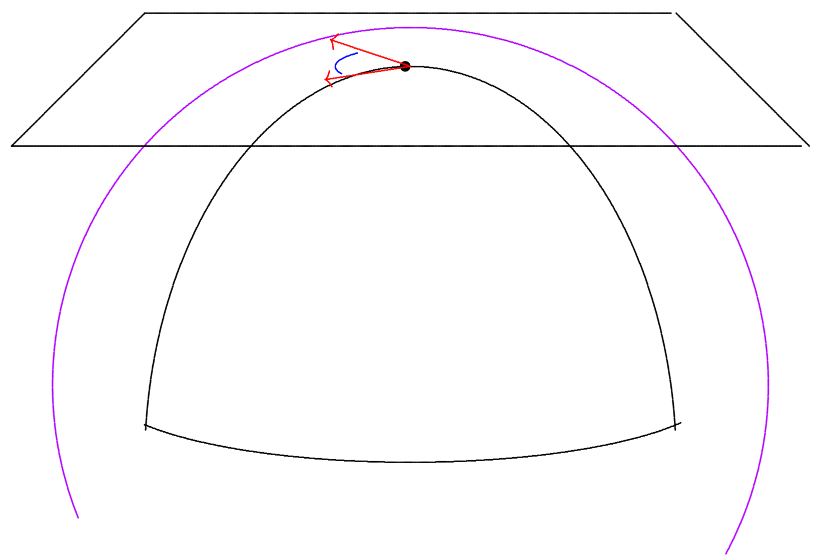

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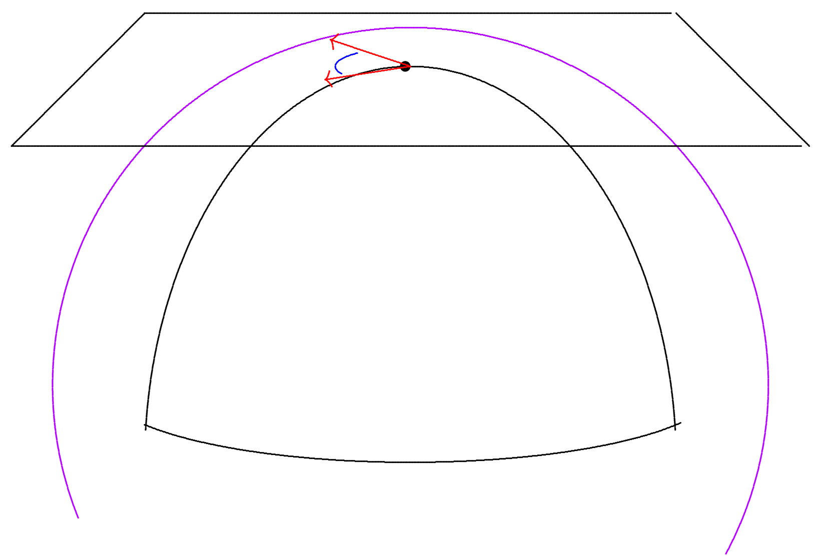
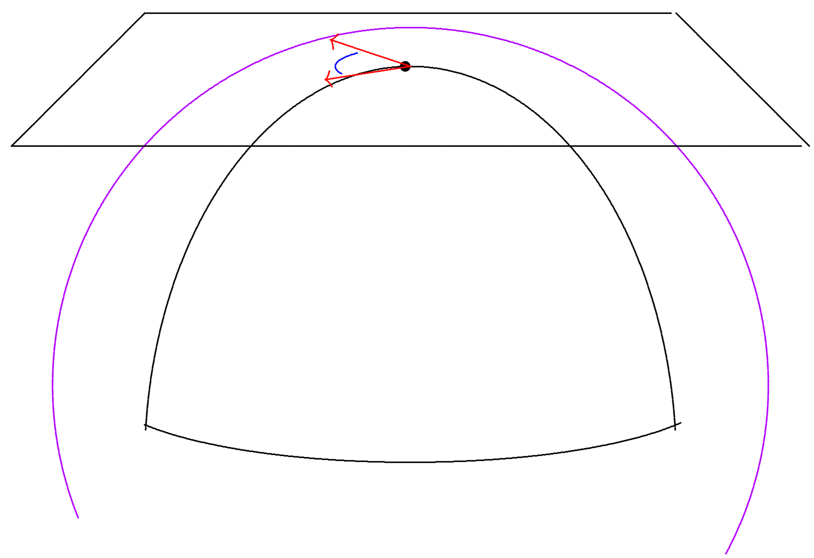


if M is simply connected.

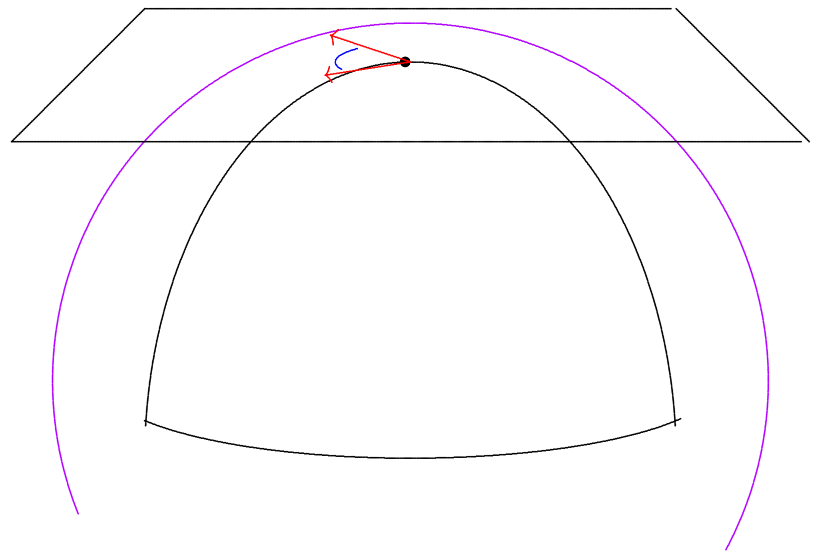
Hyper-Kähler metrics:

$(M^{4\ell}, g)$ holonomy



Hyper-Kähler metrics:

$(M^{4\ell}, g)$ hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(\ell)$



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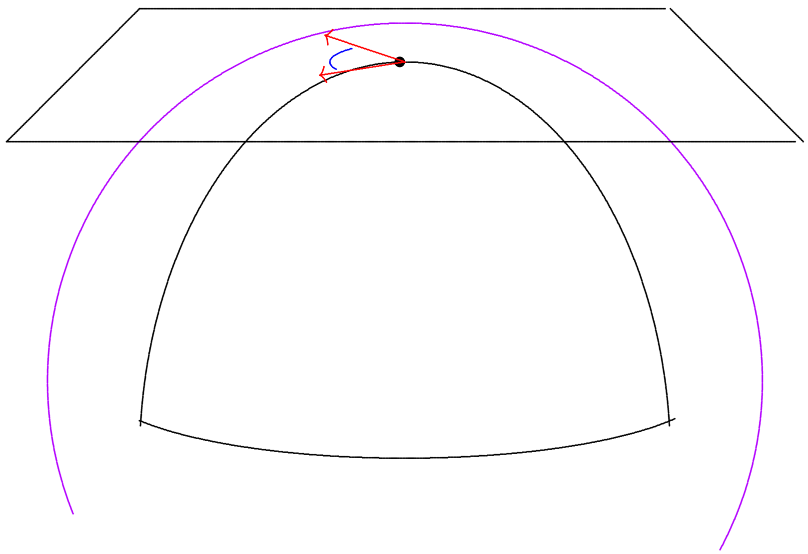

$$\mathbf{Sp}(\ell) := \mathbf{O}(4\ell) \cap \mathbf{GL}(\ell, \mathbb{H})$$

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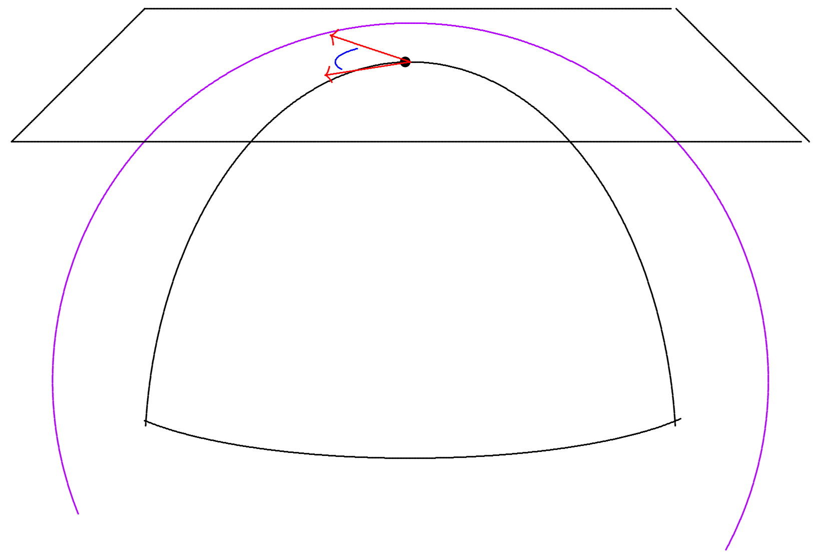

