Smooth planar maps and Laplacian determinants

Morris Ang, Minjae Park, Joshua Pfeffer, Scott Sheffield

Massachusetts Institute of Technology

March 5, 2020

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1 + 2 + 3 + \ldots = -1/12$.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That is *so* wrong.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That is *so* wrong.
- **Alice:** Also $1 * 2 * 3 * \ldots = \sqrt{2\pi}$.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That is *so* wrong.
- **Alice:** Also $1 * 2 * 3 * \ldots = \sqrt{2\pi}$.
- **Bob:** Seriously wrong.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That is *so* wrong.
- **Alice:** Also $1 * 2 * 3 * \ldots = \sqrt{2\pi}$.
- **Bob:** Seriously wrong.
- **Alice:** $\zeta'(s) = -\log(1)1^{-s} - \log(2)2^{-s} - \ldots$ so $-\zeta'(0) = \log(1) + \log(2) + \ldots$ We know $-\zeta'(0) = \log(\sqrt{2\pi})$. So $1 * 2 * \ldots = e^{-\zeta'(0)} = \sqrt{2\pi}$.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1+2+3+\ldots=-1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s)=1^{-s}+2^{-s}+3^{-s}+\ldots$ makes sense for $\mathrm{Re}(s)>1$. Analytic continuation gives $\zeta(-1)=1+2+3+\ldots=-1/12$.
- **Bob:** That is *so* wrong.
- **Alice:** Also $1*2*3*\ldots=\sqrt{2\pi}$.
- **Bob:** Seriously wrong.
- **Alice:** $\zeta'(s)=-\log(1)1^{-s}-\log(2)2^{-s}-\ldots$ so $-\zeta'(0)=\log(1)+\log(2)+\ldots$ We know $-\zeta'(0)=\log(\sqrt{2\pi})$. So $1*2*\ldots=e^{-\zeta'(0)}=\sqrt{2\pi}$.
- **Alice:** Generally if λ_j grows linearly, write $Z(s):=\sum\lambda_j^{-s}$. Then $Z'(s)=\sum-\log(\lambda_j)\lambda_j^{-s}$ and $Z'(0)=-\sum\log(\lambda_j)$. So $\prod\lambda_j=e^{-Z'(0)}$. That's obvious for fnite products. So if we define $Z(s)$ for $\mathrm{Re}(s)>1$ and analytically continue, then it also works in the infinite limit.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That is *so* wrong.
- **Alice:** Also $1 * 2 * 3 * \ldots = \sqrt{2\pi}$.
- **Bob:** Seriously wrong.
- **Alice:** $\zeta'(s) = -\log(1)1^{-s} - \log(2)2^{-s} - \ldots$ so $-\zeta'(0) = \log(1) + \log(2) + \ldots$ We know $-\zeta'(0) = \log(\sqrt{2\pi})$. So $1 * 2 * \ldots = e^{-\zeta'(0)} = \sqrt{2\pi}$.
- **Alice:** Generally if λ_j grows linearly, write $Z(s) := \sum \lambda_j^{-s}$. Then $Z'(s) = \sum -\log(\lambda_j)\lambda_j^{-s}$ and $Z'(0) = -\sum \log(\lambda_j)$. So $\prod \lambda_j = e^{-Z'(0)}$. That's obvious for fnite products. So if we define $Z(s)$ for $\mathrm{Re}(s) > 1$ and analytically continue, then it also works in the infinite limit.
- **Bob:** That's not how limits work. Converges on $\mathrm{Re}(s) > 1 \neq$ converges at 0.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1+2+3+\ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1+2+3+\ldots = -1/12$.
- **Bob:** That is *so* wrong.
- **Alice:** Also $1 * 2 * 3 * \ldots = \sqrt{2\pi}$.
- **Bob:** Seriously wrong.
- **Alice:** $\zeta'(s) = -\log(1)1^{-s} - \log(2)2^{-s} - \ldots$ so $-\zeta'(0) = \log(1) + \log(2) + \ldots$ We know $-\zeta'(0) = \log(\sqrt{2\pi})$. So $1 * 2 * \ldots = e^{-\zeta'(0)} = \sqrt{2\pi}$.
- **Alice:** Generally if λ_j grows linearly, write $Z(s) := \sum \lambda_j^{-s}$. Then $Z'(s) = \sum -\log(\lambda_j)\lambda_j^{-s}$ and $Z'(0) = -\sum \log(\lambda_j)$. So $\prod \lambda_j = e^{-Z'(0)}$. That's obvious for fnite products. So if we define $Z(s)$ for $\mathrm{Re}(s) > 1$ and analytically continue, then it also works in the infinite limit.
- **Bob:** That's not how limits work. Converges on $\mathrm{Re}(s) > 1 \neq$ converges at 0.
- **Alice:** Not in the *conventional* sense. But it is useful, for example in string theory, to interpret these identities as being true *in an alternative sense*.

A conversation that happens somewhere every five minutes

- **Alice:** I learned a cool identity: $1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That's wrong. How does that make *any* sense?
- **Alice:** The Riemann zeta function $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$ makes sense for $\mathrm{Re}(s) > 1$. Analytic continuation gives $\zeta(-1) = 1 + 2 + 3 + \ldots = -1/12$.
- **Bob:** That is *so* wrong.
- **Alice:** Also $1 * 2 * 3 * \ldots = \sqrt{2\pi}$.
- **Bob:** Seriously wrong.
- **Alice:** $\zeta'(s) = -\log(1)1^{-s} - \log(2)2^{-s} - \ldots$ so $-\zeta'(0) = \log(1) + \log(2) + \ldots$ We know $-\zeta'(0) = \log(\sqrt{2\pi})$. So $1 * 2 * \ldots = e^{-\zeta'(0)} = \sqrt{2\pi}$.
- **Alice:** Generally if λ_j grows linearly, write $Z(s) := \sum \lambda_j^{-s}$. Then $Z'(s) = \sum -\log(\lambda_j)\lambda_j^{-s}$ and $Z'(0) = -\sum \log(\lambda_j)$. So $\prod \lambda_j = e^{-Z'(0)}$. That's obvious for fnite products. So if we define $Z(s)$ for $\mathrm{Re}(s) > 1$ and analytically continue, then it also works in the infinite limit.
- **Bob:** That's not how limits work. Converges on $\mathrm{Re}(s) > 1 \neq$ converges at 0.
- **Alice:** Not in the *conventional* sense. But it is useful, for example in string theory, to interpret these identities as being true *in an alternative sense*.
- **Bob:** Look Alice, we can have a civil conversation about zeta functions, but your "cool identities" are just wrong. Maybe string theorists can get away with this nonsense, but if you keep saying stuff like $1 + 2 + 3 + \ldots = -1/12$ you will lose all your friends. You will spend your life sad and alone.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.
- **Eve:** Ramanujan described a similar discussion in a letter to Hardy.

  > Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \ldots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal.

  I think you need somebody to explain some of the *probability* behind all this.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.
- **Eve:** Ramanujan described a similar discussion in a letter to Hardy.

  Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \ldots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal.

  I think you need somebody to explain some of the *probability* behind all this.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.
- **Eve:** Ramanujan described a similar discussion in a letter to Hardy.

  > Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \ldots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal.

  I think you need somebody to explain some of the *probability* behind all this.
- **Alice:** I think we need a new cryptoprotocol. **Bob:** Yes.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.
- **Eve:** Ramanujan described a similar discussion in a letter to Hardy.

  > Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1+2+3+4+\ldots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal.

  I think you need somebody to explain some of the *probability* behind all this.
- **Alice:** I think we need a new cryptoprotocol. **Bob:** Yes.
- **Eve:** A Laplace Beltrami operator on a compact Riemannian surface (on functions mod additive constant) has eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ growing linearly. *Naively* it has trace $\sum \lambda_j = \infty$ and determinant $\prod \lambda_i = \infty$.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.
- **Eve:** Ramanujan described a similar discussion in a letter to Hardy.

  > Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \ldots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal.

  I think you need somebody to explain some of the *probability* behind all this.
- **Alice:** I think we need a new cryptoprotocol. **Bob:** Yes.
- **Eve:** A Laplace Beltrami operator on a compact Riemannian surface (on functions mod additive constant) has eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ growing linearly. *Naively* it has trace $\sum \lambda_j = \infty$ and determinant $\prod \lambda_i = \infty$.
- **Eve:** But as Alice said, an alternative (first introduced by Ray and Singer in 1973) is $\det' \Delta = e^{-Z'(0)}$ where $Z(s) = \sum \lambda_j^{-s}$ analytically continued.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.
- **Eve:** Ramanujan described a similar discussion in a letter to Hardy.

  > Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \ldots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal.

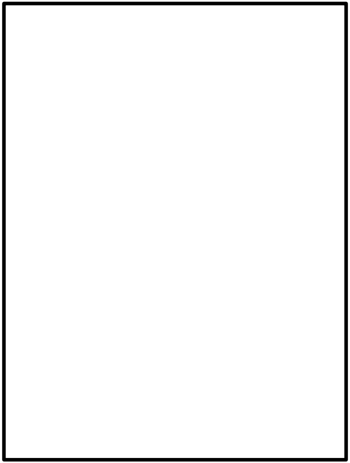
  I think you need somebody to explain some of the *probability* behind all this.
- **Alice:** I think we need a new cryptoprotocol. **Bob:** Yes.
- **Eve:** A Laplace Beltrami operator on a compact Riemannian surface (on functions mod additive constant) has eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ growing linearly. *Naively* it has trace $\sum \lambda_j = \infty$ and determinant $\prod \lambda_i = \infty$.
- **Eve:** But as Alice said, an alternative (first introduced by Ray and Singer in 1973) is $\det' \Delta = e^{-Z'(0)}$ where $Z(s) = \sum \lambda_j^{-s}$ analytically continued.
- **Eve:** To justify it, let's try to relate $\det \Delta$ to loop soups, GFFs, heat kernels, random surfaces — the things Laplacian determinants were *made* for.

- **Eve:** Pardon me, but I couldn't help overhearing your discussion.
- **Eve:** Ramanujan described a similar discussion in a letter to Hardy.

  > Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \ldots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal.

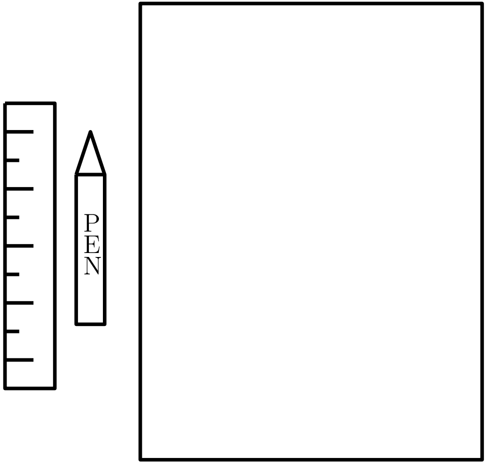
  I think you need somebody to explain some of the *probability* behind all this.
- **Alice:** I think we need a new cryptoprotocol. **Bob:** Yes.
- **Eve:** A Laplace Beltrami operator on a compact Riemannian surface (on functions mod additive constant) has eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ growing linearly. *Naively* it has trace $\sum \lambda_j = \infty$ and determinant $\prod \lambda_i = \infty$.
- **Eve:** But as Alice said, an alternative (first introduced by Ray and Singer in 1973) is $\det' \Delta = e^{-Z'(0)}$ where $Z(s) = \sum \lambda_j^{-s}$ analytically continued.
- **Eve:** To justify it, let's try to relate $\det \Delta$ to loop soups, GFFs, heat kernels, random surfaces — the things Laplacian determinants were *made* for.
- **Eve:** Story involves Riemann, Ramanujan, Ray, McKean, Singer, Polyakov, Alvarez, Sarnak, Singer, Dubédat, Kenyon, Zamolodchikov, Knizhnik, David, Distler, Kawai, Duplantier, Hoegh-Krohn, Kahane, Schaeffer, Marckert, Cori, Mokkadem, Le Gall, Vaquelin, Chassaing, Marckert, Mokkadem, Paulin....

First, remember planar maps



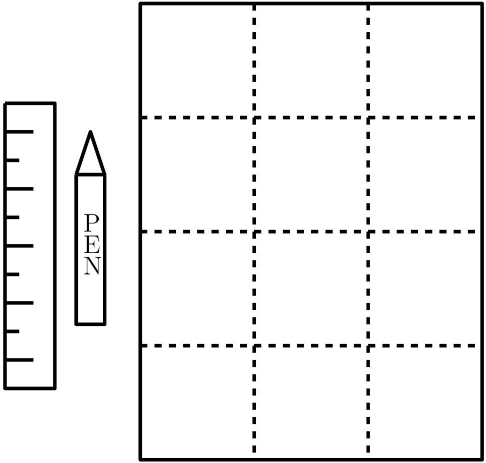
Start out with a sheet of paper

First, remember planar maps



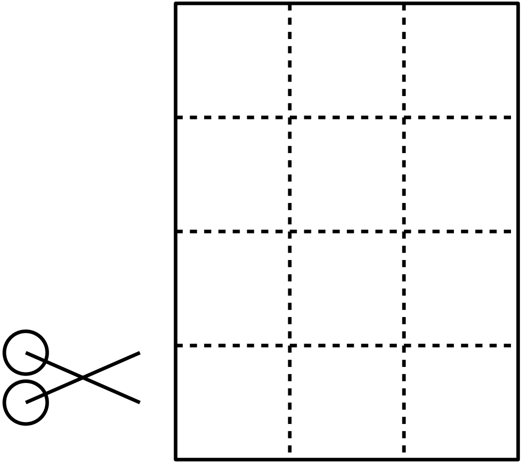
Get out pen and ruler

First, remember planar maps



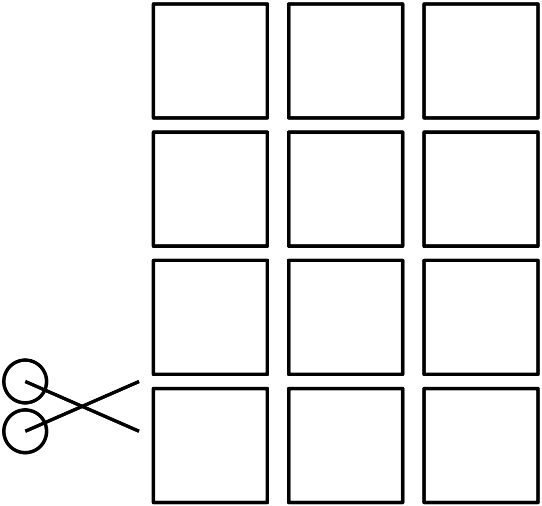
Measure and mark squares squares of equal size

