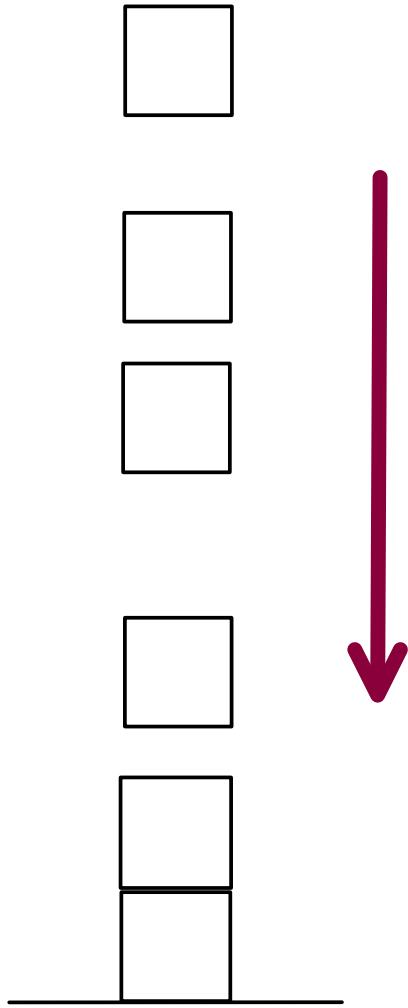# Integrable Probability

Alexei Borodin

What is integrable probability?

Imagine you are building a tower out of standard square blocks that fall down at random time moments.
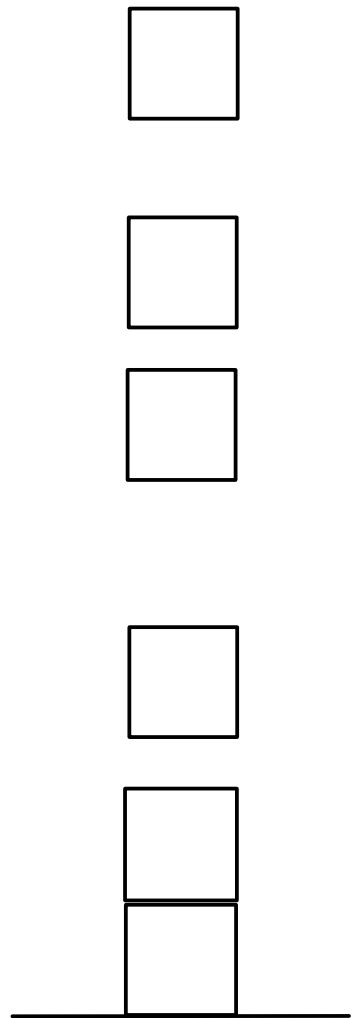
How tall will it be after a large time T?

It is natural to expect that

Height = const · T + random fluctuations

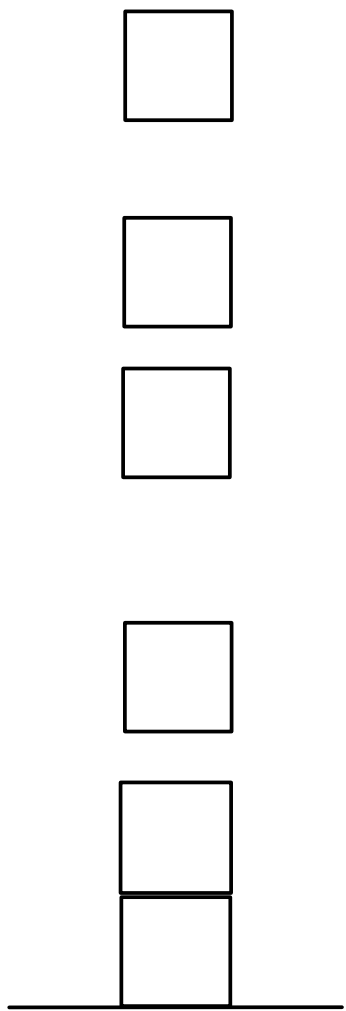What can one say about the fluctuations?

What is integrable probability?

A simple *integrable* example: The time is discrete, each second a new block falls with probability ½ (independently of what happened before). Then

$$\text{Prob}\{\text{height} = n \text{ after time } T\} = \frac{1}{2^T} \frac{T!}{n!(T-n)!}$$

Theorem *[De Moivre 1738], [Laplace 1812]*

$$\lim_{T \to \infty} \text{Prob}\left\{\text{height}(T) \leq \frac{T}{2} + \frac{s}{2} \cdot T^{1/2}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-x^2/2} \, dx$$

Universality principle: For a broad class of different possibilities for randomness, the size and distribution of the fluctuations must be the same (up to scaling constants)

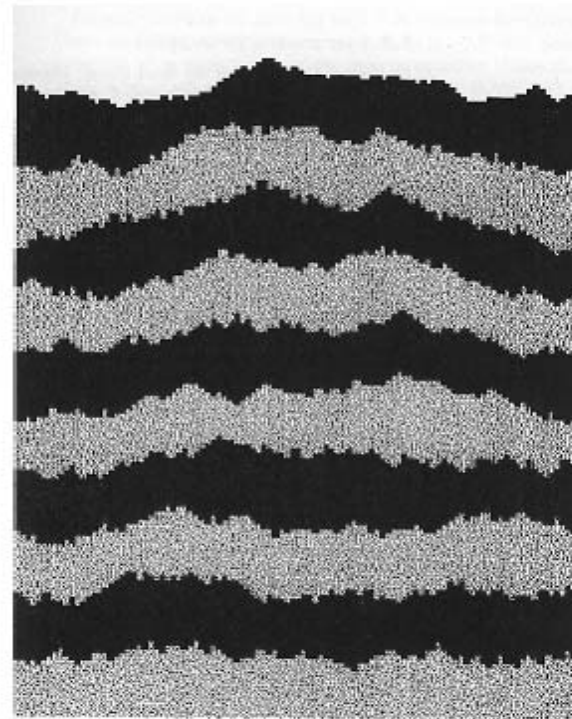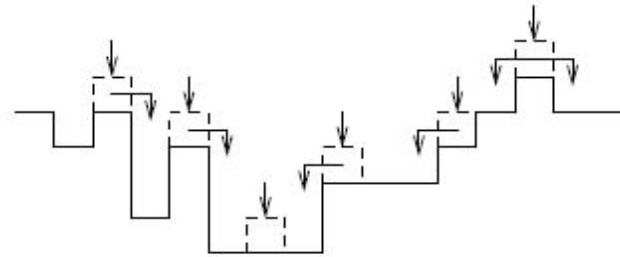This is the Central Limit Theorem, first proved by Lyapunov in 1901.

An integrable example predicts the behavior of the whole universality class.
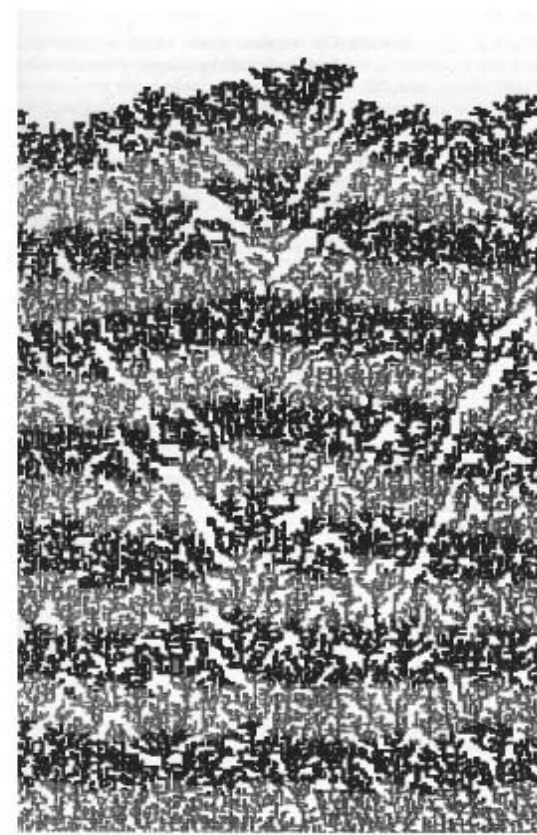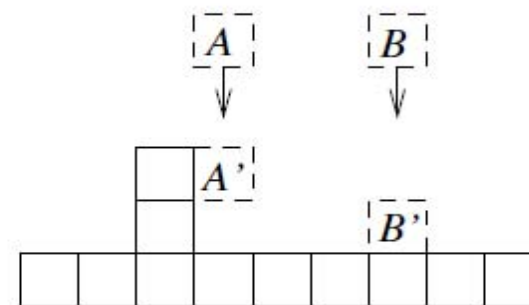
Three models of random interface growth in (1+1)d