$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

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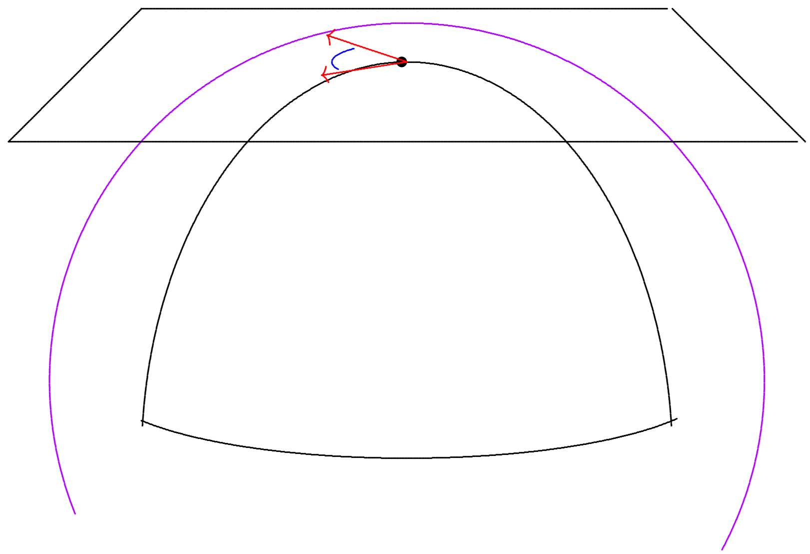

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

in many ways!

Hyper-Kähler metrics:

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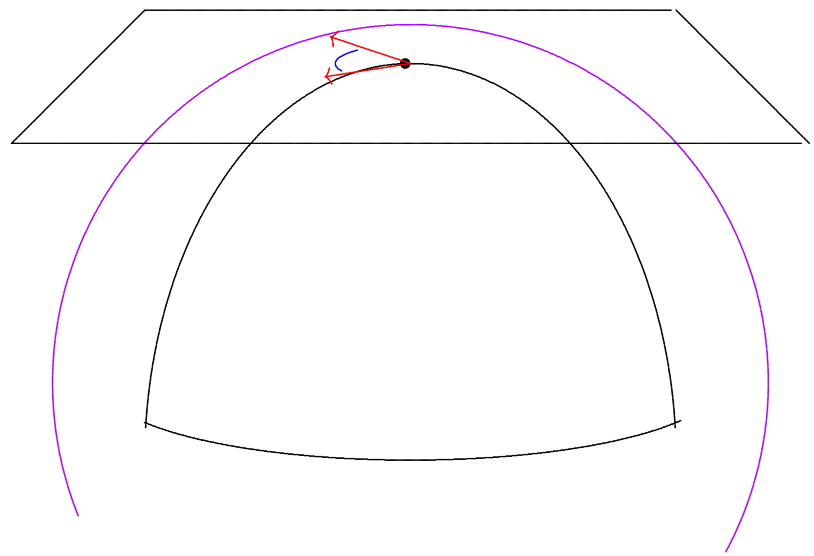

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in many ways! (For example, permute i, j, k. . .)

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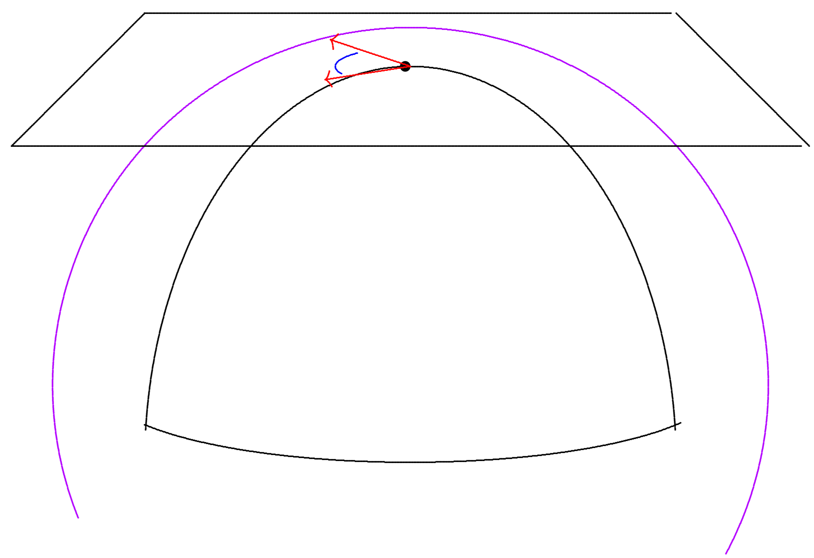

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

Ricci-flat and Kähler,

for many different complex structures!

Hyper-Kähler metrics:

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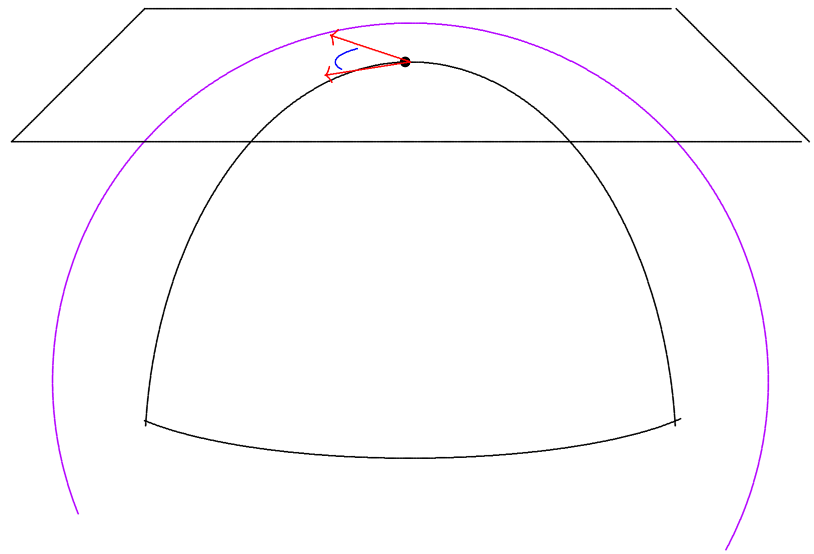
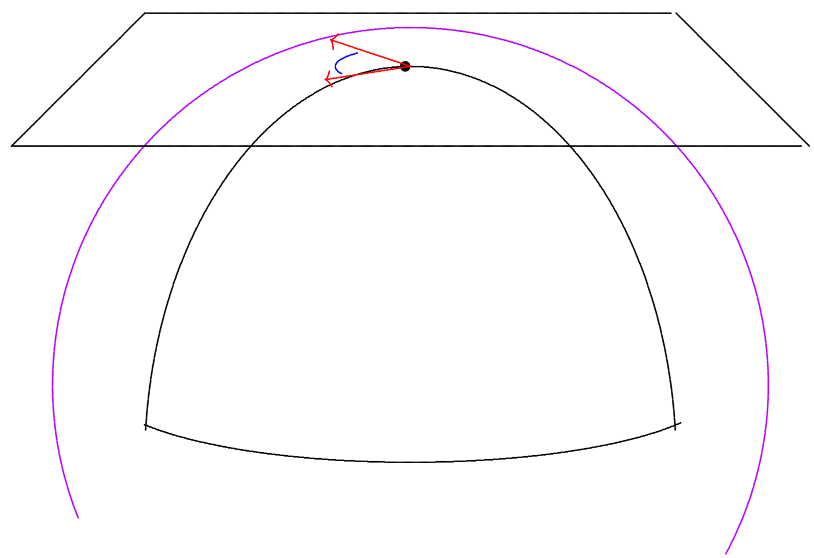


$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

Hyper-Kähler metrics:

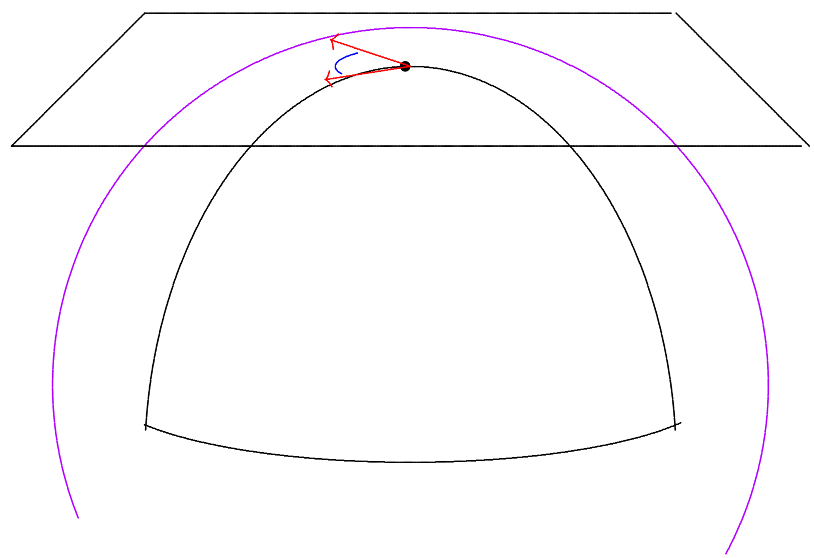
(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$\mathbf{Sp}(1) = \mathbf{SU}(2)$

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

When (M^4, g) simply connected:

hyper-Kähler $\Longleftrightarrow$ Ricci-flat Kähler.