First, remember planar maps



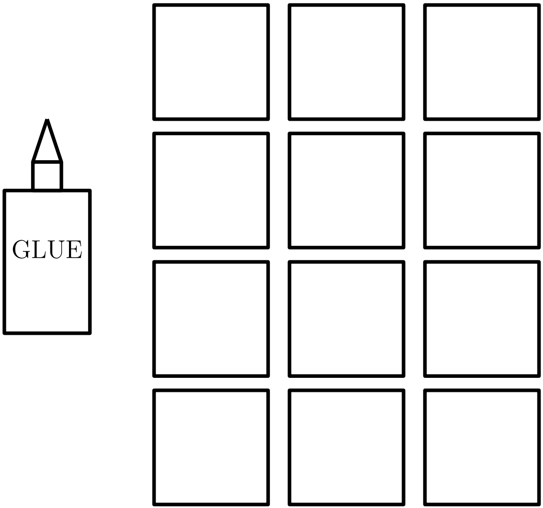
Get out scissors

First, remember planar maps



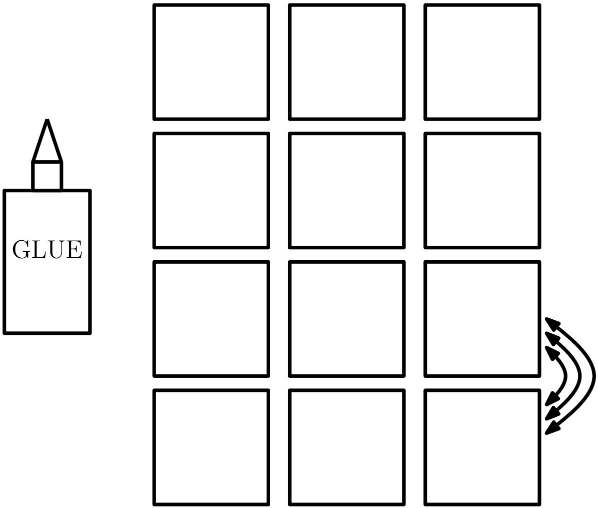
Cut into squares

First, remember planar maps

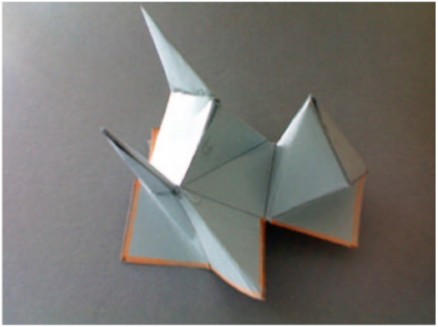


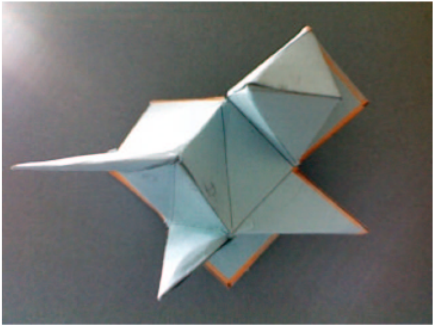
Get out bottle of glue

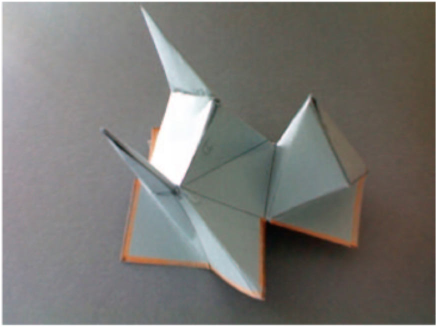
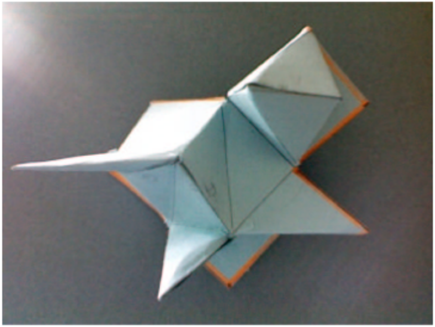
First, remember planar maps



Attach squares along boundaries with glue to form a surface “without holes.”

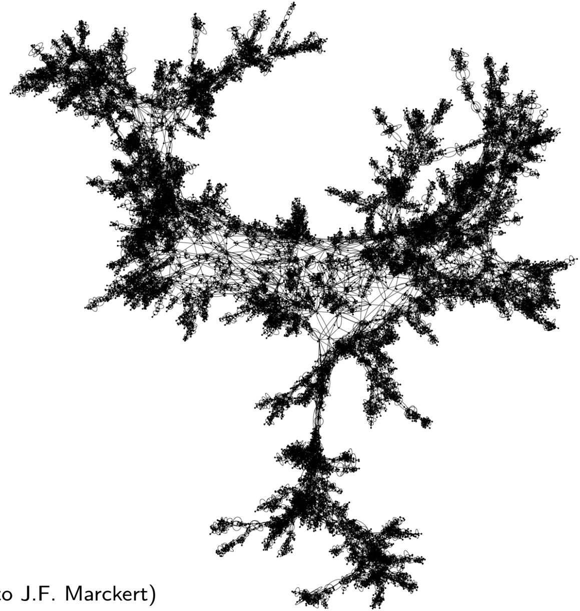




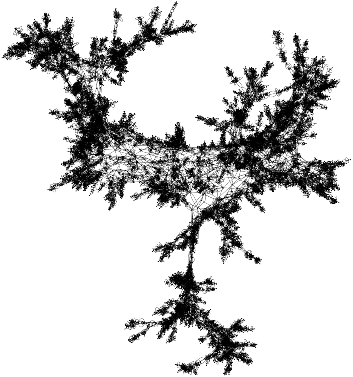
What is the structure of a typical quadrangulation when the number of faces is large?

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

More on random planar maps

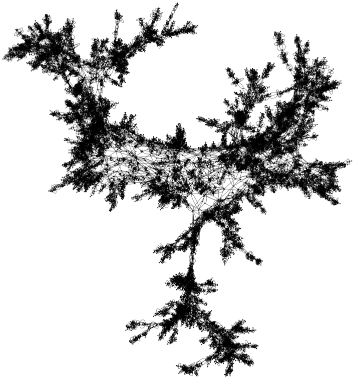


(Simulation due to J.F. Marckert)

1. First studied by Tutte in 1960s while working on the four color theorem.

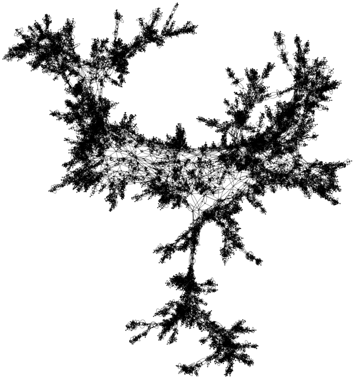
More on random planar maps

1. First studied by Tutte in 1960s while working on the four color theorem.
2. Variants: triangulations, quadrangulations, etc.



(Simulation due to J.F. Marckert)

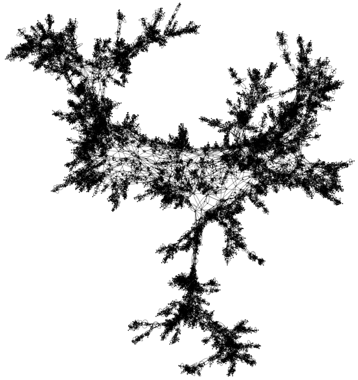
More on random planar maps



(Simulation due to J.F. Marckert)

1. First studied by Tutte in 1960s while working on the four color theorem.
2. Variants: triangulations, quadrangulations, etc.
3. Can be interpreted as Riemannian manifolds with conical singularities.

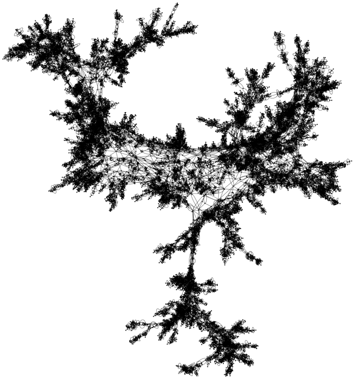
More on random planar maps



(Simulation due to J.F. Marckert)

1. First studied by Tutte in 1960s while working on the four color theorem.
2. Variants: triangulations, quadrangulations, etc.
3. Can be interpreted as Riemannian manifolds with conical singularities.
4. Converges in law in Gromov-Hausdorff sense to random metric space called Brownian map, homeomorphic to the 2-sphere, Hausdorff dimension 4 (established in several works by subsets of Chaissang, Schaefer, Le Gall, Paulin, Miermont). Higher genus work announced by Miermont and Bettinelli.

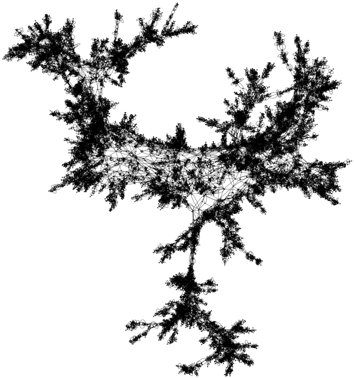
More on random planar maps



(Simulation due to J.F. Marckert)

1. First studied by Tutte in 1960s while working on the four color theorem.
2. Variants: triangulations, quadrangulations, etc.
3. Can be interpreted as Riemannian manifolds with conical singularities.
4. Converges in law in Gromov-Hausdorff sense to random metric space called Brownian map, homeomorphic to the 2-sphere, Hausdorff dimension 4 (established in several works by subsets of Chaissang, Schaefer, Le Gall, Paulin, Miermont). Higher genus work announced by Miermont and Bettinelli.
5. **String theory program:** Polyakov, etc., map *decorated* by function from vertices to $\mathbb{R}^d$, representing map *embedded* in $\mathbb{R}^d$. Can also decorate by distinguished spanning tree, edge subset, "spin" function, etc.