Random deposition
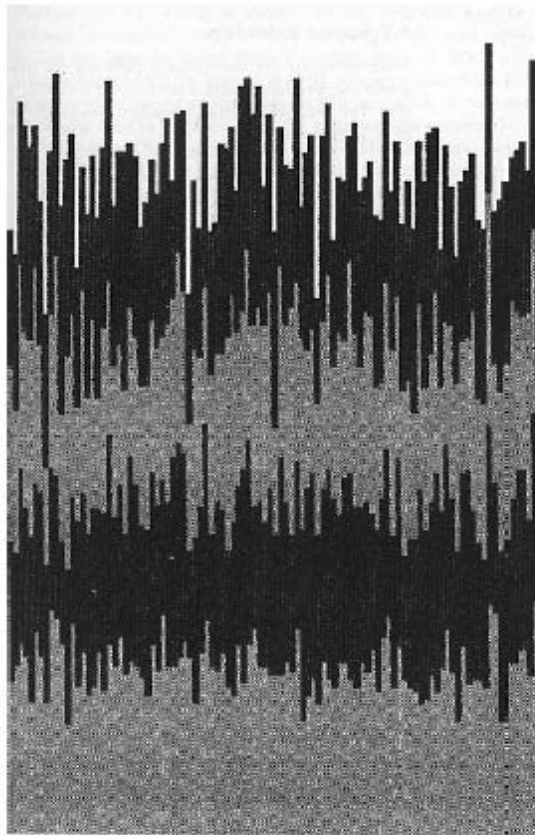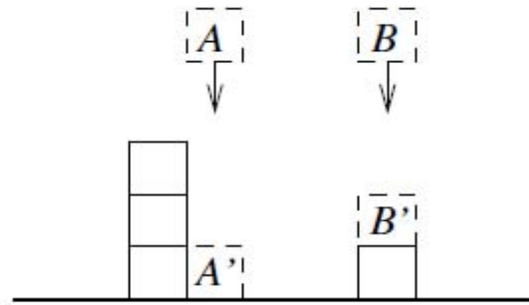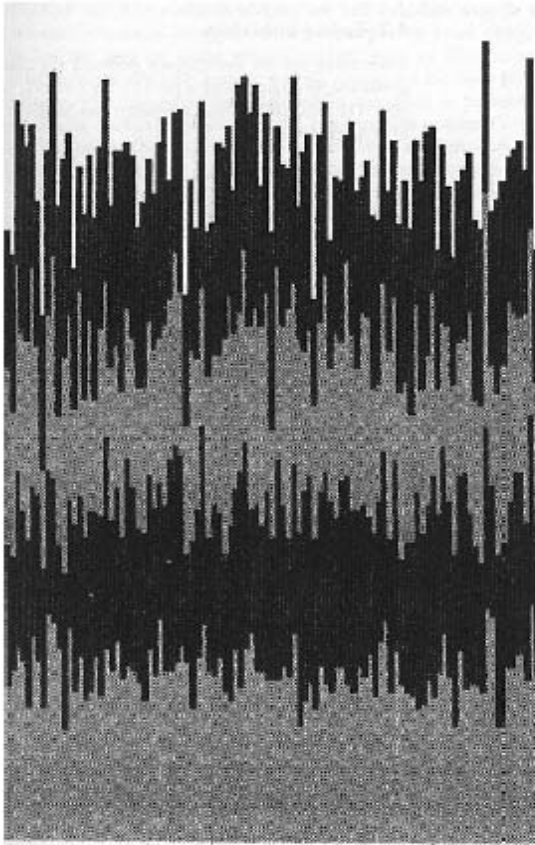
Random deposition with relaxation

Ballistic deposition

Three models of random interface growth in (1+1)d

Classical CLT	Edwards-Wilkinson eq.	Kardar-Parisi-Zhang eq.
$\partial_t h = \eta(x,t)$	$\partial_t h = \nu\, \partial_x^2 h + \eta(x,t)$	$\partial_t h = \nu\, \partial_x^2 h + \lambda(\partial_x h)^2 + \eta$
$t^{\frac{1}{2}}$ fluctuations	$t^{\frac{1}{4}}$ fluctuations	$t^{\frac{1}{3}}$ fluctuations
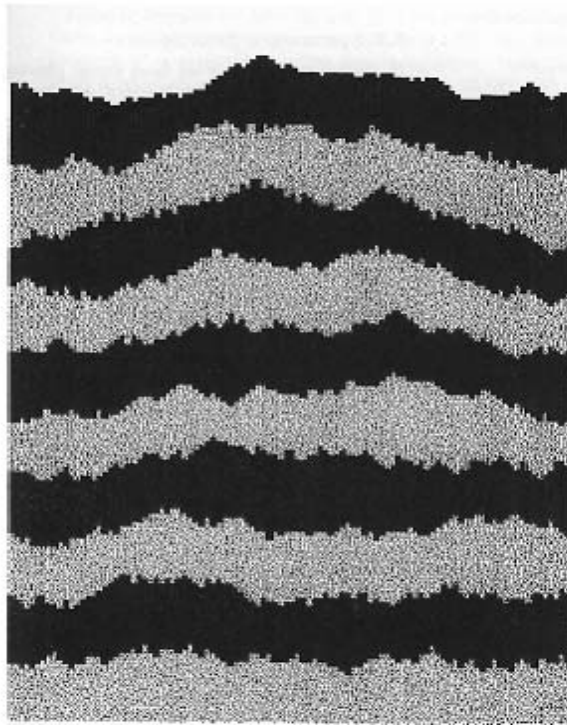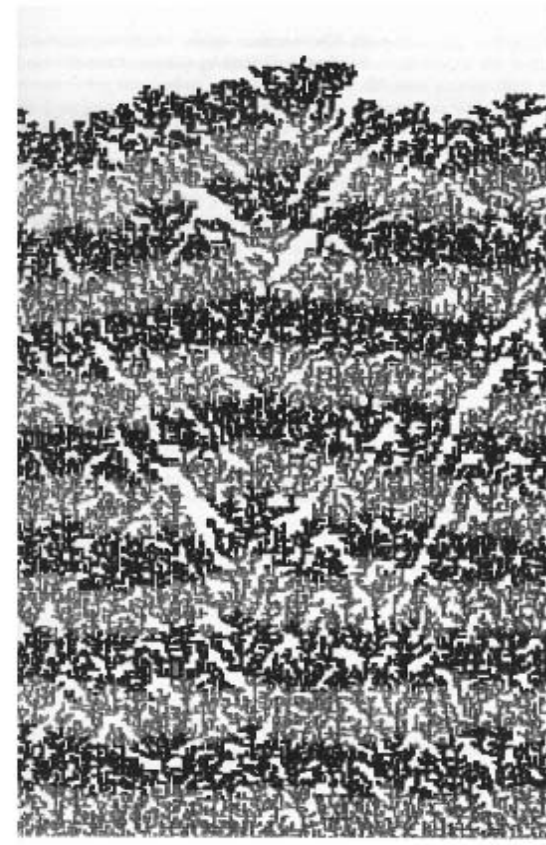
Experimental example: coffee ring effect
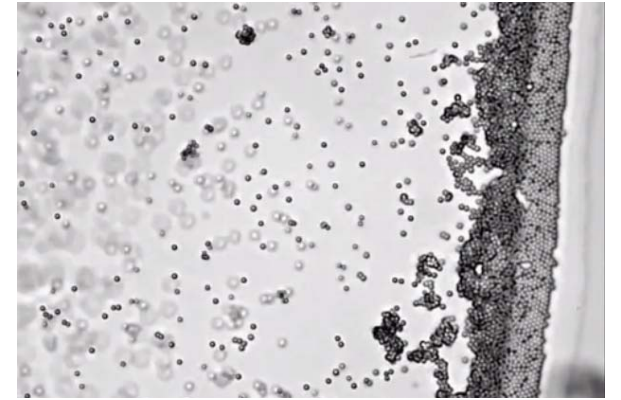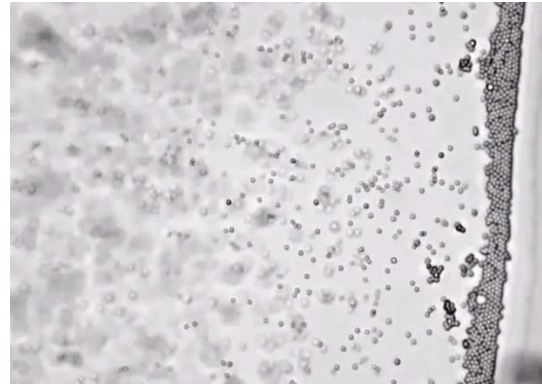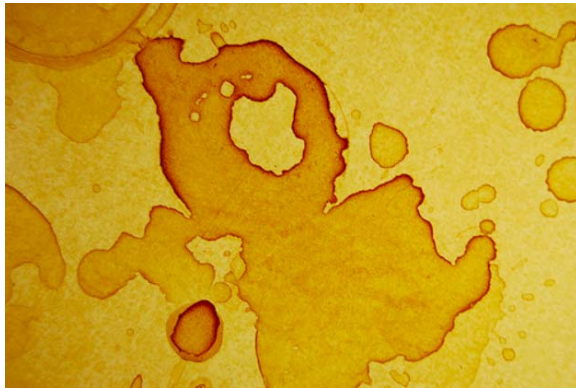
Perfectly round particles:
$t^{1/2}$ fluctuations, CLT statistics
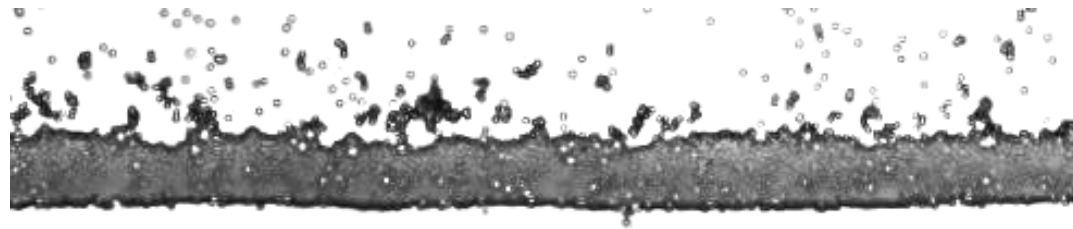
Slightly elongated particles:
$t^{1/3}$ fluctuations, KPZ statistics
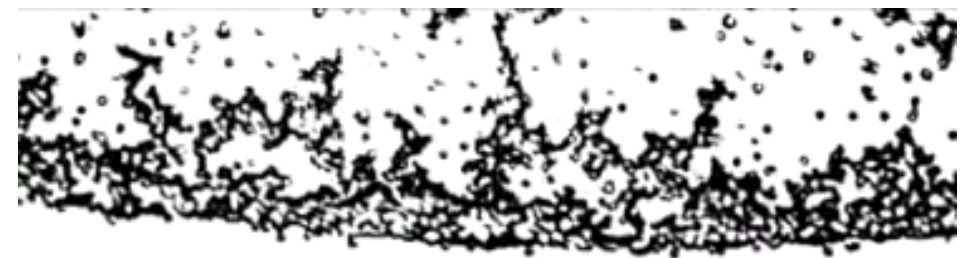
[Yunker-Lohr-Still-Borodin-Durian-Yodh, PRL 2013]