Key examples:

Term ALE coined by Gibbons & Hawking, 1979.

They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.

Their examples have just one end, with

$$\Gamma \cong \mathbb{Z}_{k+1} \subset \mathbf{SU}(2) \subset \mathbf{O}(4).$$

The G-H metrics are hyper-Kähler, and were soon independently rediscovered by Hitchin.

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Hitchin conjectured that similar metrics would exist for each finite $\Gamma \subset \mathbf{SU}(2)$.

This conjecture was proved by Kronheimer, 1986.

Felix Klein, 1884: $\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$

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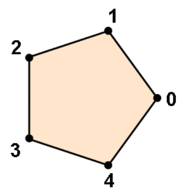

$$\mathbb{Z}_m \longleftrightarrow w^2 + x^2 + y^m = 0$$

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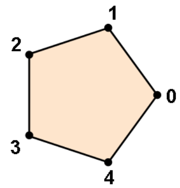

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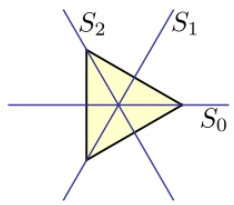

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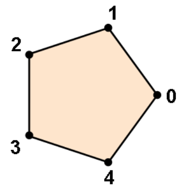

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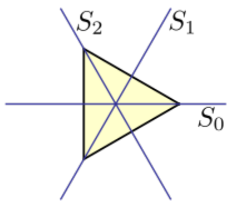

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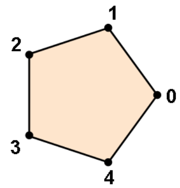

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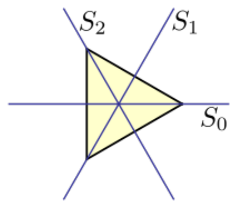

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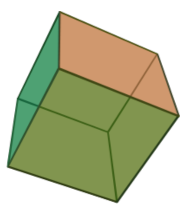
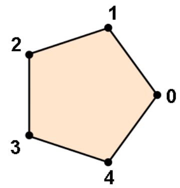


$$O^* \longleftrightarrow w^2 + x^3 + xy^3 = 0$$

Felix Klein, 1884: $\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$



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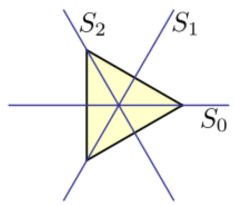

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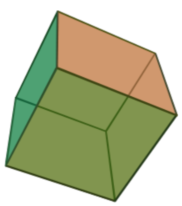
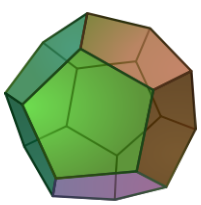
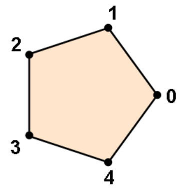


$O^* \longleftrightarrow w^2 + x^3 + xy^3 = 0$

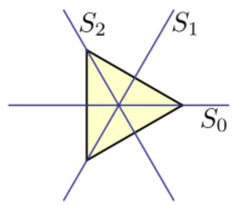


$I^* \longleftrightarrow w^2 + x^3 + y^5 = 0$

$$\Gamma \subset \mathbf{SU}(2) \longleftrightarrow \text{Gravitational Instantons}$$

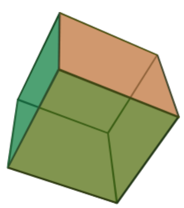


$$\mathbb{Z}_m \longleftrightarrow w^2 + x^2 + y^m = \varepsilon$$

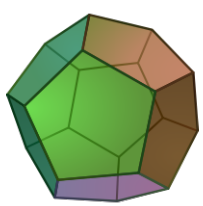


$$\mathrm{Dih}_m^* \longleftrightarrow w^2 + y(x^2 + y^m) = \varepsilon$$

$$T^* \longleftrightarrow w^2 + x^3 + y^4 = \varepsilon$$



$$O^* \longleftrightarrow w^2 + x^3 + xy^3 = \varepsilon$$



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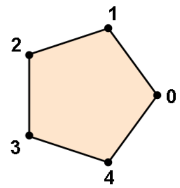
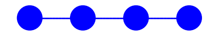
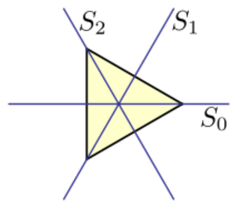
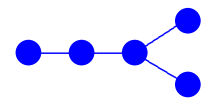


$$\mathbb{Z}_{k+1} \longleftrightarrow A_k$$

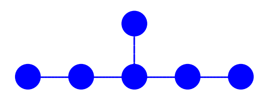


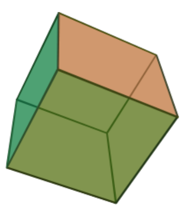


$$\mathrm{Dih}^*_{k-2} \longleftrightarrow D_k$$

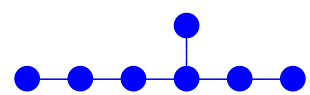


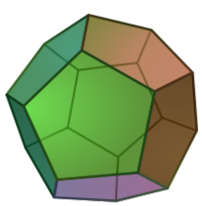
$$T^* \longleftrightarrow E_6$$



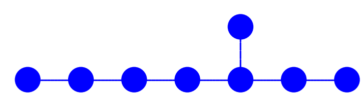


$$O^* \longleftrightarrow E_7$$



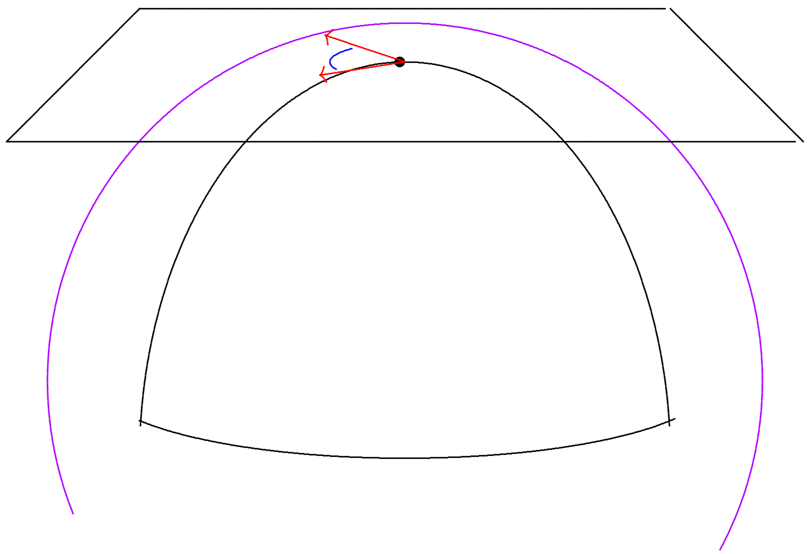


$$I^* \longleftrightarrow E_8$$



Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



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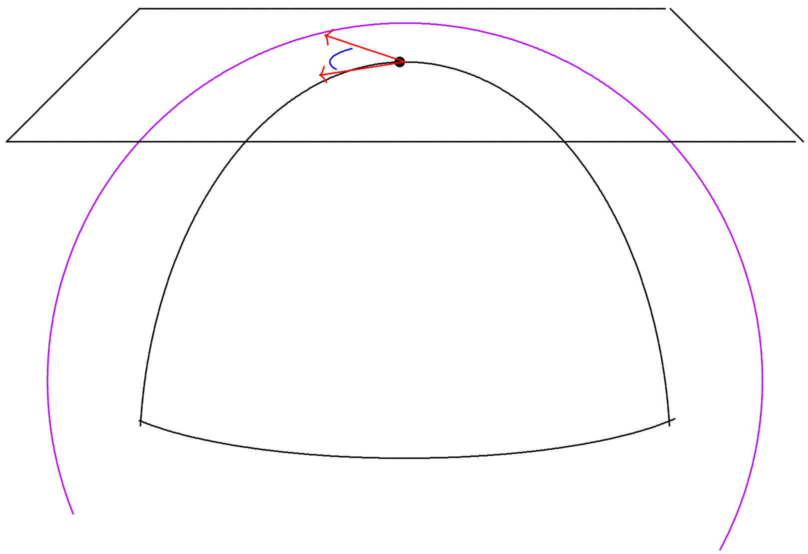
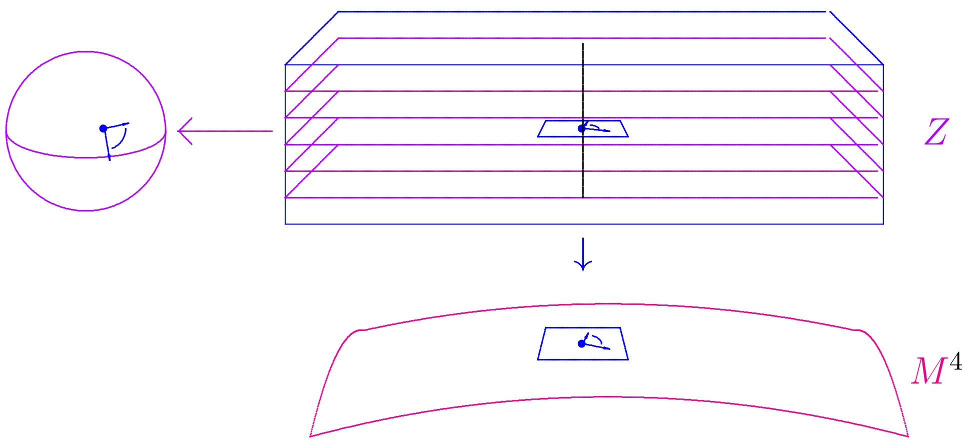