More on random planar maps



(Simulation due to J.F. Marckert)

1. First studied by Tutte in 1960s while working on the four color theorem.
2. Variants: triangulations, quadrangulations, etc.
3. Can be interpreted as Riemannian manifolds with conical singularities.
4. Converges in law in Gromov-Hausdorff sense to random metric space called Brownian map, homeomorphic to the 2-sphere, Hausdorff dimension 4 (established in several works by subsets of Chaissang, Schaefer, Le Gall, Paulin, Miermont). Higher genus work announced by Miermont and Bettinelli.
5. **String theory program:** Polyakov, etc., map *decorated* by function from vertices to $\mathbb{R}^d$, representing map *embedded* in $\mathbb{R}^d$. Can also decorate by distinguished spanning tree, edge subset, "spin" function, etc.
6. **Brownian surface program:** Understand $d = 0$ case very well, build entire theory using Brownian snakes in place of GFF.

"How many ways" are there to embed a given map in $\mathbb{R}^d$?

- Easy Gaussian integral: $\int (2\pi)^{-1/2} e^{-7x^2/2} = 7^{-1/2}$. Can write $7 = 1/\sigma^2$.

"How many ways" are there to embed a given map in $\mathbb{R}^d$?

- Easy Gaussian integral: $\int (2\pi)^{-1/2} e^{-7x^2/2} = 7^{-1/2}$. Can write $7 = 1/\sigma^2$.
- In dimension d, $\int (2\pi)^{-d/2} e^{-(x,Ax)/2} = |\det A|^{-1/2}$, which we refer to as *partition function*. Note that $|\det A|^{1/2}$ is height of normal density function at origin. Probability Gaussian is in ϵ^d box is (up to 2π factors) about $\epsilon^d |\det A|^{1/2}$.

"How many ways" are there to embed a given map in $\mathbb{R}^d$?

- Easy Gaussian integral: $\int (2\pi)^{-1/2} e^{-7x^2/2} = 7^{-1/2}$. Can write $7 = 1/\sigma^2$.
- In dimension d, $\int (2\pi)^{-d/2} e^{-(x,Ax)/2} = |\det A|^{-1/2}$, which we refer to as *partition function*. Note that $|\det A|^{1/2}$ is height of normal density function at origin. Probability Gaussian is in ϵ^d box is (up to 2π factors) about $\epsilon^d |\det A|^{1/2}$.
- Laplacian of finite connected graph (V, E) is linear operator Δ from $\mathbf{R}^V$ to itself. Its matrix is given by

$$M_{i,j} = \begin{cases} 1 & i \neq j, (v_i, v_j) \in E \\ 0 & i \neq j, (v_i, v_j) \notin E \\ -\deg(v_i) & i = j. \end{cases}.$$

"How many ways" are there to embed a given map in $\mathbb{R}^d$?

- Easy Gaussian integral: $\int (2\pi)^{-1/2} e^{-7x^2/2} = 7^{-1/2}$. Can write $7 = 1/\sigma^2$.
- In dimension d, $\int (2\pi)^{-d/2} e^{-(x,Ax)/2} = |\det A|^{-1/2}$, which we refer to as *partition function*. Note that $|\det A|^{1/2}$ is height of normal density function at origin. Probability Gaussian is in ϵ^d box is (up to 2π factors) about $\epsilon^d |\det A|^{1/2}$.
- Laplacian of finite connected graph (V, E) is linear operator Δ from $\mathbf{R}^V$ to itself. Its matrix is given by

$$M_{i,j} = \begin{cases} 1 & i \neq j, (v_i, v_j) \in E \\ 0 & i \neq j, (v_i, v_j) \notin E \\ -\deg(v_i) & i = j. \end{cases}.$$

- Let $R \subset \mathbf{R}^V$ be the set of functions with mean zero. Then $\Delta : R \to R$ is invertible, and Kirchhoff's matrix tree theorem states that $\det \Delta$ is ($|V|$ times) the number of spanning trees of (V, E).

"How many ways" are there to embed a given map in $\mathbb{R}^d$?

- Easy Gaussian integral: $\int (2\pi)^{-1/2} e^{-7x^2/2} = 7^{-1/2}$. Can write $7 = 1/\sigma^2$.
- In dimension d, $\int (2\pi)^{-d/2} e^{-(x,Ax)/2} = |\det A|^{-1/2}$, which we refer to as *partition function*. Note that $|\det A|^{1/2}$ is height of normal density function at origin. Probability Gaussian is in ϵ^d box is (up to 2π factors) about $\epsilon^d |\det A|^{1/2}$.
- Laplacian of finite connected graph (V, E) is linear operator Δ from $\mathbf{R}^V$ to itself. Its matrix is given by

$$M_{i,j} = \begin{cases} 1 & i \neq j, (v_i, v_j) \in E \\ 0 & i \neq j, (v_i, v_j) \notin E \\ -\deg(v_i) & i = j. \end{cases}.$$

- Let $R \subset \mathbf{R}^V$ be the set of functions with mean zero. Then $\Delta : R \to R$ is invertible, and Kirchhoff's matrix tree theorem states that $\det \Delta$ is ($|V|$ times) the number of spanning trees of (V, E).
- $\det \Delta$ is also the product of the (non-zero) eigenvalues of Δ. Sometimes write $\det' \Delta$ to stress on is excluding zero mode.

"How many ways" are there to embed a given map in $\mathbb{R}^d$?

- Easy Gaussian integral: $\int (2\pi)^{-1/2} e^{-7x^2/2} = 7^{-1/2}$. Can write $7 = 1/\sigma^2$.
- In dimension d, $\int (2\pi)^{-d/2} e^{-(x,Ax)/2} = |\det A|^{-1/2}$, which we refer to as *partition function*. Note that $|\det A|^{1/2}$ is height of normal density function at origin. Probability Gaussian is in ϵ^d box is (up to 2π factors) about $\epsilon^d |\det A|^{1/2}$.
- Laplacian of finite connected graph (V,E) is linear operator Δ from $\mathbf{R}^V$ to itself. Its matrix is given by

$$M_{i,j} = \begin{cases} 1 & i \neq j, (v_i, v_j) \in E \\ 0 & i \neq j, (v_i, v_j) \notin E \\ -\deg(v_i) & i = j. \end{cases}.$$

- Let $R \subset \mathbf{R}^V$ be the set of functions with mean zero. Then $\Delta : R \to R$ is invertible, and Kirchhoff's matrix tree theorem states that $\det \Delta$ is ($|V|$ times) the number of spanning trees of (V,E).
- $\det \Delta$ is also the product of the (non-zero) eigenvalues of Δ. Sometimes write $\det' \Delta$ to stress on is excluding zero mode.
- The DGFF partition function ("number of ways to embed") can be written as (power of 2π times) $\int e^{-(f,\Delta f)/2} df = (\det \Delta)^{-1/2}$.

Think about $\det \Delta$ and decorated maps

- $\det \Delta$ (the number of spanning trees) is minimized if map is itself a tree. If we fix number of edges, then intuitively, the less *tree-like* M is, the larger $\det \Delta$.

Think about $\det \Delta$ and decorated maps

- $\det \Delta$ (the number of spanning trees) is minimized if map is itself a tree. If we fix number of edges, then intuitively, the less *tree-like* M is, the larger $\det \Delta$.
- **Tree-weighted maps:** If instead of choosing a uniform planar map M, we choose a tree-decorated map (M, T) then the marginal law of M gives each M a probability proportional to $\det \Delta$. So M tends to be "less tree like" than in the unweighted case.

Think about $\det \Delta$ and decorated maps

- $\det \Delta$ (the number of spanning trees) is minimized if map is itself a tree. If we fix number of edges, then intuitively, the less *tree-like* M is, the larger $\det \Delta$.
- **Tree-weighted maps:** If instead of choosing a uniform planar map M, we choose a tree-decorated map (M, T) then the marginal law of M gives each M a probability proportional to $\det \Delta$. So M tends to be "less tree like" than in the unweighted case.
- **GFF-weighted maps:** Choosing map decorated by d instances of the GFF (interpreted as a map embedded in $\mathbf{R}^d$) corresponds to weighting by $(\det \Delta)^{-d/2}$. The higher the value of d, the more "tree-like" M should typically be.

Think about loop soup decorated maps

- If we fix a boundary vertex then for an appropriately defined loop measure μ (and p number of vertices hit) we have

$$\mu(p > 1) = -\frac{1}{2}\log(\det \Delta)$$

.