Totally Asymmetric Simple Exclusion Process (TASEP):
An integrable random interface growth model
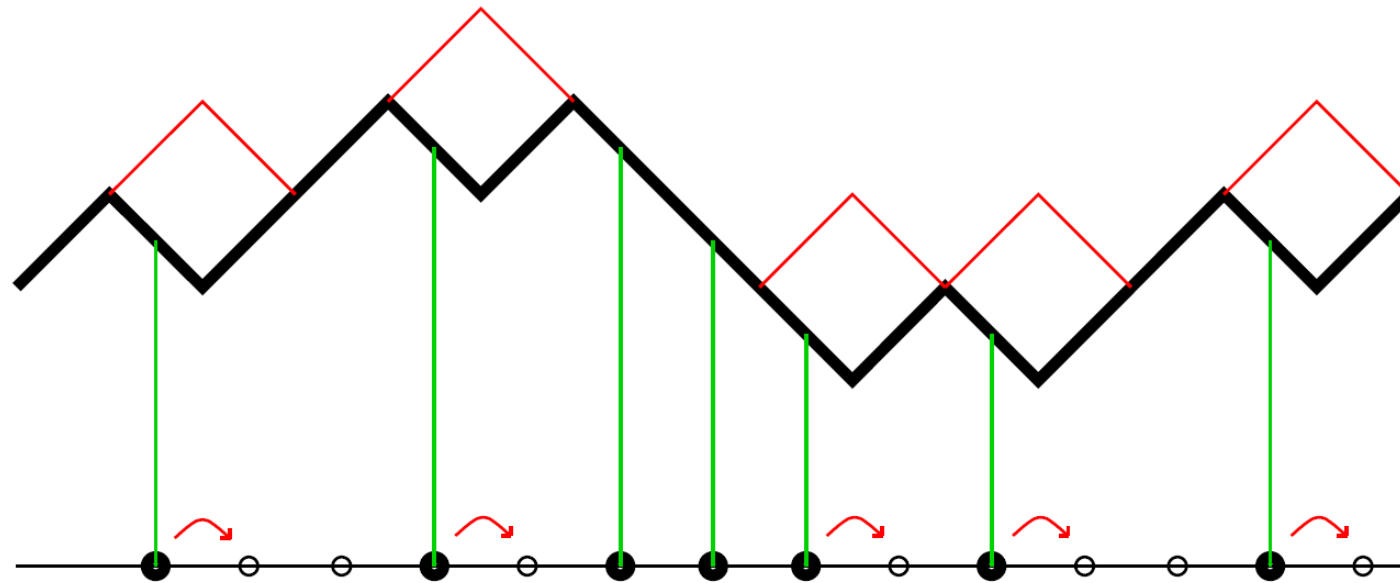
TASEP animation

Red boxes are added independently at rate 1. Equivalently, particles with no right neighbor jump independently with waiting time distributed as $e^{-x}dx$.

TASEP - hydrodynamic limit

At large time, over distances comparable with time (macroscopic or hydrodynamic limit), the evolution of the interface is deterministic with high probability.

In terms of the average density of particles ρ it is described by the inviscid Burgers equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho(1-\rho)\right)$$

It is known to develop shocks that correspond to traffic jams of the particle system.

TASEP – fluctuations

Step initial data

$h^{TASEP}(s,x)$

+2

Rate 1

time L

$L^{1/3}$

$L^{2/3}$
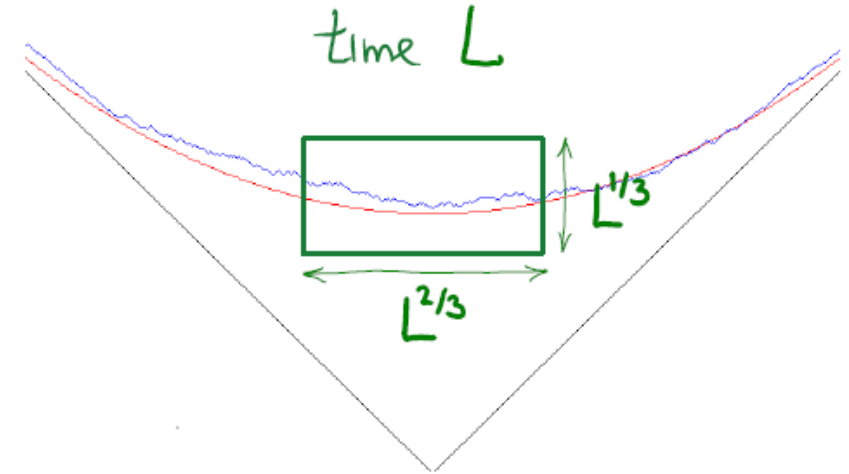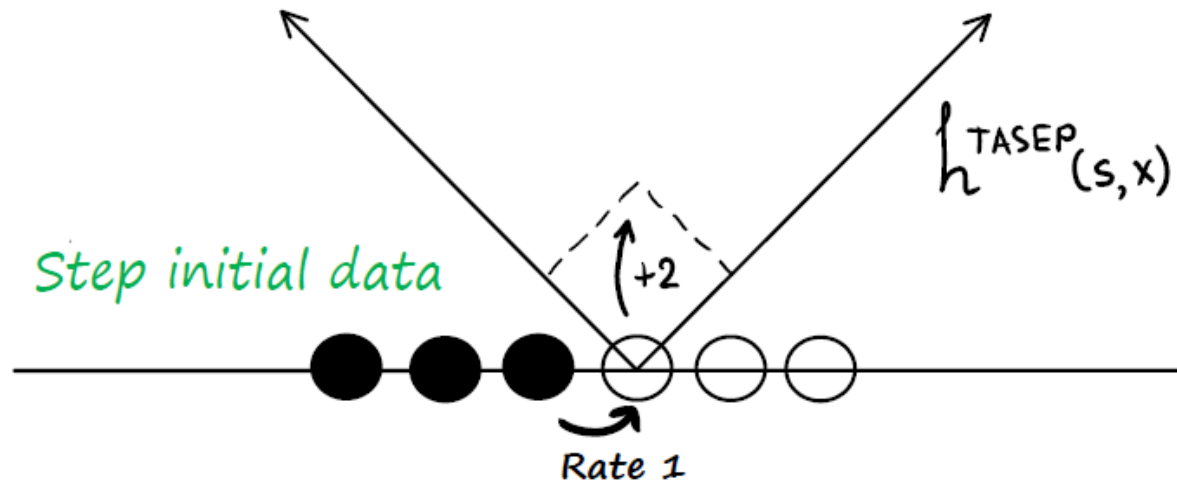
$$h_L(t,x) := \frac{1}{L^{1/3}} h^{TASEP}(Lt, L^{2/3}x) - L^{2/3} \cdot \frac{t}{2}$$

