From Laplacian growth to competitive erosion

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September 8, 2016

Joint work with Lionel Levine

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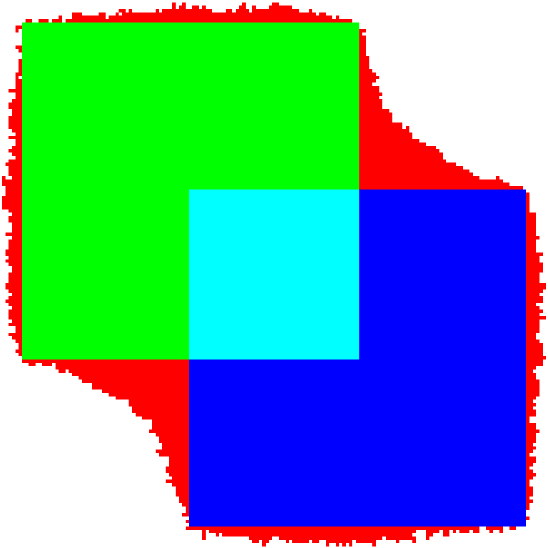
  where y_j is the endpoint of a random walk started at x_j and stopped on exiting C_{j-1}.
- Define $A + B = C_k$.
- Abelian property: the law of $A + B$ does not depend on the ordering of $x_1, \ldots, x_k$.



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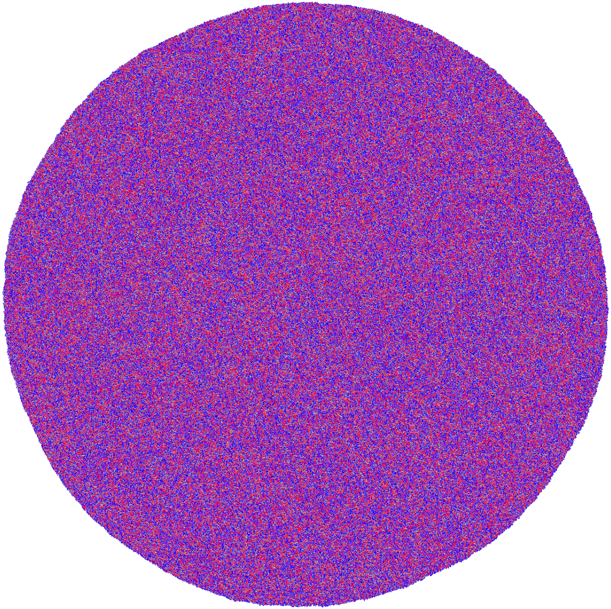
  for all sufficiently large r.
- Here $B_r = \{x \in \mathbb{Z}^d \ : \ |x| < r\}$, and ω_d is the volume of the unit ball in $\mathbb{R}^d$.
- Logarithmic error bounds recently proved by Assaleh-Gaudilierre and by Jerison-Levine-Sheffield.



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- Each site $x \in \mathbb{Z}^2$ has a **rotor** pointing North, South, East or West.
  (Start all rotors pointing North, say.)
- A particle starts at the origin. At each site it comes to, it
  1. Turns the rotor clockwise by 90 degrees;
  2. Takes a step in direction of the rotor.

Rotor-Router Aggregation (Proposed by Jim Propp)

- Sequence of lattice regions

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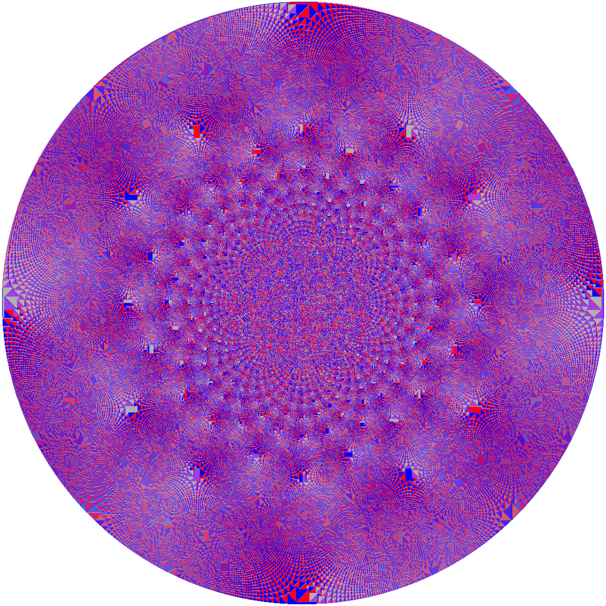
- Makes sense in $\mathbb{Z}^d$ for any d.
- Choices of which particles to route in what order don't affect the final shape generated or the final rotor directions.