Think about loop soup decorated maps

- If we fix a boundary vertex then for an appropriately defined loop measure μ (and p number of vertices hit) we have

$$\mu(p > 1) = -\frac{1}{2}\log(\det\Delta)$$

- Requiring $p > 1$ effectively truncates small loops. Since paths are killed when they hit the boundary vertex, long loops are exponentially rare. (See Lupu, Dynkin, Le Jan, Lawler, Werner...)

Think about loop soup decorated maps

- If we fix a boundary vertex then for an appropriately defined loop measure μ (and p number of vertices hit) we have

$$\mu(p > 1) = -\frac{1}{2}\log(\det \Delta)$$

- Requiring $p > 1$ effectively truncates small loops. Since paths are killed when they hit the boundary vertex, long loops are exponentially rare. (See Lupu, Dynkin, Le Jan, Lawler, Werner...)
- **Loop soup weighted maps of intensity c:** Random loop-decorated map. Given map, the loops are Poisson point process; total number Poisson with parameter $c\mu(p > 1)$. Given that there are k loops, map probabilities are proportional to $(c\mu(p > 1))^k$. Overall probability of a map is proportional to its "intensity c loop soup partition function" which is $e^{c\mu(p>1)} = (\det \Delta)^{-c/2}$

Think about loop soup decorated maps

- If we fix a boundary vertex then for an appropriately defined loop measure μ (and p number of vertices hit) we have

$$\mu(p > 1) = -\frac{1}{2}\log(\det \Delta)$$

- Requiring $p > 1$ effectively truncates small loops. Since paths are killed when they hit the boundary vertex, long loops are exponentially rare. (See Lupu, Dynkin, Le Jan, Lawler, Werner...)
- **Loop soup weighted maps of intensity c:** Random loop-decorated map. Given map, the loops are Poisson point process; total number Poisson with parameter $c\mu(p > 1)$. Given that there are k loops, map probabilities are proportional to $(c\mu(p > 1))^k$. Overall probability of a map is proportional to its "intensity c loop soup partition function" which is $e^{c\mu(p>1)} = (\det \Delta)^{-c/2}$
- Can increase c (make M more tree-like) or decrease c (make M less tree-like).

Think about loop soup decorated maps

- If we fix a boundary vertex then for an appropriately defined loop measure μ (and p number of vertices hit) we have
$$\mu(p > 1) = -\frac{1}{2}\log(\det \Delta)$$
- Requiring $p > 1$ effectively truncates small loops. Since paths are killed when they hit the boundary vertex, long loops are exponentially rare. (See Lupu, Dynkin, Le Jan, Lawler, Werner...)
- **Loop soup weighted maps of intensity c:** Random loop-decorated map. Given map, the loops are Poisson point process; total number Poisson with parameter $c\mu(p > 1)$. Given that there are k loops, map probabilities are proportional to $(c\mu(p > 1))^k$. Overall probability of a map is proportional to its "intensity c loop soup partition function" which is $e^{c\mu(p>1)} = (\det \Delta)^{-c/2}$
- Can increase c (make M more tree-like) or decrease c (make M less tree-like).
- **Belief:** If $c \leq 1$ then weighting by $(\det \Delta)^{-c/2}$ makes the scaling limit a Liouville quantum gravity surface with parameter γ whose value is fixed by $Q = 2/\gamma + \gamma/2$ where $c = 25 - 6Q^2$.

Think about loop soup decorated maps

- If we fix a boundary vertex then for an appropriately defined loop measure μ (and p number of vertices hit) we have

$$\mu(p > 1) = -\frac{1}{2}\log(\det \Delta)$$

- Requiring $p > 1$ effectively truncates small loops. Since paths are killed when they hit the boundary vertex, long loops are exponentially rare. (See Lupu, Dynkin, Le Jan, Lawler, Werner...)
- **Loop soup weighted maps of intensity c:** Random loop-decorated map. Given map, the loops are Poisson point process; total number Poisson with parameter $c\mu(p > 1)$. Given that there are k loops, map probabilities are proportional to $(c\mu(p > 1))^k$. Overall probability of a map is proportional to its "intensity c loop soup partition function" which is $e^{c\mu(p>1)} = (\det \Delta)^{-c/2}$
- Can increase c (make M more tree-like) or decrease c (make M less tree-like).
- **Belief:** If $c \leq 1$ then weighting by $(\det \Delta)^{-c/2}$ makes the scaling limit a Liouville quantum gravity surface with parameter γ whose value is fixed by $Q = 2/\gamma + \gamma/2$ where $c = 25 - 6Q^2$.
- **Note:** As c increases from $-\infty$ to 1 the value γ increases from 0 to 2.

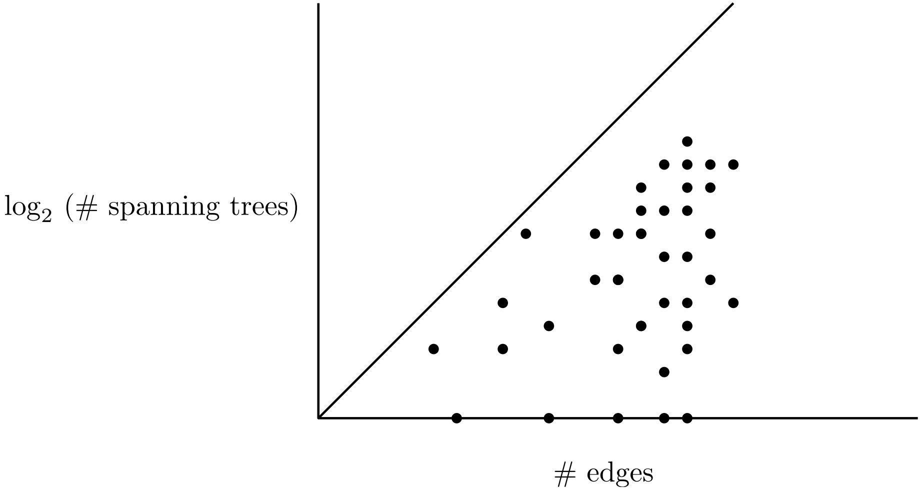
Think about loop soup decorated maps

- If we fix a boundary vertex then for an appropriately defined loop measure μ (and p number of vertices hit) we have

$$\mu(p > 1) = -\frac{1}{2}\log(\det\Delta)$$

- Requiring $p > 1$ effectively truncates small loops. Since paths are killed when they hit the boundary vertex, long loops are exponentially rare. (See Lupu, Dynkin, Le Jan, Lawler, Werner...)
- **Loop soup weighted maps of intensity c:** Random loop-decorated map. Given map, the loops are Poisson point process; total number Poisson with parameter $c\mu(p > 1)$. Given that there are k loops, map probabilities are proportional to $(c\mu(p > 1))^k$. Overall probability of a map is proportional to its "intensity c loop soup partition function" which is $e^{c\mu(p>1)} = (\det\Delta)^{-c/2}$
- Can increase c (make M more tree-like) or decrease c (make M less tree-like).
- **Belief:** If $c \leq 1$ then weighting by $(\det\Delta)^{-c/2}$ makes the scaling limit a Liouville quantum gravity surface with parameter γ whose value is fixed by $Q = 2/\gamma + \gamma/2$ where $c = 25 - 6Q^2$.
- **Note:** As c increases from $-\infty$ to 1 the value γ increases from 0 to 2.
- When $c > 1$, naive construction seems to give embedded continuum random tree as scaling limit. But as often happens in life, there is an alternative...

Two measures of (sphere-embedded) planar map “size”



Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.

Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.
- Ising model: $c = 1/2$

Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.
- Ising model: $c = 1/2$
- FK random cluster model: $c \in [-2, 1]$

Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.
- Ising model: $c = 1/2$
- FK random cluster model: $c \in [-2, 1]$
- Bipolar orientations: $c = -7$ (Kenyon, Miller, S. ,Wilson)

Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.
- Ising model: $c = 1/2$
- FK random cluster model: $c \in [-2, 1]$
- Bipolar orientations: $c = -7$ (Kenyon, Miller, S. ,Wilson)
- Schnyder Woods: $c = -25/2$ (Li, Sun, Watson)

Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.
- Ising model: $c = 1/2$
- FK random cluster model: $c \in [-2, 1]$
- Bipolar orientations: $c = -7$ (Kenyon, Miller, S. ,Wilson)
- Schnyder Woods: $c = -25/2$ (Li, Sun, Watson)
- And, again, spanning tree: $c = -2$

Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.
- Ising model: $c = 1/2$
- FK random cluster model: $c \in [-2, 1]$
- Bipolar orientations: $c = -7$ (Kenyon, Miller, S. ,Wilson)
- Schnyder Woods: $c = -25/2$ (Li, Sun, Watson)
- And, again, spanning tree: $c = -2$
- Discrete GFF (or some similar discrete height function): $c = 1$

Other statistical physics models

- Adding other critical statistical physics structure thought to be morally like adding loop soups of some intensity c. Either way, scaling limit should be LQG surface with parameter γ fixed by $Q = 2/\gamma + \gamma/2$ and $c = 25 - 6Q^2$.
- Ising model: $c = 1/2$
- FK random cluster model: $c \in [-2, 1]$
- Bipolar orientations: $c = -7$ (Kenyon, Miller, S. ,Wilson)
- Schnyder Woods: $c = -25/2$ (Li, Sun, Watson)
- And, again, spanning tree: $c = -2$
- Discrete GFF (or some similar discrete height function): $c = 1$
- Intensity c loop soup: c

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.
- **Bob:** And loop-soup-decorated maps should scale to “LQG surfaces” that are “rougher or smoother” (i.e., more or less tree-like) than the Brownian map as c is respectively positive or negative. Big c means super rough. c close to $-\infty$ means smoother (but still fractal).

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.
- **Bob:** And loop-soup-decorated maps should scale to "LQG surfaces" that are "rougher or smoother" (i.e., more or less tree-like) than the Brownian map as c is respectively positive or negative. Big c means super rough. c close to $-\infty$ means smoother (but still fractal).
- **Alice:** The exact Hausdorff dimension as a function of c is a big open problem, but it should increase from 2 to 4.8 or so as c increases from $-\infty$ to 1. But there's some *truly amazing* new work about the metric structure of these LQG surfaces (Gwynne, Miller, Ding, Dubédat, Falconet, Pfeffer, Sun, Ang, Remy, others I'm forgetting)

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.
- **Bob:** And loop-soup-decorated maps should scale to "LQG surfaces" that are "rougher or smoother" (i.e., more or less tree-like) than the Brownian map as c is respectively positive or negative. Big c means super rough. c close to $-\infty$ means smoother (but still fractal).
- **Alice:** The exact Hausdorff dimension as a function of c is a big open problem, but it should increase from 2 to 4.8 or so as c increases from $-\infty$ to 1. But there's some *truly amazing* new work about the metric structure of these LQG surfaces (Gwynne, Miller, Ding, Dubédat, Falconet, Pfeffer, Sun, Ang, Remy, others I'm forgetting)
- **Eve:** Right.

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.
- **Bob:** And loop-soup-decorated maps should scale to "LQG surfaces" that are "rougher or smoother" (i.e., more or less tree-like) than the Brownian map as c is respectively positive or negative. Big c means super rough. c close to $-\infty$ means smoother (but still fractal).
- **Alice:** The exact Hausdorff dimension as a function of c is a big open problem, but it should increase from 2 to 4.8 or so as c increases from $-\infty$ to 1. But there's some *truly amazing* new work about the metric structure of these LQG surfaces (Gwynne, Miller, Ding, Dubédat, Falconet, Pfeffer, Sun, Ang, Remy, others I'm forgetting)
- **Eve:** Right.
- **Alice:** But you haven't mentioned continuum loop soups or zeta functions.

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.
- **Bob:** And loop-soup-decorated maps should scale to "LQG surfaces" that are "rougher or smoother" (i.e., more or less tree-like) than the Brownian map as c is respectively positive or negative. Big c means super rough. c close to $-\infty$ means smoother (but still fractal).
- **Alice:** The exact Hausdorff dimension as a function of c is a big open problem, but it should increase from 2 to 4.8 or so as c increases from $-\infty$ to 1. But there's some *truly amazing* new work about the metric structure of these LQG surfaces (Gwynne, Miller, Ding, Dubédat, Falconet, Pfeffer, Sun, Ang, Remy, others I'm forgetting)
- **Eve:** Right.
- **Alice:** But you haven't mentioned continuum loop soups or zeta functions.
- **Eve:** That's coming.

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.
- **Bob:** And loop-soup-decorated maps should scale to "LQG surfaces" that are "rougher or smoother" (i.e., more or less tree-like) than the Brownian map as c is respectively positive or negative. Big c means super rough. c close to $-\infty$ means smoother (but still fractal).
- **Alice:** The exact Hausdorff dimension as a function of c is a big open problem, but it should increase from 2 to 4.8 or so as c increases from $-\infty$ to 1. But there's some *truly amazing* new work about the metric structure of these LQG surfaces (Gwynne, Miller, Ding, Dubédat, Falconet, Pfeffer, Sun, Ang, Remy, others I'm forgetting)
- **Eve:** Right.
- **Alice:** But you haven't mentioned continuum loop soups or zeta functions.
- **Eve:** That's coming.
- **Bob:** Can you just show us a theorem?

The conversation

- **Alice:** You are taking longer to get the point than I anticipated.
- **Bob:** Yes, I mean I think we already believed that loop soups, Laplacian determinants, and GFFs were related on *discrete* maps.
- **Alice:** We know uniform random planar maps scale to Brownian surfaces.
- **Bob:** And loop-soup-decorated maps should scale to "LQG surfaces" that are "rougher or smoother" (i.e., more or less tree-like) than the Brownian map as c is respectively positive or negative. Big c means super rough. c close to $-\infty$ means smoother (but still fractal).
- **Alice:** The exact Hausdorff dimension as a function of c is a big open problem, but it should increase from 2 to 4.8 or so as c increases from $-\infty$ to 1. But there's some *truly amazing* new work about the metric structure of these LQG surfaces (Gwynne, Miller, Ding, Dubédat, Falconet, Pfeffer, Sun, Ang, Remy, others I'm forgetting)
- **Eve:** Right.
- **Alice:** But you haven't mentioned continuum loop soups or zeta functions.
- **Eve:** That's coming.
- **Bob:** Can you just show us a theorem?
- **Alice:** Or maybe first some more pictures and then a theorem?

Pre-theorem definitions

- For a Riemannian manifold (M, g) (possibly with boundary) and a curve η on it, define its *duration* $\nu(\eta)$ to be half its quadratic variation.

Pre-theorem definitions

- For a Riemannian manifold (M, g) (possibly with boundary) and a curve η on it, define its *duration* $\nu(\eta)$ to be half its quadratic variation.
- Suppose that g_0 is a metric and $g = e^{2\sigma} g_0$ for some smooth $\sigma : M \to \mathbf{R}$, and let ν_0, ν be their respective duration functions. If $\eta : [0, T] \to \mathbf{R}$ is parametrized by ν_0, then we have

$$\nu(\eta) = \int_0^T e^{2\sigma(\eta_t)}\, dt.$$

Pre-theorem definitions

- For a Riemannian manifold (M, g) (possibly with boundary) and a curve η on it, define its *duration* $\nu(\eta)$ to be half its quadratic variation.
- Suppose that g_0 is a metric and $g = e^{2\sigma} g_0$ for some smooth $\sigma : M \to \mathbf{R}$, and let ν_0, ν be their respective duration functions. If $\eta : [0, T] \to \mathbf{R}$ is parametrized by ν_0, then we have

$$\nu(\eta) = \int_0^T e^{2\sigma(\eta_t)} \, dt.$$

- Recall Brownian loop measure, introduced by Lawler and Werner for planar domains. Definition extends easily to the general setting of a Riemannian manifold (M, g), possibly with boundary, and in this case we write $\mu^{\mathrm{loop}}_{M,g}$ for the Brownian loop measure.

Pre-theorem definitions

- For a Riemannian manifold (M, g) (possibly with boundary) and a curve η on it, define its *duration* $\nu(\eta)$ to be half its quadratic variation.
- Suppose that g_0 is a metric and $g = e^{2\sigma} g_0$ for some smooth $\sigma : M \to \mathbf{R}$, and let ν_0, ν be their respective duration functions. If $\eta : [0, T] \to \mathbf{R}$ is parametrized by ν_0, then we have

$$\nu(\eta) = \int_0^T e^{2\sigma(\eta_t)}\, dt.$$

- Recall Brownian loop measure, introduced by Lawler and Werner for planar domains. Definition extends easily to the general setting of a Riemannian manifold (M, g), possibly with boundary, and in this case we write $\mu_{M,g}^{\text{loop}}$ for the Brownian loop measure.
- On any 2-dimensional Riemannian manifold (M, g), the loop measure $\mu_{M,g}^{\text{loop}}$ is infinite because of the many loops of short duration, so when we study the loop mass we need to perform a regularization procedure to handle the infinitude of small loops. We will sometimes truncate loops shorter than a constant δ.