$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

Ricci-flat and Kähler,

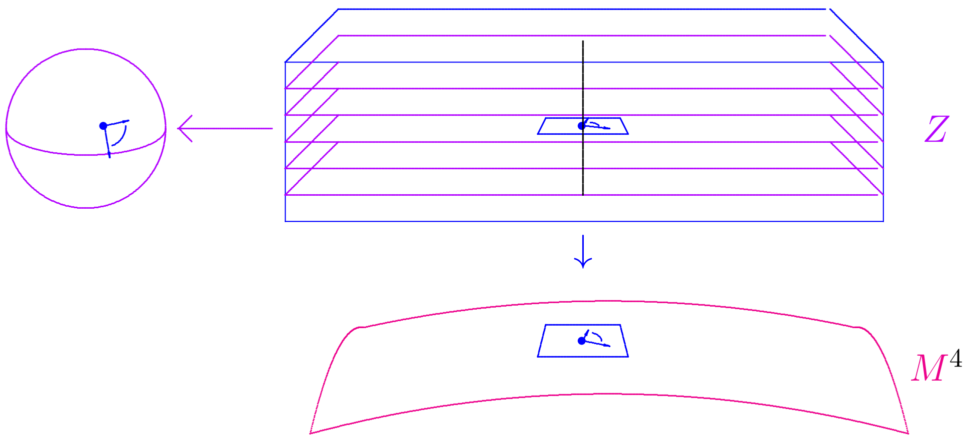
for many different complex structures!

All these complex structures can be repackaged



All these complex structures can be repackaged as

Penrose Twistor Space (Z^6, J),



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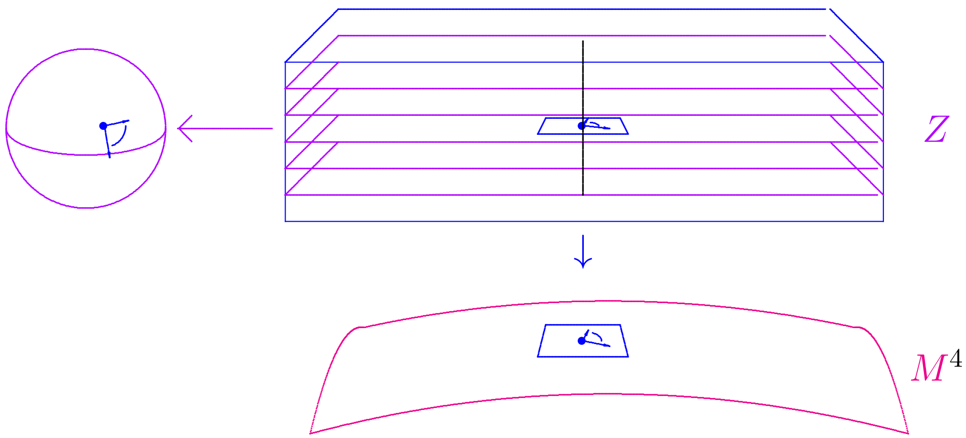

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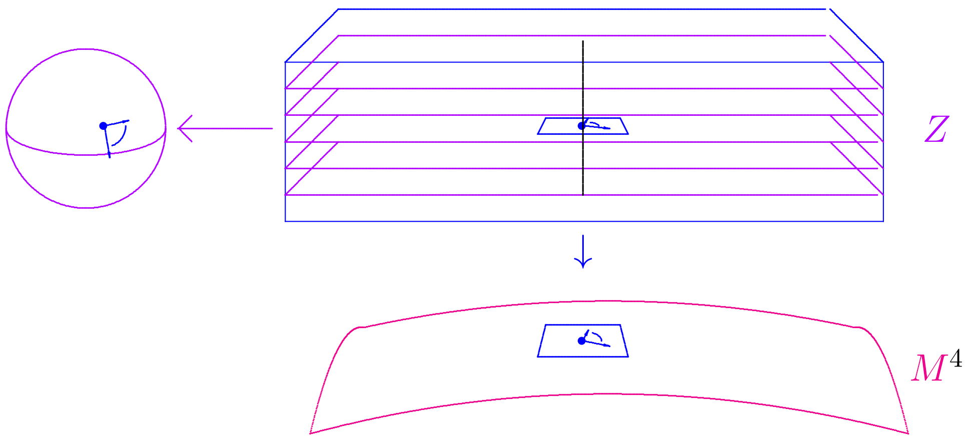

But similar for scalar-flat Kähler surfaces (M^4, g, J)!

Any scalar-flat Kähler surface (M^4, g, J) has a

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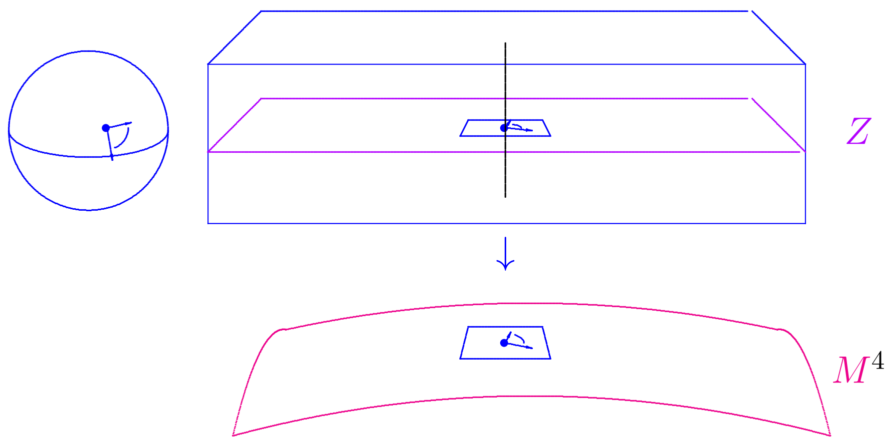

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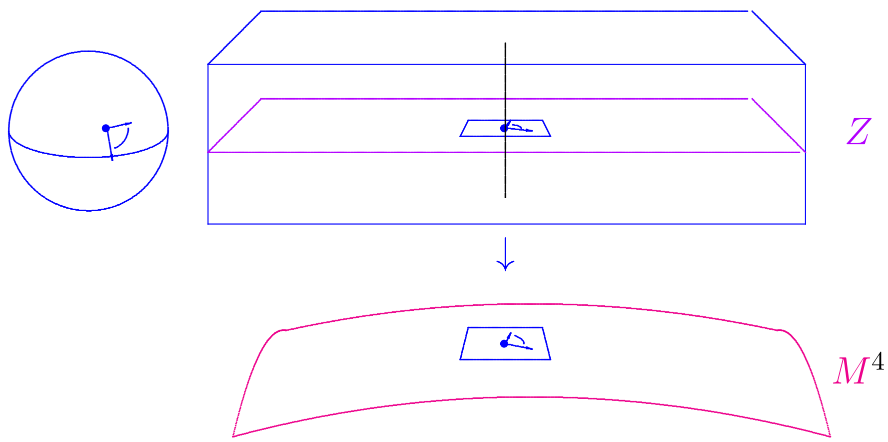