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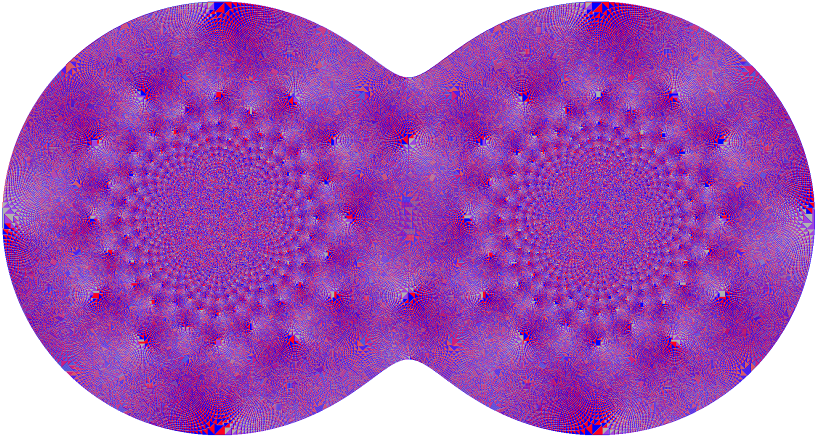
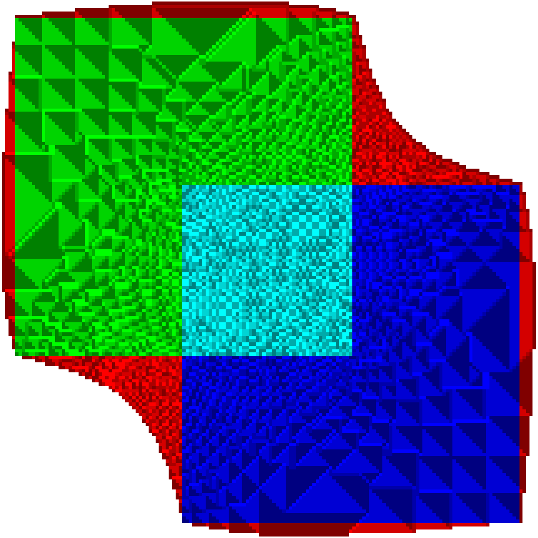
  - c, c' depend only on d.
- **Corollary**: Inradius/Outradius $\to 1$ as $n \to \infty$.