Theorem *[Johansson, 1999]* For TASEP with step initial data

$$\lim_{L \to \infty} \mathbb{P}\left\{ h_L(1,0) \geq -s \right\} = F_{GUE}(s)$$

Tracy–Widom limit distribution for the largest eigenvalue of large Hermitian matrices

Edge scaling limit of random matrices

Gaussian Unitary Ensemble (GUE) consists of Hermitian $N \times N$ matrices $H = H^*$ distributed as $const \cdot e^{-Trace(H^2)} dH$ *[Wigner, 1955]*.
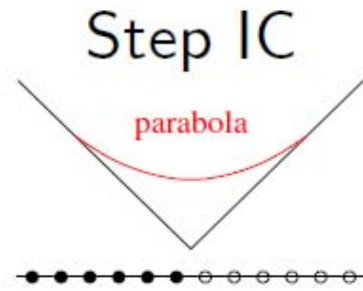
Theorem *[Tracy-Widom, 1993]*

$$\lim_{N \to \infty} Prob\left\{ \frac{(\text{top eigenvalue of } H) - \sqrt{2N}}{N^{-1/6}} \leq x \right\} = F_2(x)$$

where $u = \sqrt{-(\log F_2)''}$ satisfies $u'' = xu + 2u^3$ with appropriate initial conditions.

For real symmetric matrices one similarly defines the *Gaussian Orthogonal Ensemble (GOE)* and $F_1(x)$.

TASEP – fluctuations depend on hydrodynamic profile

Step IC

$(t^{2/3}, t^{1/3})$-scaling leads to $Airy_2$ process; one-point fluctuations are those of the edge of GUE

Johansson 1999, 2002
Prähofer-Spohn 2002

Flat IC
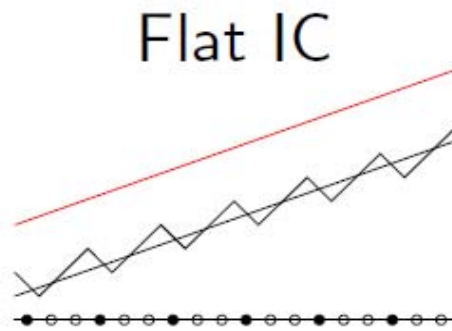
$Airy_1$ process and edge of GOE

Sasamoto 2005
Borodin-Ferrari-
Prähofer-Sasamoto
2006, 2007

Half-flat IC

$Airy_1$ in flat part,
$Airy_2$ in curved part,
$Airy_{1\to2}$ in between.

Borodin-Ferrari-
Sasamoto
2007

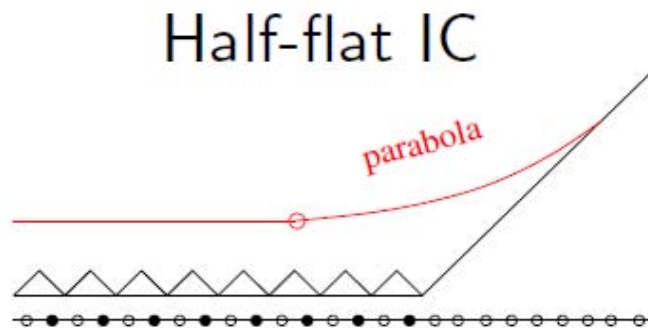# Kardar-Parisi-Zhang (KPZ) universality class

TASEP is an integrable representative of the (conjectural) KPZ universality class of growth models in 1+1 dimensions that is characterized by

◆ Locality of growth (no long-range interaction)
◆ A smoothing mechanism (a.k.a. relaxation, no fractal structures)
◆ Lateral growth (speed of growth depends on the slope)

If the speed of growth does not depend on the slope than the model is in the Edwards-Wilkinson class with different fluctuations ($t^{1/4}$ instead of $t^{1/3}$ and Gaussian distributions)

TASEP is one of a few growth models in the KPZ class that can be analyzed via the techniques of determinantal point processes (or free fermions, nonintersecting paths, Schur processes).

Other examples include

♦ Discrete time TASEPs with sequential/parallel update

♦ PushASEP or long range TASEP

♦ Directed last passage percolation in 2d with geometric/Bernoulli/exponential weights

♦ Polynuclear growth processes

More recently, a variety of non-determinantal, yet still _integrable_ models have been analyzed in the large time limit:

- ASEP   [Tracy-Widom, 2007-09], [Borodin-Corwin-Sasamoto, 2012]

- KPZ eq., or Stochastic Heat eq., or continuous Brownian polymer   [Amir-Corwin-Quastel, 2010], [Sasamoto-Spohn, 2010], [Dotsenko, 2010+], [Calabrese-Le Doussal-Rosso, 2010+], [Borodin-Corwin-Ferrari, 2012]

- q-TASEP   [Borodin-Corwin, 2011+]

- Semi-discrete Brownian polymer

  [O'Connell, 2009], [Borodin-Corwin, 2011], [Borodin-Corwin-Ferrari, 2012]