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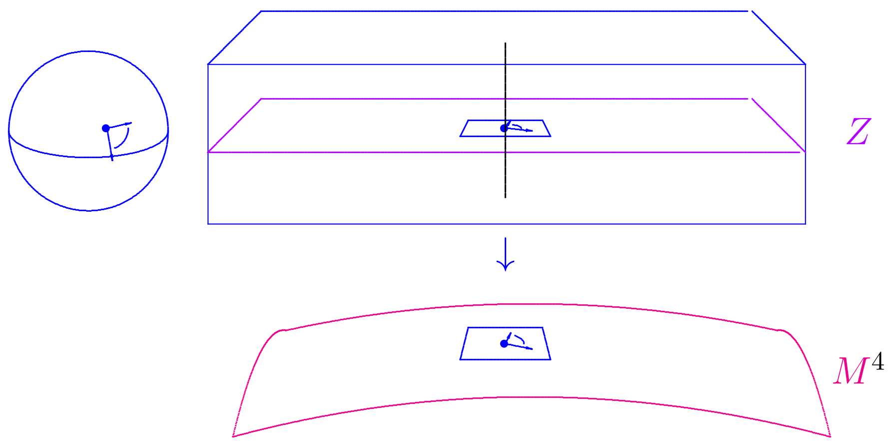

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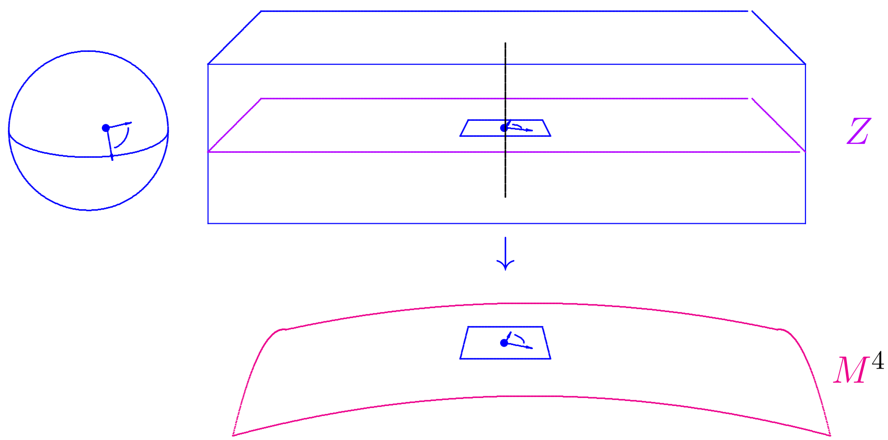

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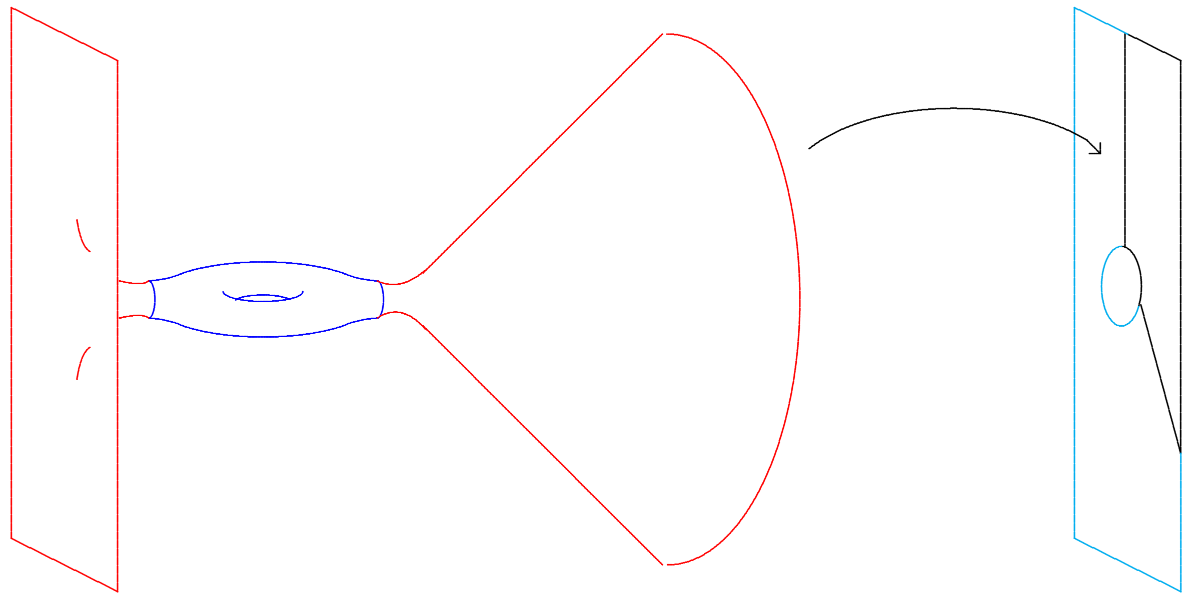
Many of the resulting examples are AE or ALE, but corresponding classification problem is still open.

Definition. *Complete, non-compact n-manifold* (M^n, g) *is* asymptotically locally Euclidean *(ALE)* *if* $\exists$ *compact set* $K \subset M$ *such that* $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\Gamma_i$, *where* $\Gamma_i \subset \mathbf{O}(\mathbf{n})$, *such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Definition. *The* mass *(at a given end) of an ALE* n*-manifold is defined to be*

$$\mathfrak{m}(M, g) := \qquad [g_{ij,i} - g_{ii,j}]$$

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$$m(M, g) := \qquad\qquad \left[g_{ij,i} - g_{ii,j}\right] \nu^j$$

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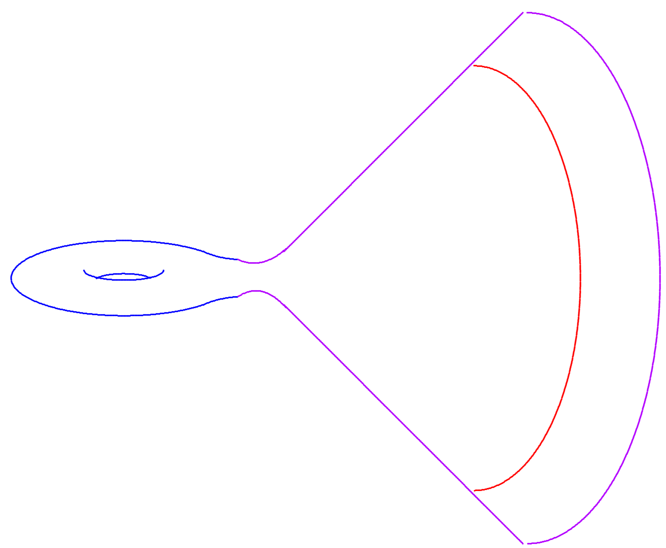

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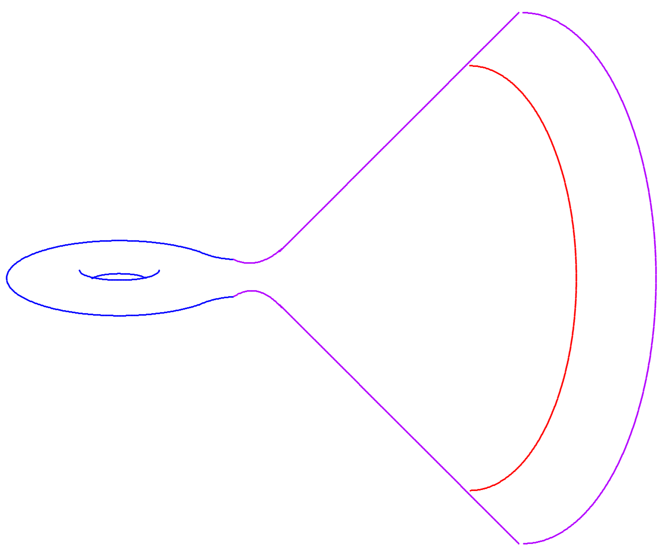

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Seems to depend on choice of coordinates!