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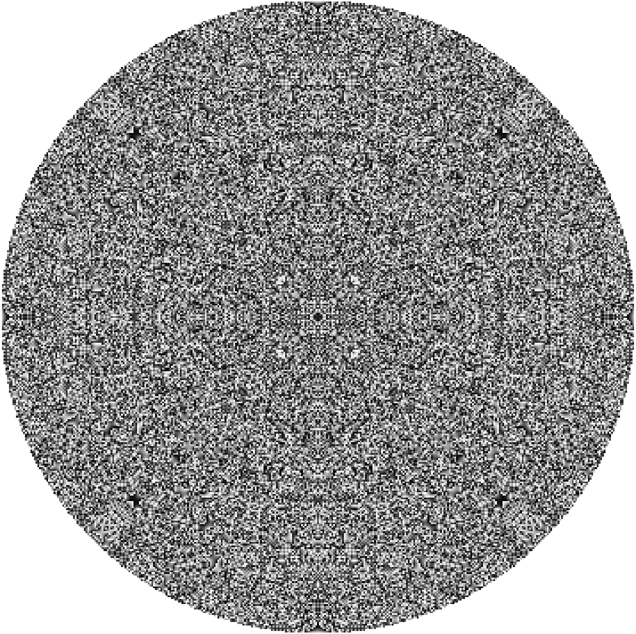
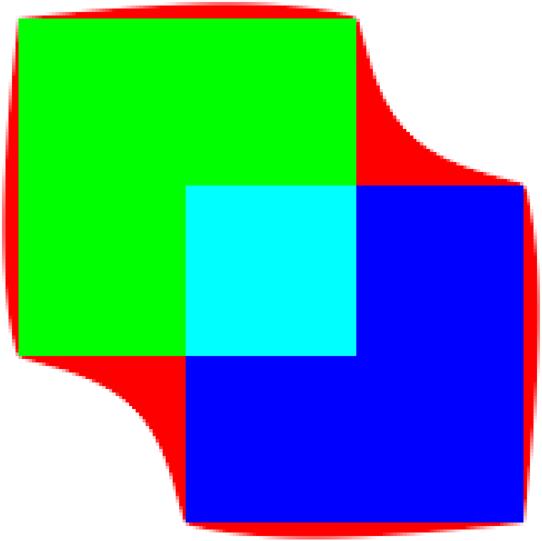
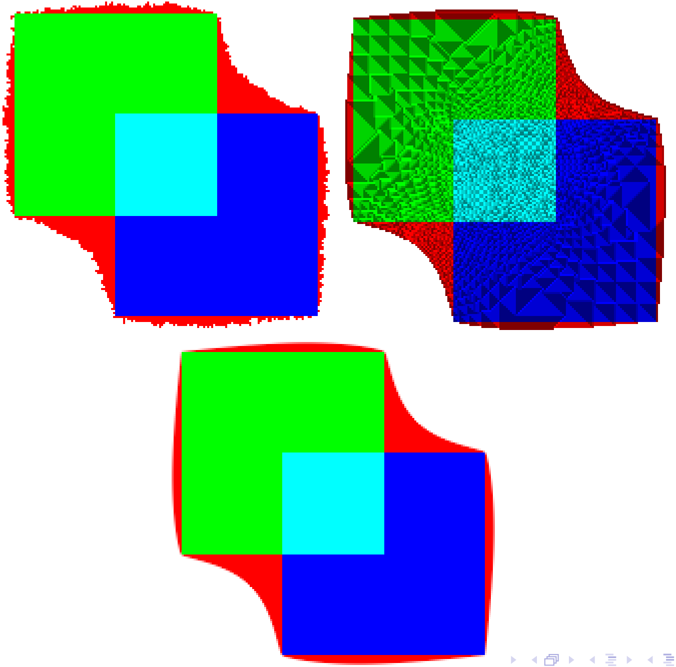
- **Theorem** (Levine-P.): There are constants c and c' depending only on d, such that

$$B_{r-c} \subset A_m \subset B_{r+c'}$$

where $m = \omega_d r^d$.







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- If so, can we describe the limiting shape?
- Is it the same for all three models?
- Not clear how to define dynamics in $\mathbb{R}^d$.

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$$= \begin{cases} -1 & x \in A \cap B \\ 0 & x \in A \cup B - A \cap B \\ 1 & x \in A \oplus B - A \cup B. \end{cases}$$