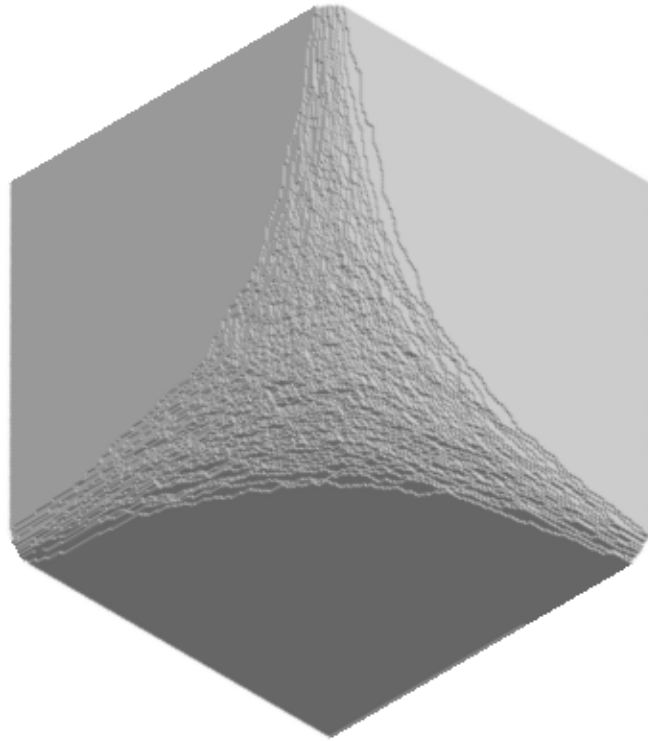- Fully discrete log-Gamma polymer [Corwin-O'Connell-Seppalainen-Zygouras, 2011], [Borodin-Corwin-Remenik, 2012]
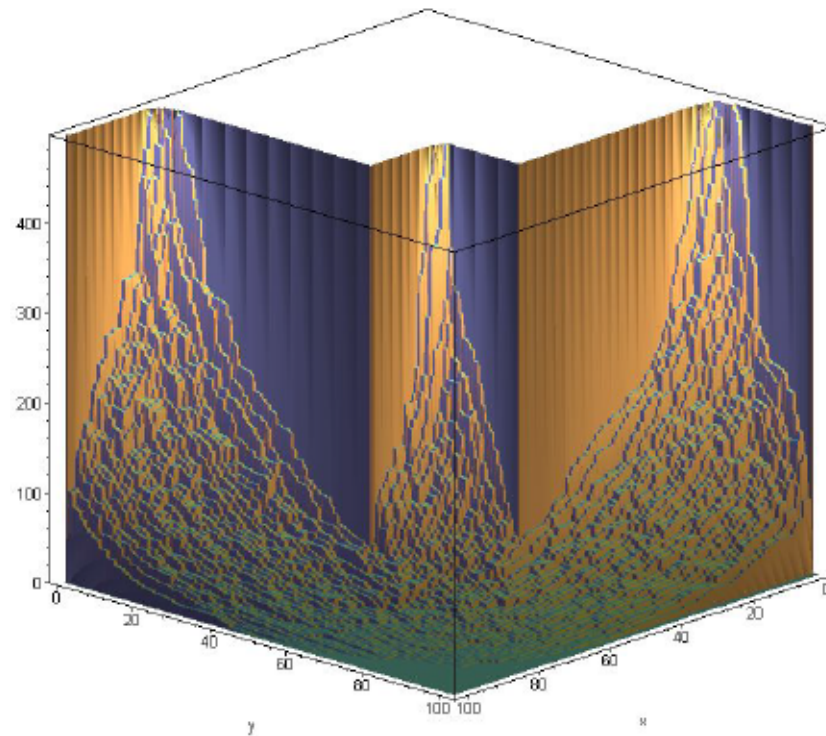
How about random interfaces in the 3d space?
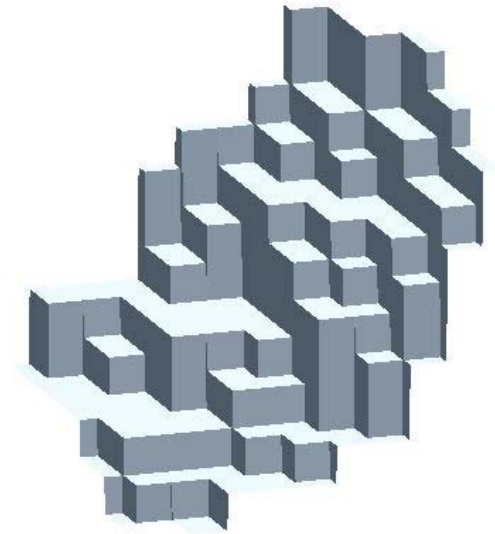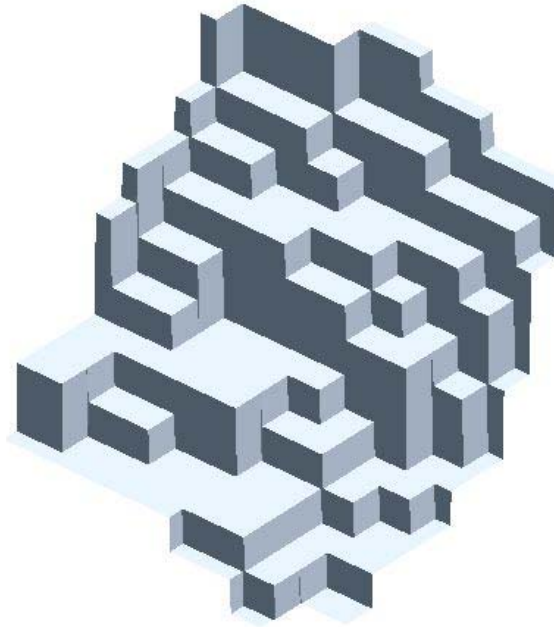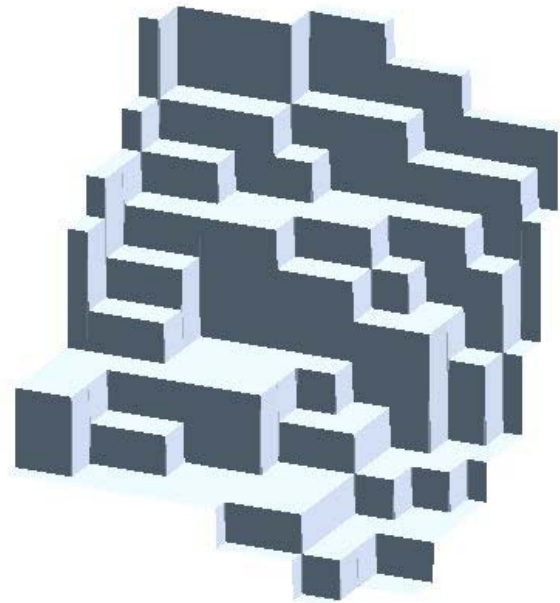
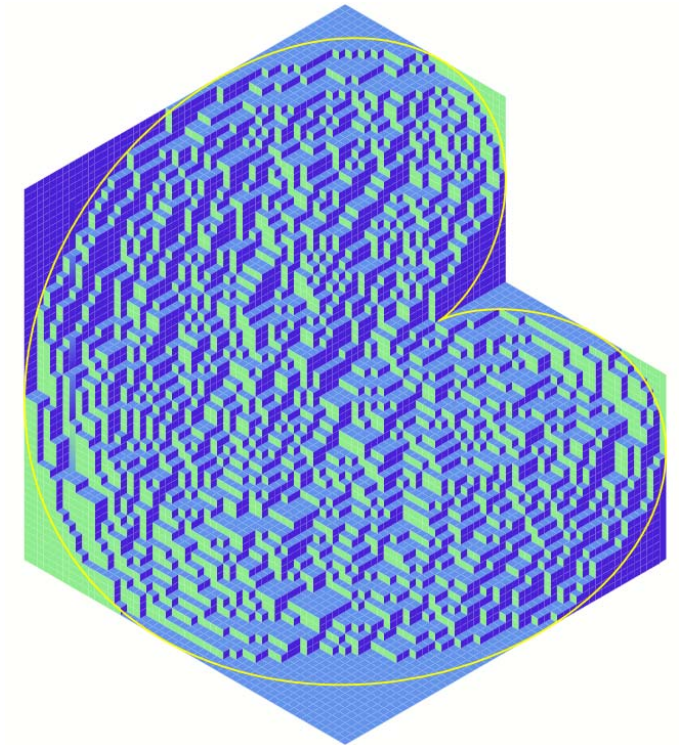We will consider stepped surfaces built from 1x1x1 cubes.