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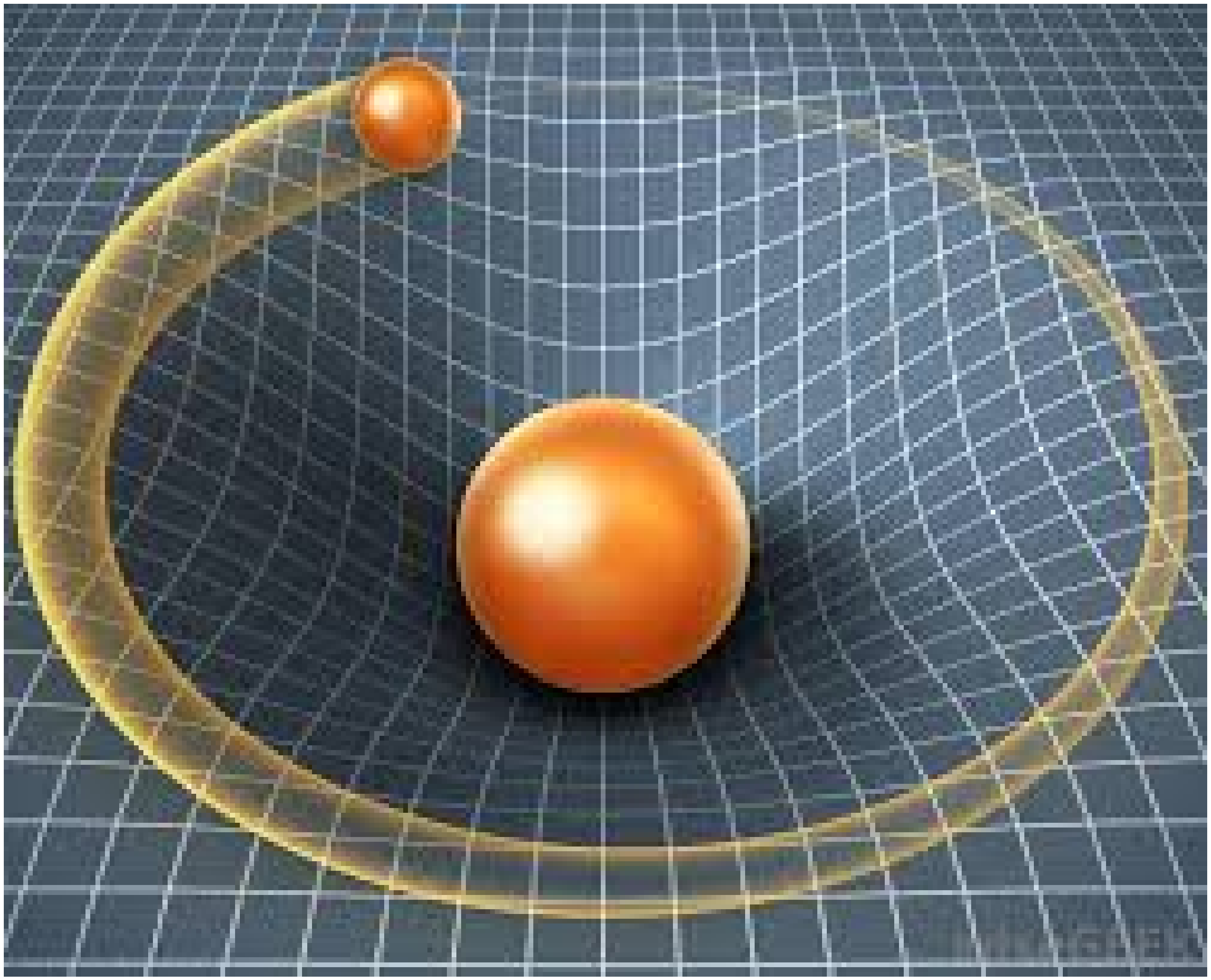

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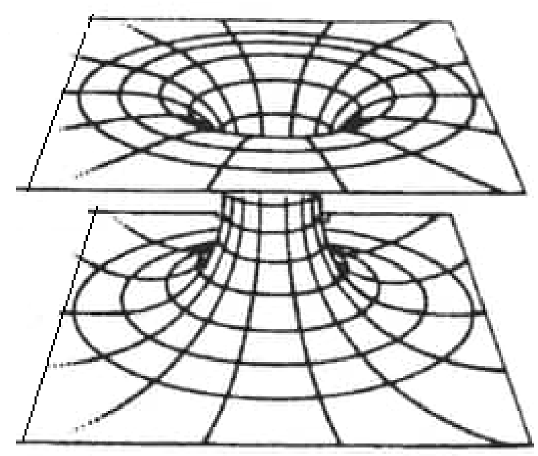
Two such regions fit together to form the wormhole metric.



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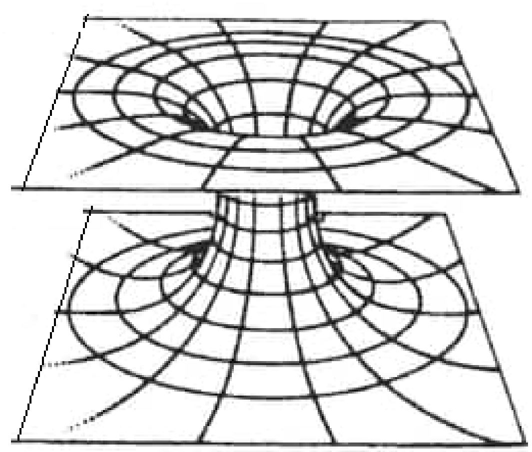
Two such regions fit together to form the wormhole metric. Scalar-flat, AE, two ends.



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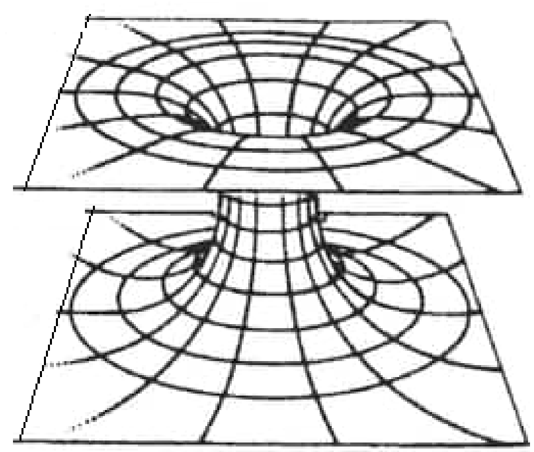

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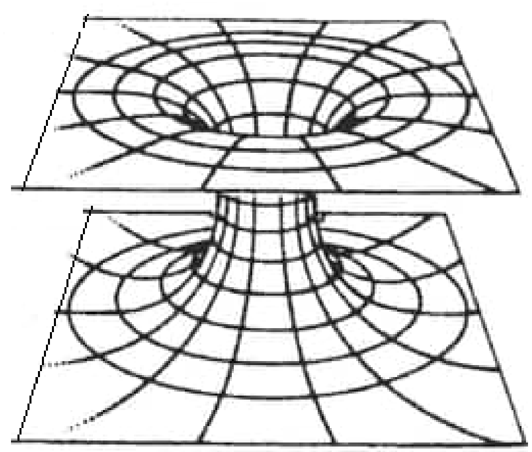

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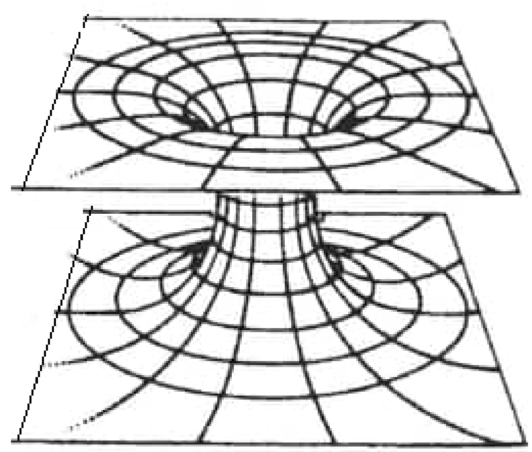
Two such regions fit together to form the wormhole metric. Scalar-flat, AE, two ends. Not Ricci-flat, but conformally flat. Same mass m at both ends: "size of throat."



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Burns metric on $\widetilde{\mathbb{C}^2} \subset \mathbb{C}^2 \times \mathbb{CP}_1$

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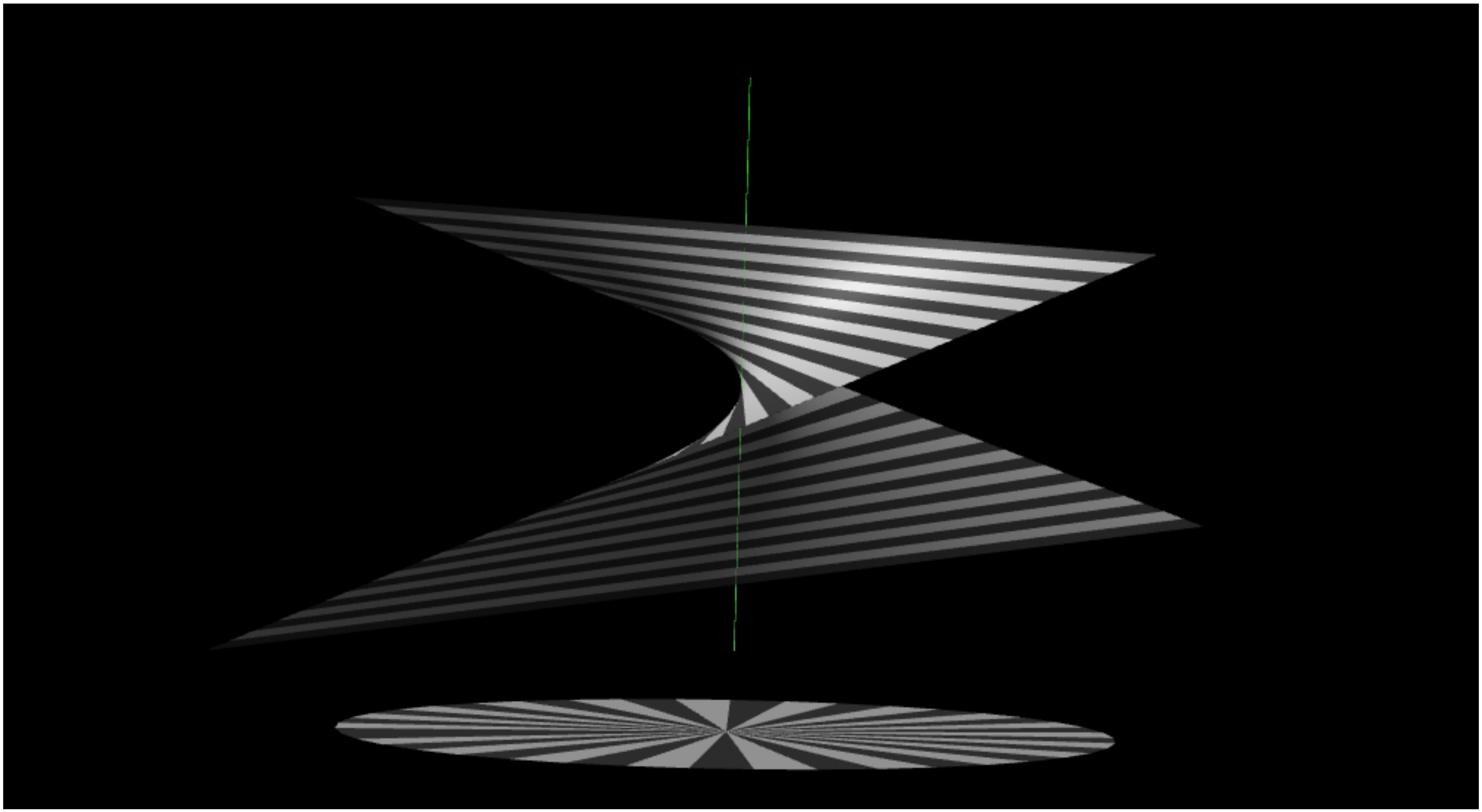