Least Superharmonic Majorant

- Let
$$\gamma(x) = -|x|^2 - \sum_{y \in A} g(x,y) - \sum_{y \in B} g(x,y),$$
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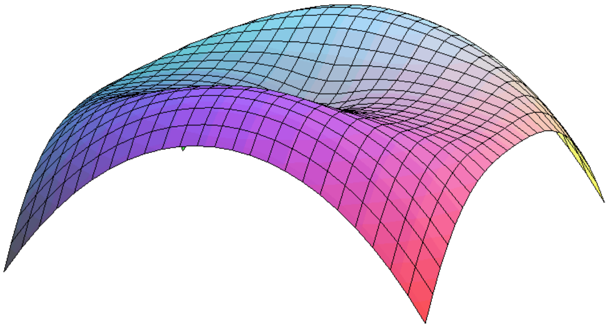
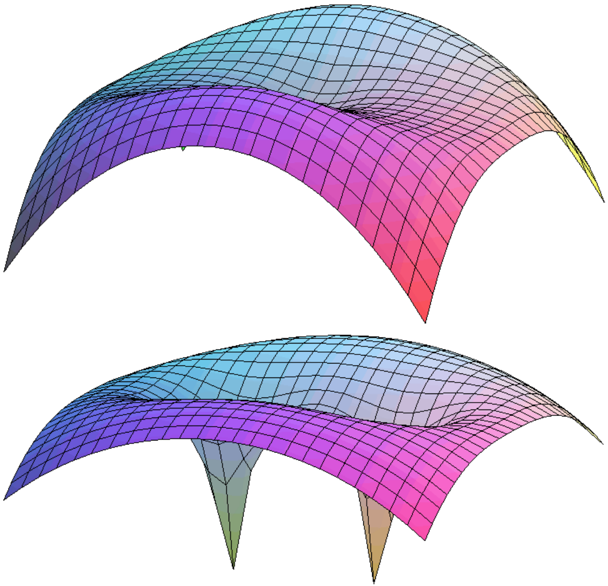
  - In dimension two, we use the negative of the potential kernel in place of g.
- Let $s(x) = \inf\{\phi(x) \mid \phi \text{ superharmonic}, \phi \geq \gamma\}$.
- **Claim**: odometer $= s - \gamma$.





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- Reverse inequality: $s-\gamma-u$ is superharmonic on $A \oplus B$ and nonnegative outside $A \oplus B$, hence nonnegative inside as well.

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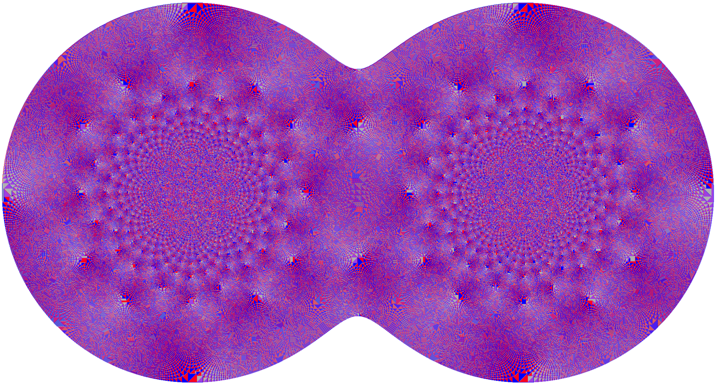

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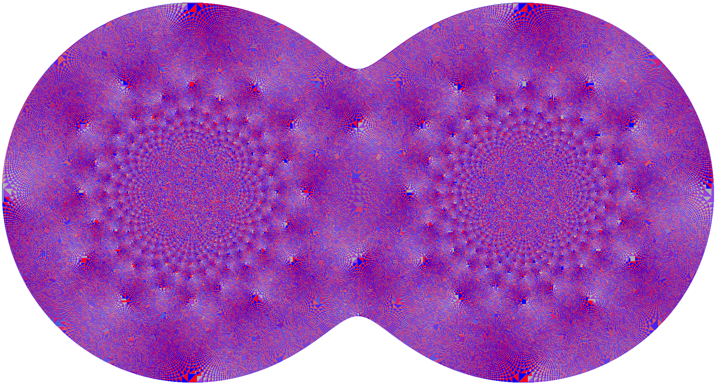

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The boundary ∂D is given by the algebraic curve

$$\left(x^2+y^2\right)^2 - 2r^2\left(x^2+y^2\right) - 2(x^2-y^2) = 0.$$

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- Follows from the main result and the case of a single point source.

Steps of the Proof

convergence of densities

$$\Downarrow$$

convergence of obstacles

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convergence of domains.

Adapting the Proof for Rotors

- Rotor-router odometer:

$$u(x) = \text{total number of particles emitted from } x.$$

- Instead of $\Delta u = 1$, we only know $-2 \leq \Delta u \leq 4$.
- Repeating the argument only gives

$$B_{cr} \subset A_n \subset B_{c'r}.$$

Smoothing

- To do better, let

$$v(x) = \frac{1}{4k^2} \sum_{y \in S_k(x)} u(y)$$

where $S_k(x)$ is a box of side length $2k$ centered at x.