A theorem about surfaces with boundary

- **THEOREM (Ang, Park, Pfeffer, S.):** Let (M, g) be a compact orientable dimension 2 Riemannian manifold with boundary. Then for small $\delta > 0$ we have $\mu^{\text{loop}}_{M,g}(\mathcal{L}(M, g, \delta))$ is equal to

$$\frac{\mathrm{Vol}_g(M)}{2\pi\delta} - \frac{\mathrm{Len}_g(\partial M)}{\sqrt{8\pi\delta}} - \log\det\Delta_g - \frac{\chi(M)}{6}\log\delta + (\gamma + \log 2)\frac{\chi(M)}{6} + O(\delta^{1/2}),$$

where γ is the Euler-Mascheroni constant and $\chi(M)$ is the Euler characteristic.

Theorem for compact surfaces without boundary

- **THEOREM (Ang, Park, Pfeffer, S.):** Let (M, g) be a compact orientable surface. Then for $\delta > 0$ small and $C > 0$ large we have, with γ the Euler-Mascheroni constant,

$$\mu^{\text{loop}}_{M,g}(\mathcal{L}(M, g, \delta)\backslash\mathcal{L}(M, g, C)) = \frac{\mathrm{Vol}_g(M)}{2\pi\delta} - \frac{\chi(M)}{6}\log\delta + \log C - \log\det{}'\Delta_g + \\ + O(\delta^{1/2}) + O(e^{-\alpha C}),$$

where $\alpha > 0$ depends on the manifold (M, g).

Theorem for compact surfaces without boundary

- **THEOREM (Ang, Park, Pfeffer, S.):** Let (M, g) be a compact orientable surface. Then for $\delta > 0$ small and $C > 0$ large we have, with γ the Euler-Mascheroni constant,

$$\mu^{\text{loop}}_{M,g}(\mathcal{L}(M,g,\delta)\backslash\mathcal{L}(M,g,C)) = \frac{\mathrm{Vol}_g(M)}{2\pi\delta} - \frac{\chi(M)}{6}\log\delta + \log C - \log\det{}'\Delta_g + + O(\delta^{1/2}) + O(e^{-\alpha C}),$$

  where $\alpha > 0$ depends on the manifold (M, g).
- **COROLLARY:** Let (S^2, g) be a sphere and η a simple smooth closed curve on the sphere. Then the mass of loops hitting γ of size between δ and C is given by

$$\frac{\mathrm{Len}_g(\eta)}{\sqrt{2\pi\delta}} + \log C - \log\mathrm{Vol}_g(S^2) - \frac{1}{12}I_L(\eta)$$
$$-\mathcal{H}(S^1, g) - \gamma - \log 2 + O(\delta^{1/2}) + O(e^{-\alpha C}),$$

  where $I_L(\eta)$ is the Loewner energy of the curve η, and γ is the Euler-Mascheroni constant.

Theorem for compact surfaces without boundary

- **THEOREM (Ang, Park, Pfeffer, S.):** Let (M, g) be a compact orientable surface. Then for $\delta > 0$ small and $C > 0$ large we have, with γ the Euler-Mascheroni constant,

$$\mu^{\text{loop}}_{M,g}(\mathcal{L}(M,g,\delta)\setminus\mathcal{L}(M,g,C)) = \frac{\mathrm{Vol}_g(M)}{2\pi\delta} - \frac{\chi(M)}{6}\log\delta + \log C - \log\det{}'\Delta_g + + O(\delta^{1/2}) + O(e^{-\alpha C}),$$

where $\alpha > 0$ depends on the manifold (M, g).

- **COROLLARY:** Let (S^2, g) be a sphere and η a simple smooth closed curve on the sphere. Then the mass of loops hitting γ of size between δ and C is given by

$$\frac{\mathrm{Len}_g(\eta)}{\sqrt{2\pi\delta}} + \log C - \log \mathrm{Vol}_g(S^2) - \frac{1}{12} I_L(\eta)$$
$$-\mathcal{H}(S^1, g) - \gamma - \log 2 + O(\delta^{1/2}) + O(e^{-\alpha C}),$$

where $I_L(\eta)$ is the Loewner energy of the curve η, and γ is the Euler-Mascheroni constant.

- Take loop mass on sphere, subtract loop mass in each half of $S^2\setminus\eta$ applying Yilin-Wang (the quantity $\mathcal{H}(S^1, g)$ there is a nonexplicit constant).

Polyakov-Alvarez

- For the case of a simply connected domain $D \subset \mathbf{C}$ with smooth boundary, we can rewrite the above result using the Polyakov-Alvarez conformal anomaly formula. Let σ be a smooth function on D with derivatives extending continuously to ∂D. Then, with respect to the Brownian loop measure on D, the mass of loops having duration at least δ with respect to the metric $g = e^{2\sigma}(dx^2 + dy^2)$ is given by

$$\mu_D^{\text{loop}}(\mathcal{L}(D, g, \delta)) = \frac{\mathrm{Vol}_g(D)}{2\pi\delta} - \frac{\mathrm{Len}_g(\partial D)}{\sqrt{8\pi\delta}} - \frac{1}{6}\log\delta + \frac{1}{12\pi}\iint_D |\nabla\sigma(z)|^2 \, dz + \frac{1}{4\pi}\int_{\partial D} \sigma_{\mathbf{n}}(w)\, dw + \frac{1}{6\pi}\int_{\partial D} k_0 \sigma(w)\, dw + \tilde{c} + o(1),$$

where we write $\sigma_{\mathbf{n}}$ to denote the derivative of σ in the outward normal direction along ∂D, and k_0 for the geodesic curvature on ∂D with respect to the Euclidean metric $dx^2 + dy^2$. Here, $\tilde{c}$ is constant not depending on σ.

Exponentially discount long loops (has discrete analog)

- For $\kappa > 0$, define the *loop measure with κ-decay* $\mu^{\mathrm{loop}}_{M,g,\kappa}$ to be the loop measure such that for any Brownian loop η, we have the Radon-Nikodym derivative

$$\frac{d\mu^{\mathrm{loop}}_{M,g,\kappa}}{d\mu^{\mathrm{loop}}_{M,g}}(\eta) = e^{-\kappa\nu(\eta)}.$$

Exponentially discount long loops (has discrete analog)

- For $\kappa > 0$, define the *loop measure with κ-decay* $\mu^{\text{loop}}_{M,g,\kappa}$ to be the loop measure such that for any Brownian loop η, we have the Radon-Nikodym derivative
$$\frac{d\mu^{\text{loop}}_{M,g,\kappa}}{d\mu^{\text{loop}}_{M,g}}(\eta) = e^{-\kappa\nu(\eta)}.$$
- **THEOREM:** Let (M, g) be a closed orientable dimension 2 Riemannian manifold. For $\kappa > 0$, we have $\mu^{\text{loop}}_{M,g,2\kappa}(\mathcal{L}(M, g, \delta)) =$
$$\frac{\mathrm{Vol}_g(M)}{2\pi\delta} - \frac{\chi(M)}{6}\log\delta - \log\kappa - \log\det{}'\Delta_g + (\gamma + \log 2)\frac{\chi(M)}{6} + \text{lower order terms.}$$

Defining Brownian loop measure

- The *rooted Brownian loop measure* on (M, g), denoted $\mu^{rooted}_{M,g}$, is a measure on rooted loops, i.e., paths $\gamma : [0, L] \to M$ with $\gamma(0) = \gamma(L)$, given by

$$\mu^{rooted}_{M,g} := \int_M \frac{1}{\nu_{M,g}(\gamma)} \mu^{z,z}_{M,g} \mathrm{Vol}_g(dz).$$

  Recall that $\nu_{M,g}(\gamma)$ is the duration of the loop γ, which equals L for almost every γ in the support of $\mu_{M,g}$.