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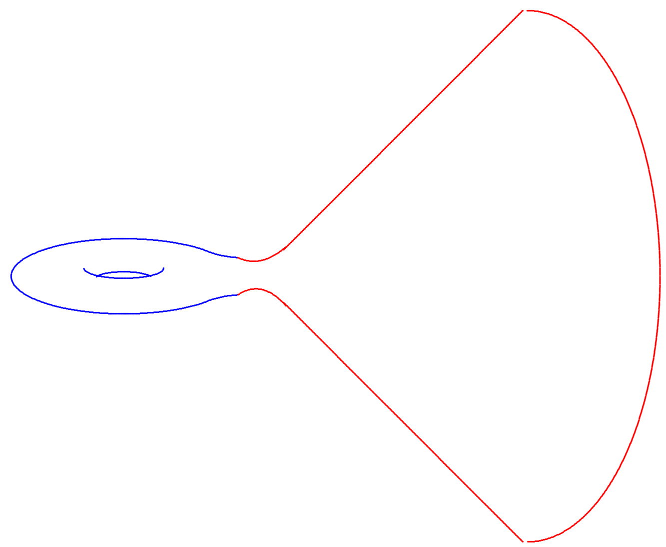

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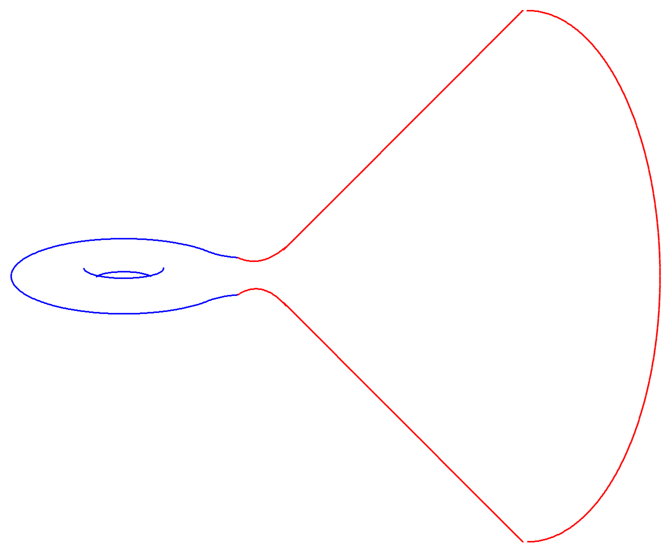

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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Suffices to examine solutions I had constructed in 1989, for which mass had never been calculated!

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}}.$$

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So **Theorem A** is an immediate consequence!

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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However, since $s = 0$,

$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4}\omega^2 = 0.$$

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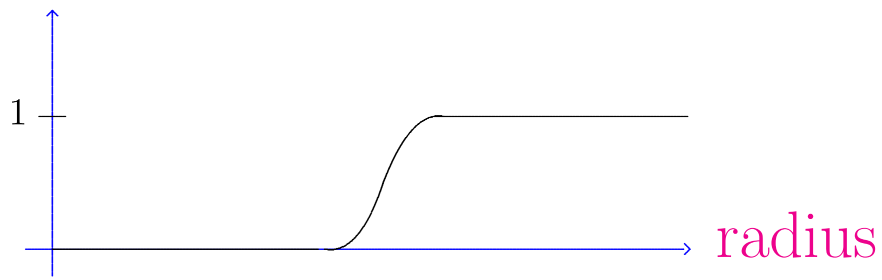

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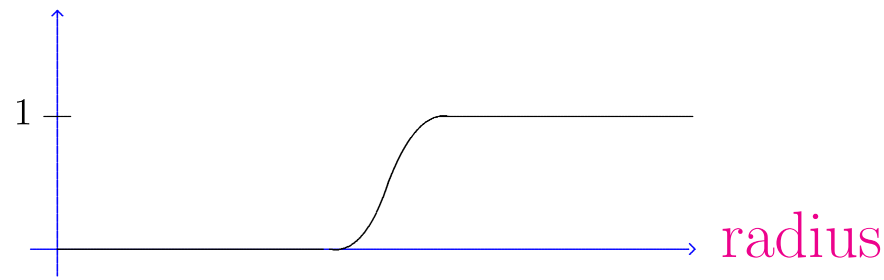

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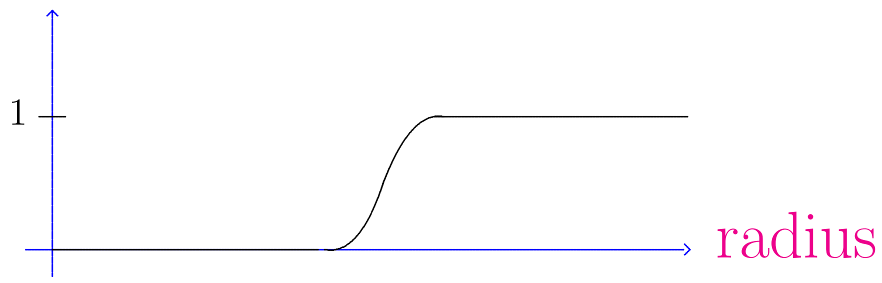

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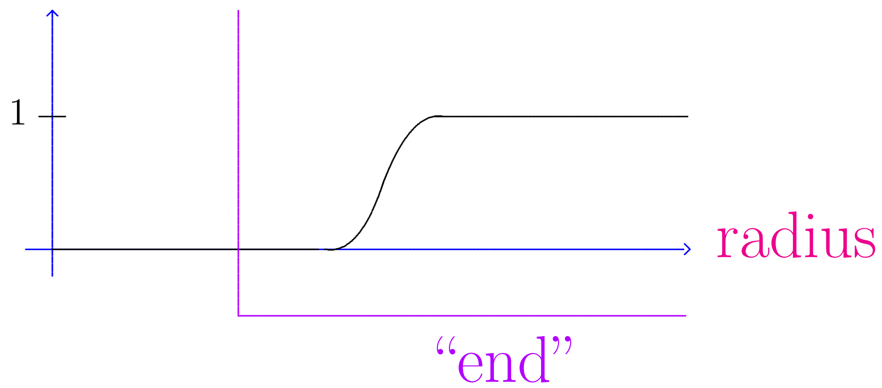

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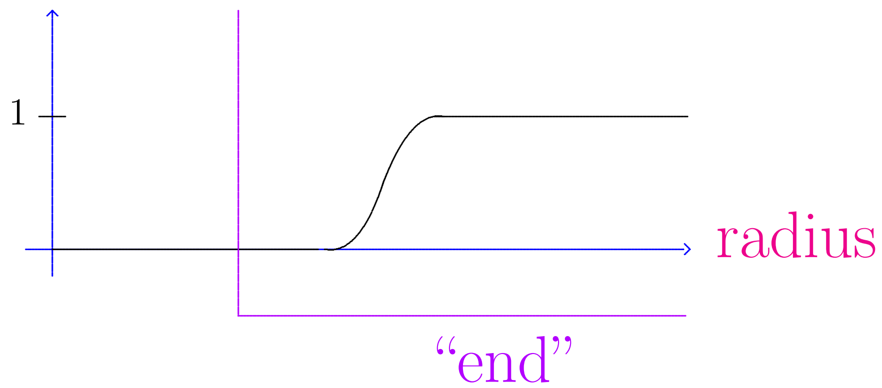