- Using $\Delta =$ div grad, we get

$$\begin{aligned} \Delta v(x) &= \frac{1}{4k^2} \sum_{(y,z) \in \partial S_k(x)} \frac{u(z) - u(y)}{4} \\ &= 1 + O\left(\frac{1}{k}\right) \end{aligned}$$

if $o \notin S_k(x)$ and all sites in $S_k(x)$ are occupied.

A Quadrature Identity

- If h is harmonic on $\delta_n \mathbb{Z}^d$, then

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- Optional stopping:

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- Therefore if $I_n \to D$, we expect the limiting domain $D \subset \mathbb{R}^d$ to satisfy
$$\int_D h(x) dx = \sum_{i=1}^{k} \lambda_i h(x_i).$$
for all harmonic functions h on D.

Quadrature Domains

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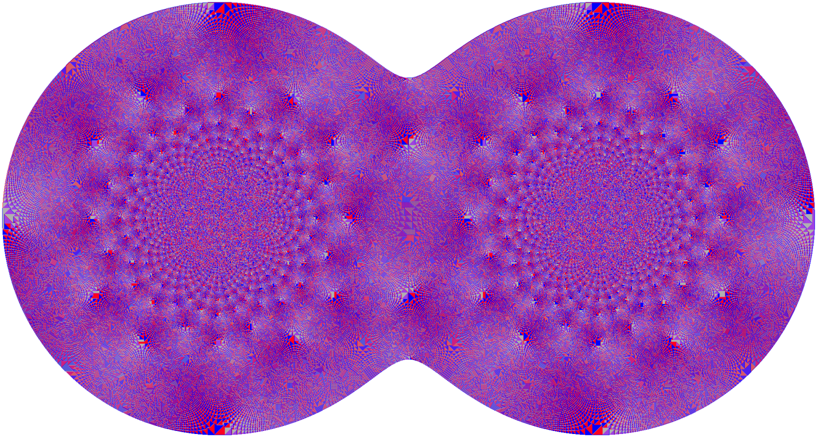

$$\iint_D h(x,y)\,dx\,dy = h(-1,0) + h(1,0)$$

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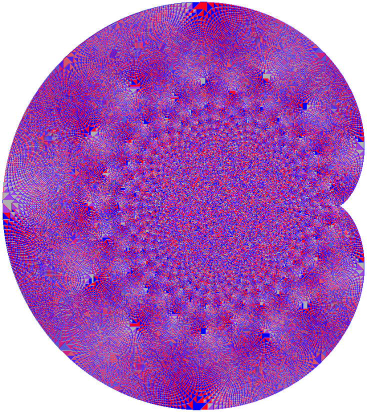 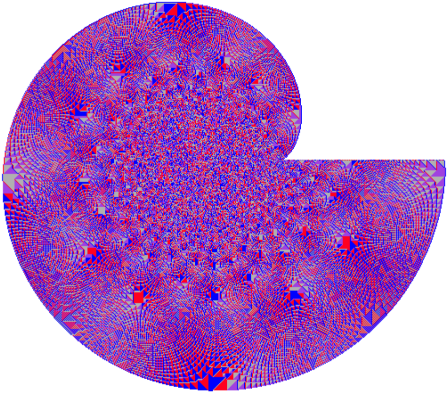