Uniformly random stepped surfaces produce beautiful algebraic limit shapes that vary depending on the boundary conditions
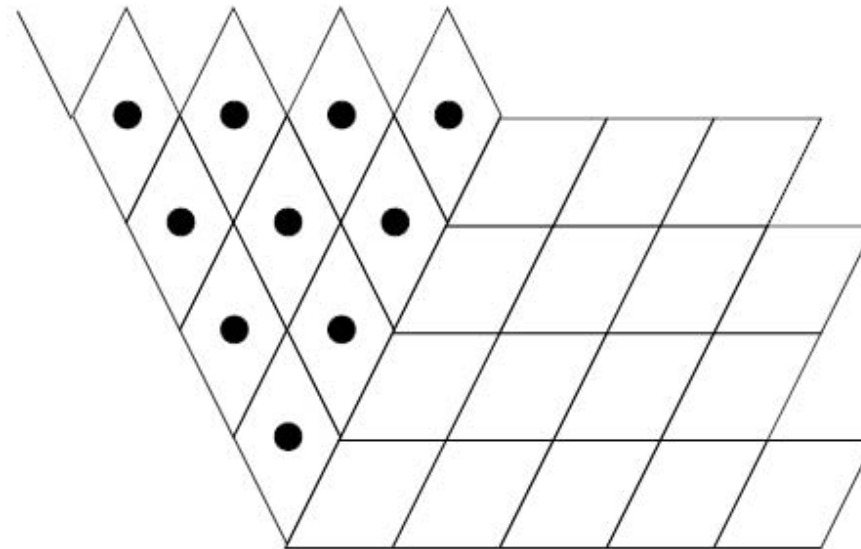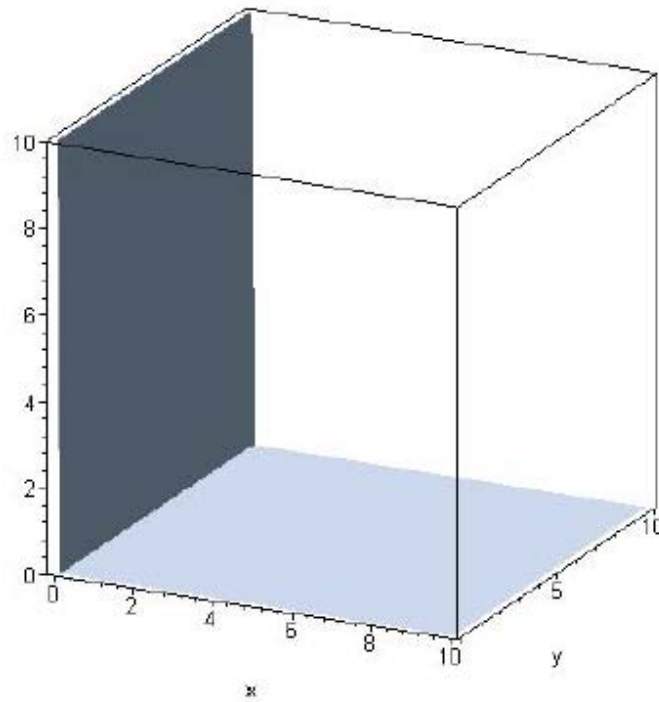

corner of a room

a room with several corners

a cardioid

[Okounkov-Reshetikhin, 2001, 2005], [Kenyon-Okounkov, 2006]

Can one grow them?

An integrable random growth model
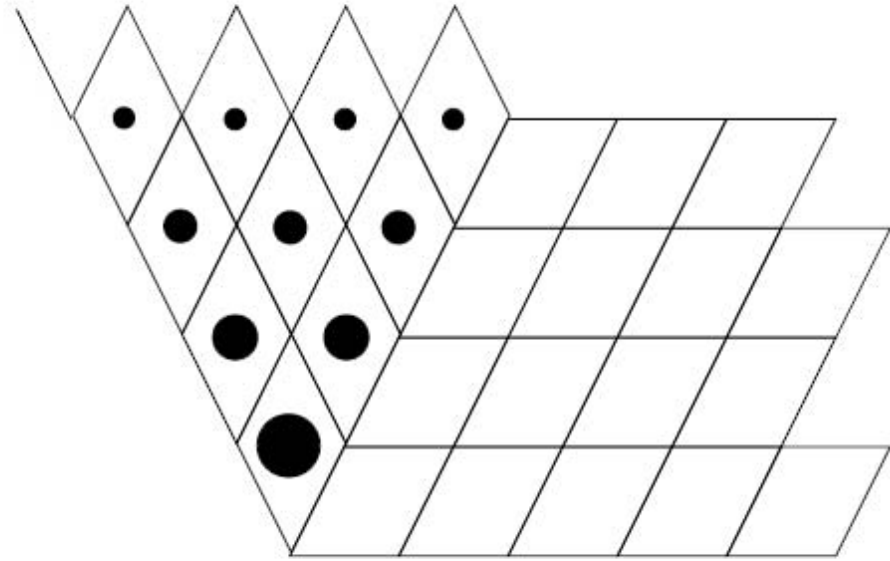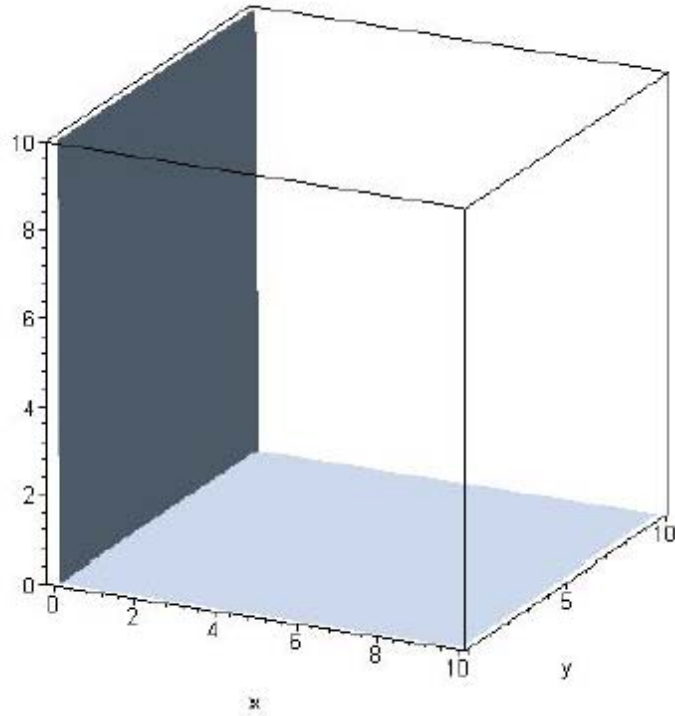
Consider the `empty' initial condition

It is often convenient to represent a stepped surface as a flat
tiling by rhombi of three types (a.k.a. `lozenges').
Place particles inside rhombi of a fixed type.
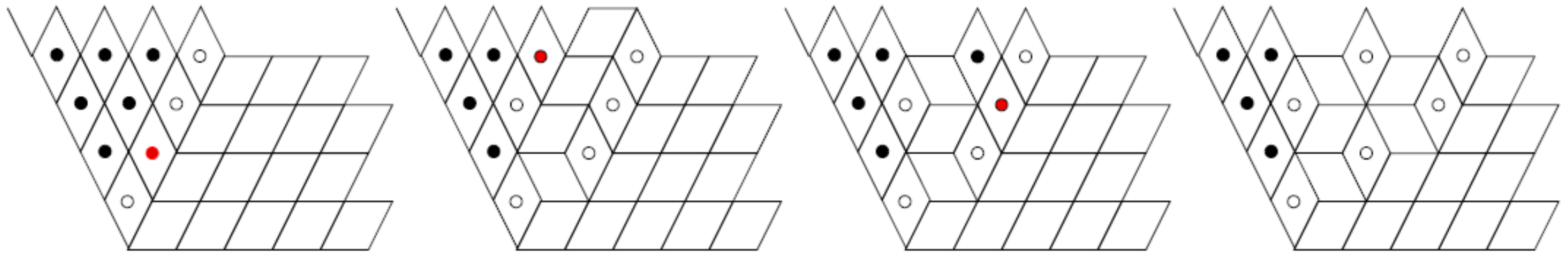
An integrable random growth model
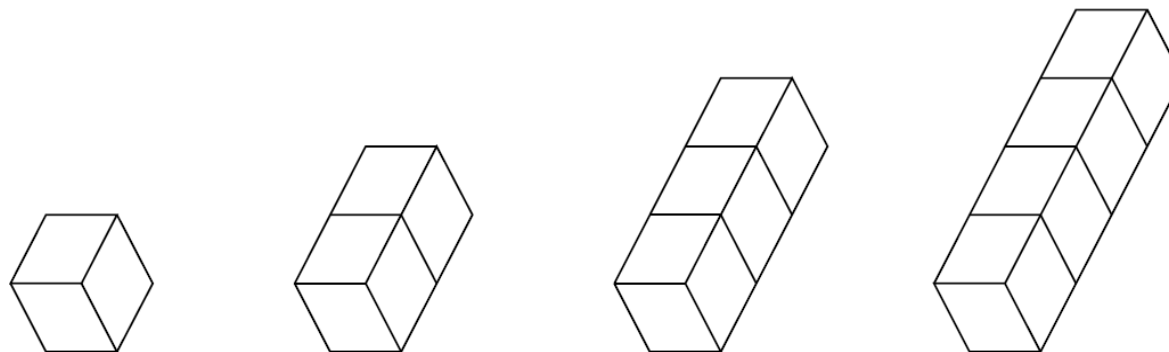
Consider the `empty' initial condition

Imagine that particles have weights that decrease upwards.

An integrable random growth model

Each particle jumps to the right independently with rate 1.
It is blocked by heavier particles and it pushes lighter particles.