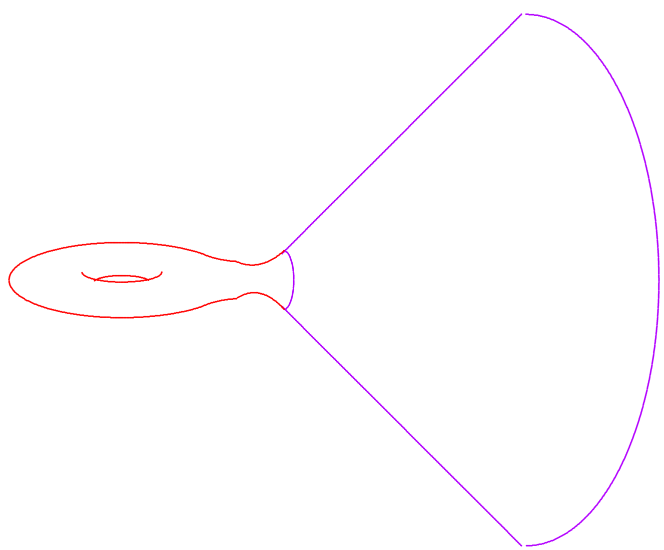

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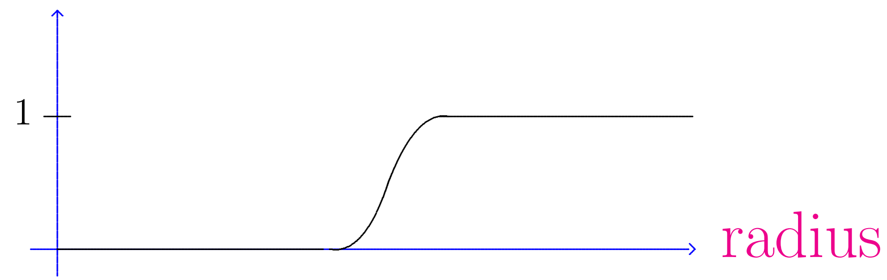

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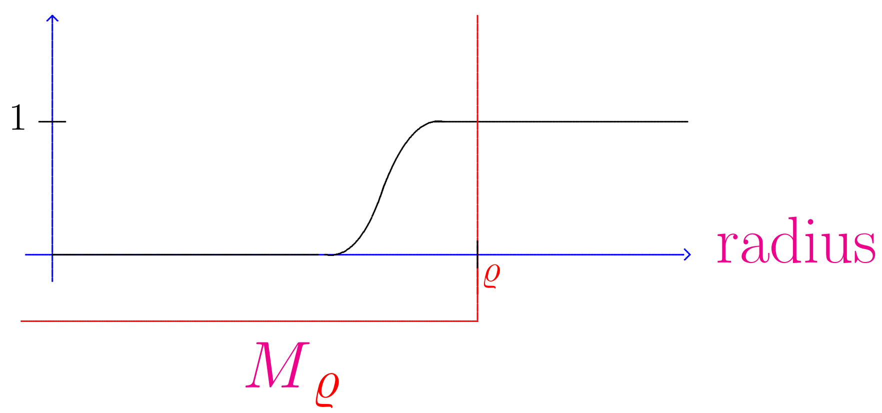

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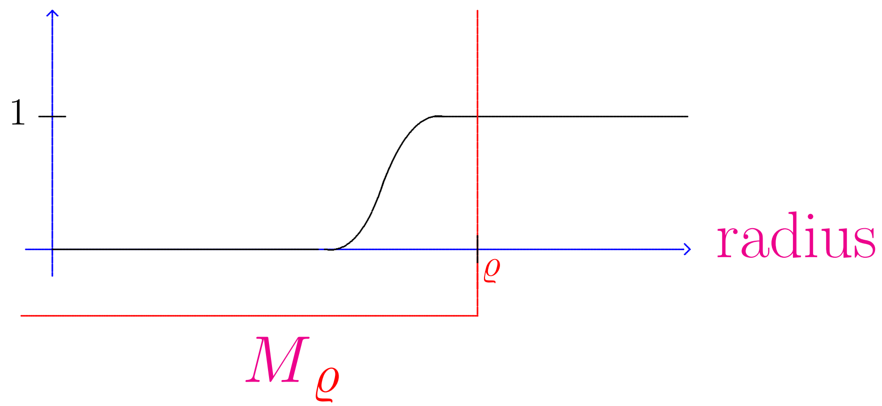

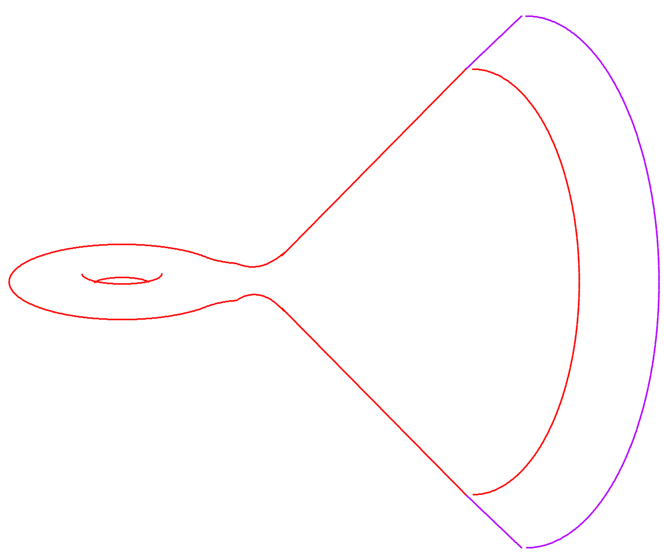

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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The last point is serious.

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Seen in "gravitational instantons"

and other explicit examples.

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One argument proceeds by osculation:

$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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Complete analytic family encodes info about J.

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This has some interesting consequences. . .

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$$AE \;\&\; \text{Kähler} \;\&\; s \geq 0 \quad \Longrightarrow \quad m(M, g) \geq 0.$$

Theorem D (Positive Mass Theorem). *Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:*

$$AE \ \& \ Kähler \ \& \ s \geq 0 \quad \Longrightarrow \quad \mathfrak{m}(M, g) \geq 0.$$

Moreover, $\mathfrak{m} = 0 \Longleftrightarrow$

Theorem D (Positive Mass Theorem). *Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:*

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Proof actually shows something stronger!

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with $=$ $\Longleftrightarrow$ (M, g, J) *is scalar-flat Kähler.*

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so the mass formula implies the claim.

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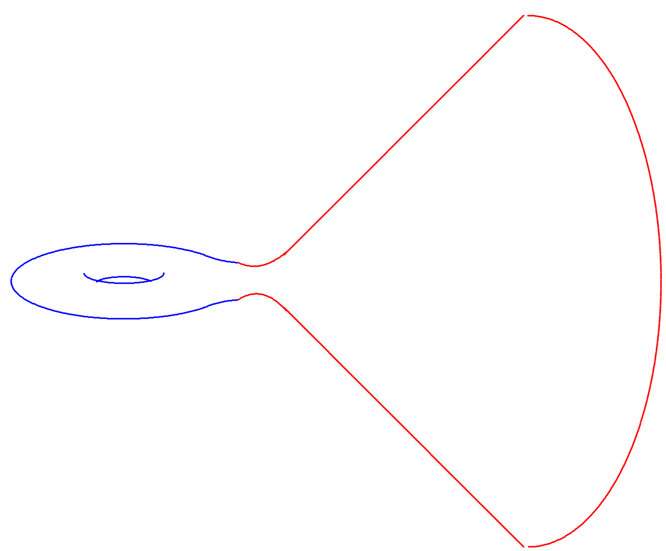

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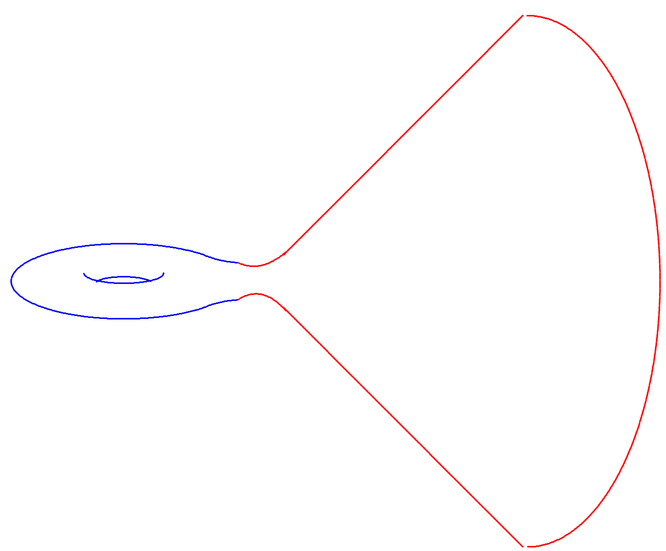

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