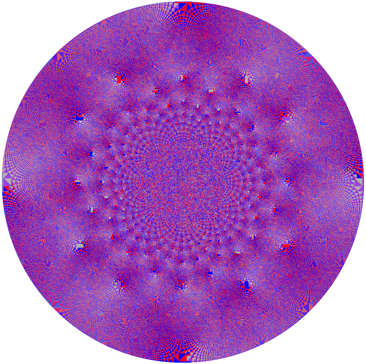

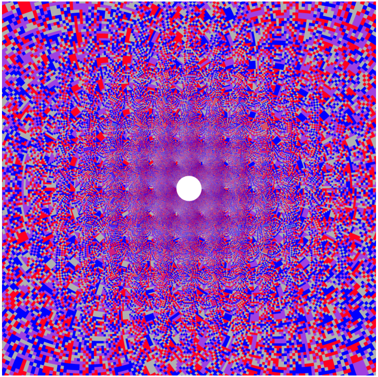

$$z \mapsto 1/z^2$$

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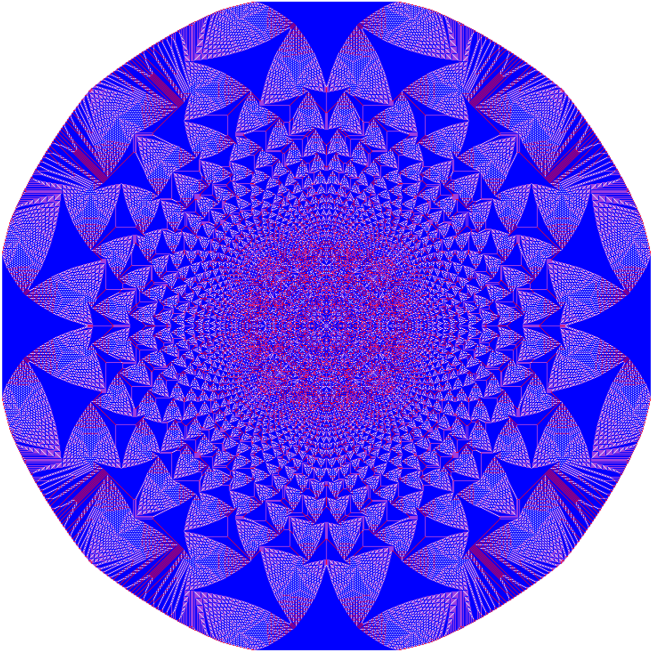
  - **Rotor**: Send extra particles according to the usual rotor rule.



Bounds for the Abelian Sandpile

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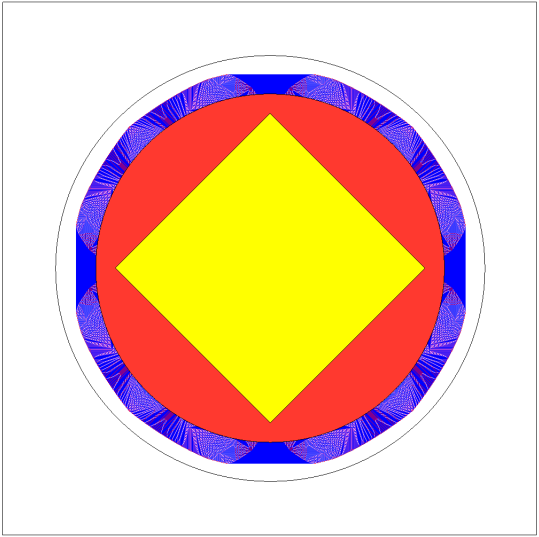
- Improves the bounds of Le Borgne and Rossin, Fey and Redig.



(Disk of area $n/3$) $\subset S_n \subset$ (Disk of area $n/2$)

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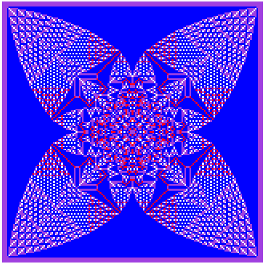
- Even for $h = 2$, the rate of growth of the square was not known; it was determined recently by Fey-Levine-P.(2009) to have edge length of order $\sqrt{n}$.



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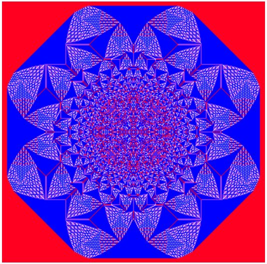
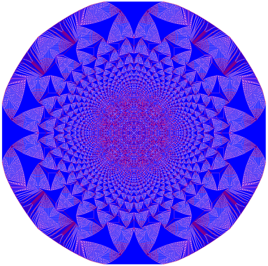


$h = 1$



$h = 0$