In 3d, this can be viewed as adding directed columns

Column deposition - Animation

An integrable random growth model

Each particle jumps to the right independently with rate 1.
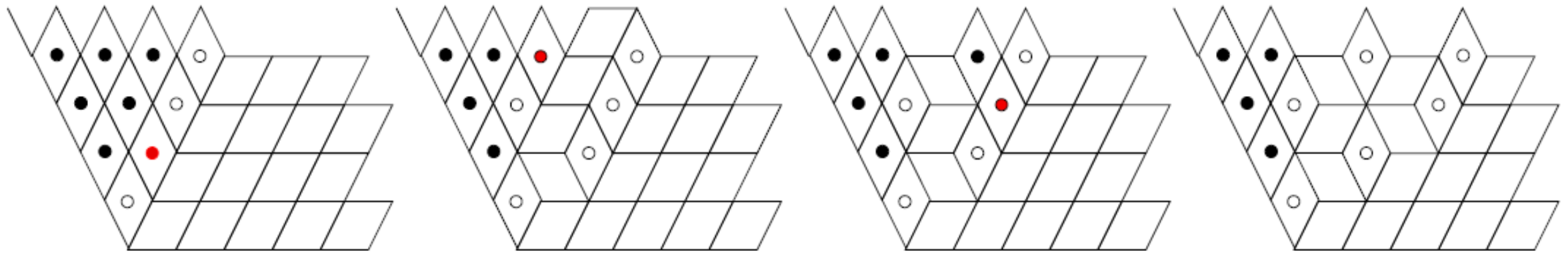It is blocked by heavier particles and it pushes lighter particles.

- Left-most particles form TASEP
- Right-most particles form PushTASEP
- Large time (diffusive) limit of the evolution of n particles on the n-th horizontal level is Dyson's Brownian motion for GUE

Growing plane partitions [Borodin, 2010]

A very similar stochastic evolution can be used to grow stepped random interfaces with given boundary conditions by growing their support
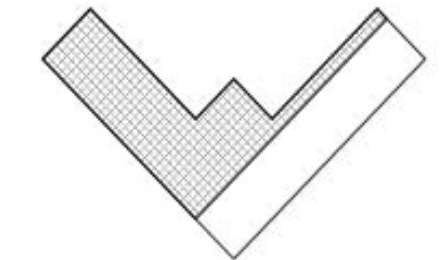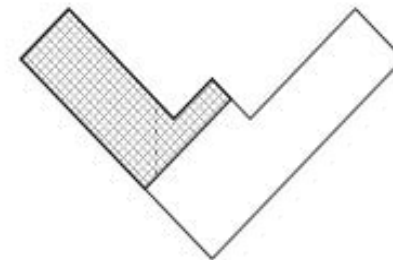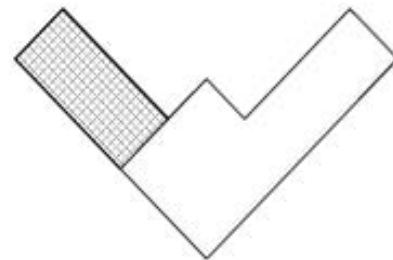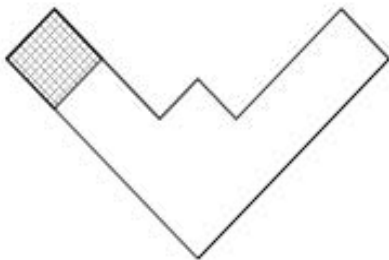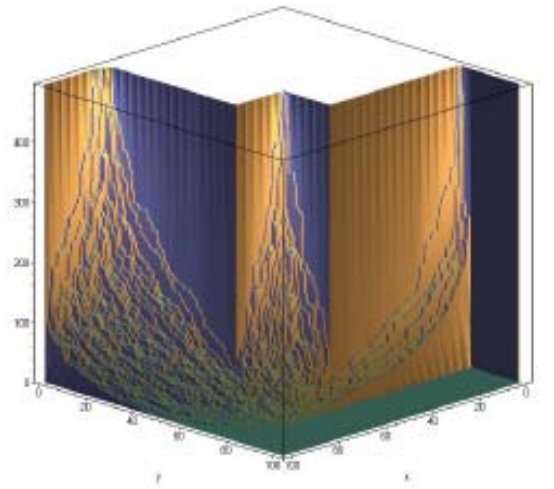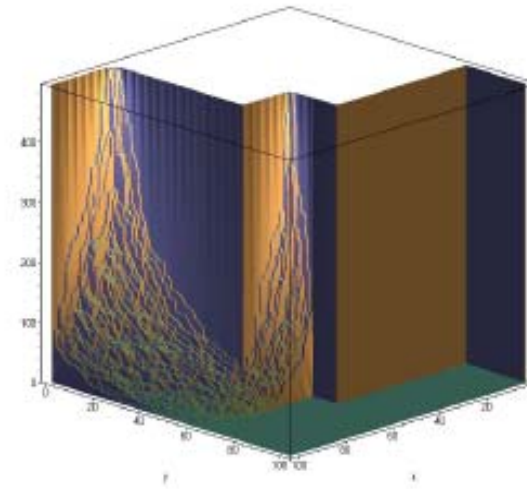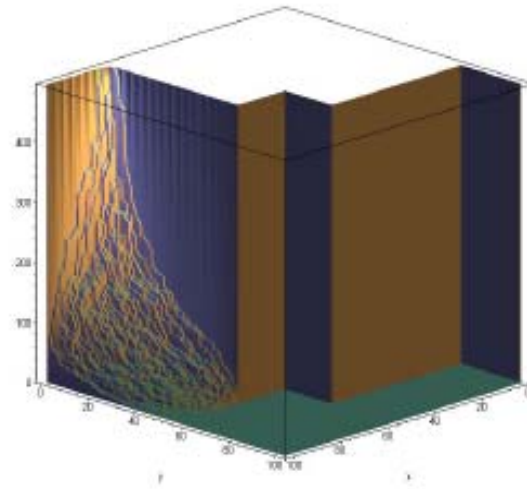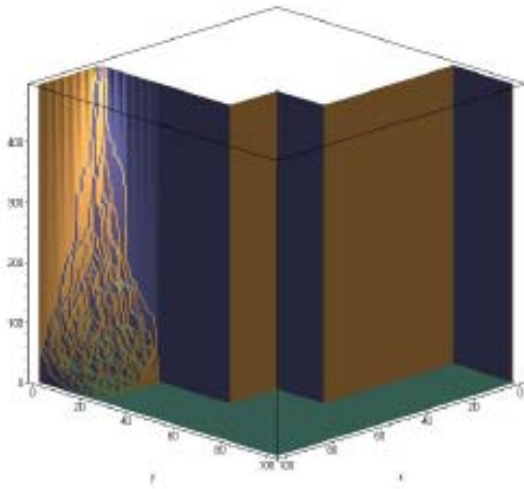
Large time behaviour

The following are broadly expected and proved in a few cases:

- In the hydrodynamic scaling, a *deterministic limit shape* arises. It is described by $\partial_t h = f(x, \nabla h)$.
- The models belong to the anisotropic KPZ universality class associated with the (formal) equation $\partial_t h = \Delta h + (\partial_x h)^2 - (\partial_y h)^2 + \text{white noise}$.
- One-point fluctuations in the bulk are Gaussian with log(t) variance (predicted in [Wolf, 1991])
- Multi-point fluctuation are described by the two-dimensional Gaussian Free Field.
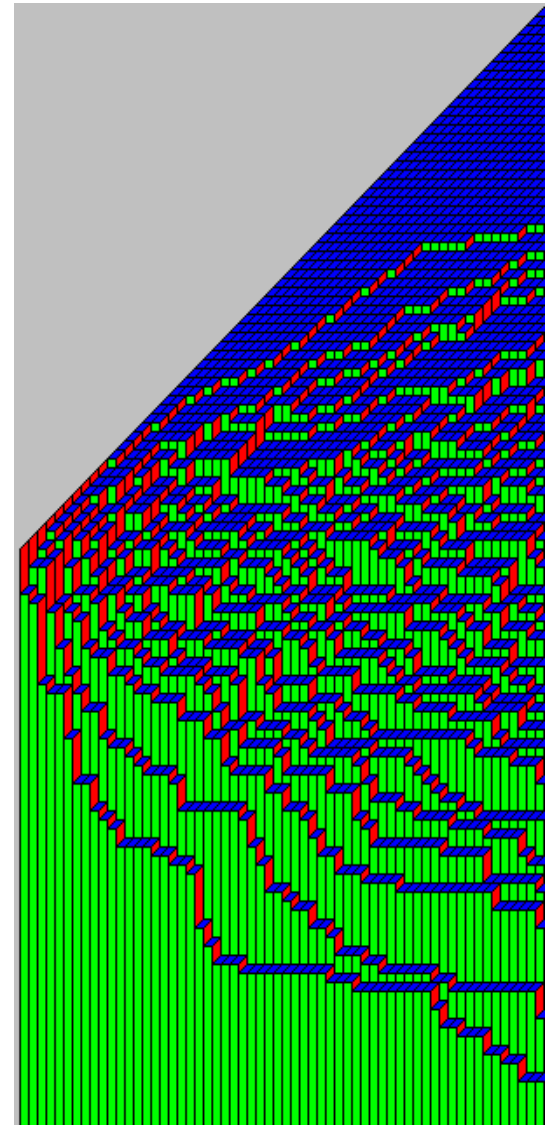
Column deposition – multipoint fluctuations
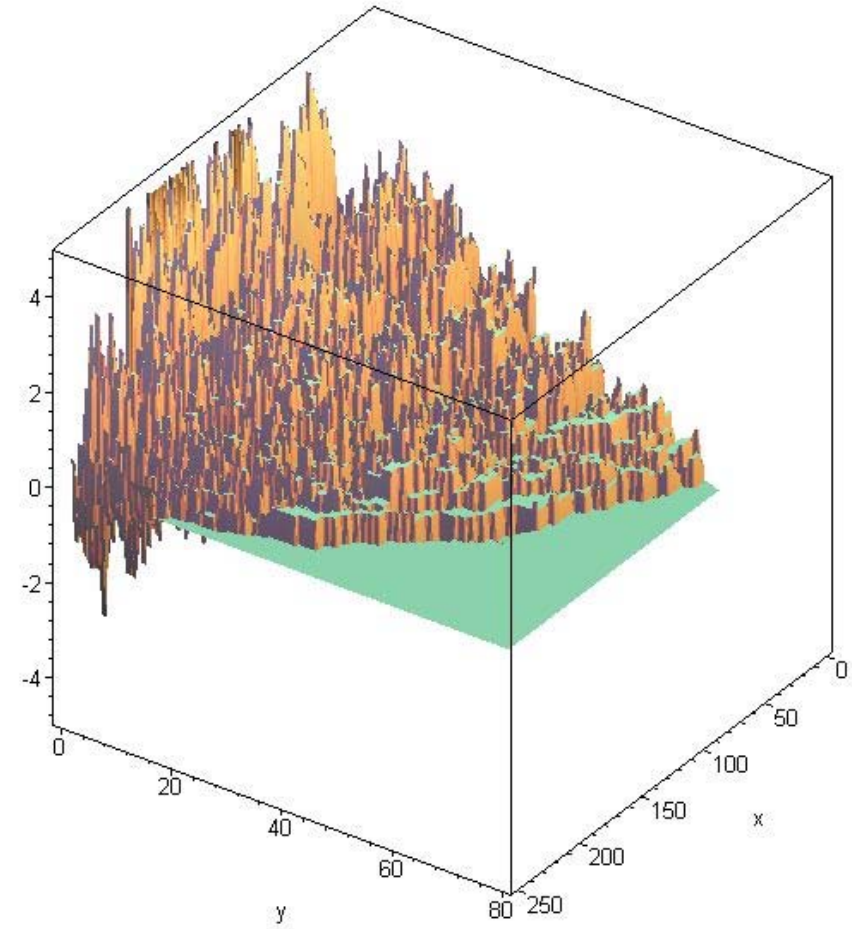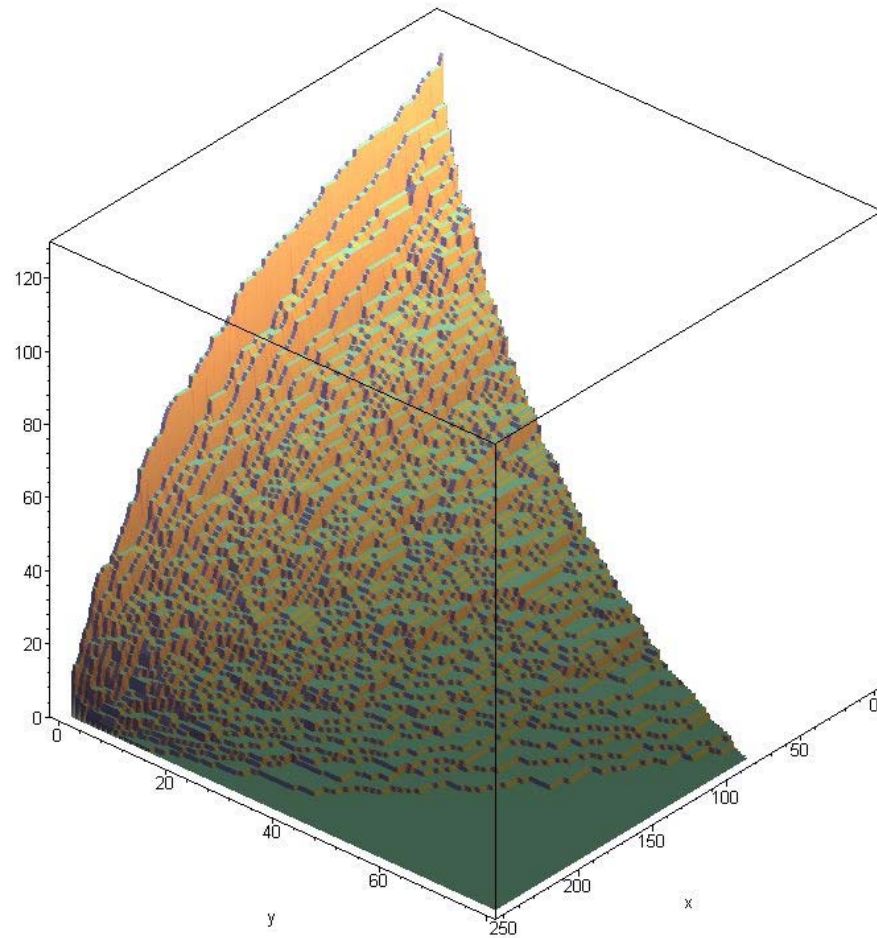
The (unscaled) fluctuations are two rough to have values at given locations.

Column deposition – multipoint fluctuations

Under a bijection $\Omega : \{\text{limit shape}\} \longrightarrow \mathbb{H} = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$

the fluctuations converge to the Gaussian generalized function

$$GFF(\Omega) = \sum_k \xi_k \cdot \frac{\varphi_k(\Omega)}{\sqrt{\lambda_k}}$$

where φ_k's are the eigenfunctions of $-\Delta$ on $\mathbb{H}$ with zero boundary conditions, λ_k is the corresponding eigenvalue, and ξ_k's are i.i.d. standard Gaussians.

The GFF was first obtained, and hence conjectured to be universal for (2+1)d anisotropic KPZ growth, via a rigorous analysis of the integrable column deposition model.

The approach that leads to large time/space analysis of the integrable probabilistic models is largely algebraic.

The hierarchy of integrable models shadows that of multivariate special functions that originate from representation theory and integrable systems as characters/zonal spherical functions for Lie groups/symmetric spaces over real/complex, finite, and p-adic fields, and as eigenfunctions for integrable quantum many body systems.

Representation theoretic tools are essential in our approach.

Macdonald processes $q, t \in [0, 1)$

Ruijsenaars-Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker processes $t = 0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$

General β RMT $t = q^{\beta/2} \to 1$

Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Calogero-Sutherland, Jack polynomials

Spherical functions for Riem. Symm. Sp.

Hall-Littlewood processes $q = 0$

Random matrices over finite fields

Spherical functions for p-adic groups

Whittaker processes $t = 0$ $q \to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$

Kingman partition structures $q = 0$ $t = 1$

Cycles of random permutations

Poisson-Dirichlet distributions

Schur processes $q = t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups

[B-Corwin, Macdonald processes, arXiv:1111.4408]

[B-Gorin, Lectures on integrable probability, arXiv:1212.3351]

[B-Petrov, Integrable probability: From representation theory to Macdonald processes, arXiv:1310.8007]