Defining Brownian loop measure

- The *rooted Brownian loop measure* on (M, g), denoted $\mu^{rooted}_{M,g}$, is a measure on rooted loops, i.e., paths $\gamma : [0, L] \to M$ with $\gamma(0) = \gamma(L)$, given by

$$\mu^{rooted}_{M,g} := \int_M \frac{1}{\nu_{M,g}(\gamma)} \mu^{z,z}_{M,g} \mathrm{Vol}_g(dz).$$

  Recall that $\nu_{M,g}(\gamma)$ is the duration of the loop γ, which equals L for almost every γ in the support of $\mu_{M,g}$.

- **LEMMA:** If g, g' are conformally equivalent metrics on a manifold M, then the measures $\mu^{rooted}_{M,g}$ and $\mu^{rooted}_{M,g'}$ induce the same measure on unrooted loops; i.e., on equivalence classes of rooted loops $\gamma : [0, L] \to M$ under the equivalence relation identifying γ with

$$\theta_r \gamma(s) := \begin{cases} \gamma(s + r), & \text{if } s \leq L - r \\ \gamma(s + r - L), & \text{if } s > L - r \end{cases}$$

zeta regularized determinant and Brownian loop mass

- For $s \in \mathbf{C}$ with $\Re s > 1$, define the *Selberg zeta function*

$$\zeta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}.$$

zeta regularized determinant and Brownian loop mass

- For $s \in \mathbf{C}$ with $\Re s > 1$, define the *Selberg zeta function*

$$\zeta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}.$$

- By Weyl's formula we know that $\lambda_n \asymp n$, so the condition that $\Re s > 1$ guarantees convergence and furthermore

$$\zeta'(s) = -\sum_{\lambda_j \neq 0} \lambda_j^{-s} \log \lambda_j.$$

zeta regularized determinant and Brownian loop mass

- For $s \in \mathbf{C}$ with $\Re s > 1$, define the *Selberg zeta function*

$$\zeta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}.$$

- By Weyl's formula we know that $\lambda_n \asymp n$, so the condition that $\Re s > 1$ guarantees convergence and furthermore

$$\zeta'(s) = -\sum_{\lambda_j \neq 0} \lambda_j^{-s} \log \lambda_j.$$

- One can show that ζ can be meromorphically continued to the complex plane, and is holomorphic at $s = 0$. One can formally interpret $\zeta'(0)$ as $-\sum_{\lambda_j \neq 0} \log \lambda_j$, motivating the definition of the *zeta regularized determinant* $\det' \Delta$ (Ray, Singer).

$$\log \det{}' \Delta := -\zeta'(0).$$

zeta regularized determinant and Brownian loop mass

- For $s \in \mathbf{C}$ with $\Re s > 1$, define the *Selberg zeta function*

$$\zeta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}.$$

- By Weyl's formula we know that $\lambda_n \asymp n$, so the condition that $\Re s > 1$ guarantees convergence and furthermore

$$\zeta'(s) = -\sum_{\lambda_j \neq 0} \lambda_j^{-s} \log \lambda_j.$$

- One can show that ζ can be meromorphically continued to the complex plane, and is holomorphic at $s = 0$. One can formally interpret $\zeta'(0)$ as $-\sum_{\lambda_j \neq 0} \log \lambda_j$, motivating the definition of the *zeta regularized determinant* $\det' \Delta$ (Ray, Singer).

$$\log \det' \Delta := -\zeta'(0).$$

- We write $\det'$ to indicate the removal of the zero eigenvalue. For the Laplace-Beltrami operator on a two dimensional compact orientable manifold with smooth boundary, we can similarly define its zeta regularized determinant $\det \Delta$ (no zero eigenvalue is removed).

zeta regularized determinant and Brownian loop mass

- (M, g) is an orientable two dimensional Riemannian manifold which is either closed or compact with boundary; let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be its nonzero eigenvalues, and let $N \geq 0$ be the multiplicity of the zero eigenvalue.

zeta regularized determinant and Brownian loop mass

- (M, g) is an orientable two dimensional Riemannian manifold which is either closed or compact with boundary; let $0 < \lambda_1 \le \lambda_2 \le \dots$ be its nonzero eigenvalues, and let $N \ge 0$ be the multiplicity of the zero eigenvalue.
- The following is valid for $\Re(s) > 1$ since $\mathrm{tr}(e^{-t\Delta}) - N$ decays exponentially in t (this follows from the presence of the boundary):

$$\zeta(s) = \sum_{j>0} \lambda_j^{-s} = \sum_{j>0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda_j}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\mathrm{tr}(e^{-t\Delta}) - N)\, dt.$$

zeta regularized determinant and Brownian loop mass

- (M, g) is an orientable two dimensional Riemannian manifold which is either closed or compact with boundary; let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be its nonzero eigenvalues, and let $N \geq 0$ be the multiplicity of the zero eigenvalue.
- The following is valid for $\Re(s) > 1$ since $\mathrm{tr}(e^{-t\Delta}) - N$ decays exponentially in t (this follows from the presence of the boundary):

$$\zeta(s) = \sum_{j>0} \lambda_j^{-s} = \sum_{j>0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda_j}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\mathrm{tr}(e^{-t\Delta}) - N)\, dt.$$

- In other words, $\zeta(s)$ is $\frac{1}{\Gamma(s)}$-times the *Mellin transform* of $\mathrm{tr}(e^{-t\Delta})$. Notice that the above integral does not makes sense when $\Re(s) \leq 1$ since it blows up near $t = 0$. Nevertheless, if we understand the behavior of $\mathrm{tr}(e^{-t\Delta})$ near 0 and ∞, we can try to to meromorphically extend the above function to the whole complex plane.

zeta regularized determinant and Brownian loop mass

- (M, g) is an orientable two dimensional Riemannian manifold which is either closed or compact with boundary; let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be its nonzero eigenvalues, and let $N \geq 0$ be the multiplicity of the zero eigenvalue.
- The following is valid for $\Re(s) > 1$ since $\mathrm{tr}(e^{-t\Delta}) - N$ decays exponentially in t (this follows from the presence of the boundary):

$$\zeta(s) = \sum_{j>0} \lambda_j^{-s} = \sum_{j>0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda_j}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\mathrm{tr}(e^{-t\Delta}) - N)\, dt.$$

- In other words, $\zeta(s)$ is $\frac{1}{\Gamma(s)}$-times the *Mellin transform* of $\mathrm{tr}(e^{-t\Delta})$. Notice that the above integral does not makes sense when $\Re(s) \leq 1$ since it blows up near $t = 0$. Nevertheless, if we understand the behavior of $\mathrm{tr}(e^{-t\Delta})$ near 0 and ∞, we can try to to meromorphically extend the above function to the whole complex plane.
- We can interpret $\eta(s)$ as measure of loops where loops are weighted by length to s power. If s is large, this penalizes small loops enough to make measure finite.

Loop mass computations: proof overview

- **McKean and Singer (1967):** short time expansion of heat kernel trace:

$$\mathrm{tr}(e^{-t\Delta}) = \frac{\mathrm{Vol}_g(M)}{4\pi t} - \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi t}} + \frac{\chi(M)}{6} + O(t^{1/2}) \quad \text{as } t \to 0^+.$$

Loop mass computations: proof overview

- **McKean and Singer (1967):** short time expansion of heat kernel trace:

$$\mathrm{tr}(e^{-t\Delta}) = \frac{\mathrm{Vol}_g(M)}{4\pi t} - \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi t}} + \frac{\chi(M)}{6} + O(t^{1/2}) \quad \text{as } t \to 0^+.$$

- Using this estimate, we have for $\Re(s) > 1$ that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_{\delta/2}^{\infty} t^{s-1}\, \mathrm{tr}(e^{-t\Delta}) dt$$

$$+ \frac{1}{\Gamma(s)} \int_0^{\delta/2} t^{s-1} \left(\mathrm{tr}(e^{-t\Delta}) - \frac{\mathrm{Vol}_g(M)}{4\pi t} + \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi t}} - \frac{\chi(M)}{6} \right) dt$$

$$+ \frac{1}{\Gamma(s)} \left(\frac{\mathrm{Vol}_g(M)}{4\pi(s-1)} \left(\frac{\delta}{2}\right)^{s-1} - \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi}(s-\frac{1}{2})} \left(\frac{\delta}{2}\right)^{s-\frac{1}{2}} \right) + \frac{1}{\Gamma(s+1)} \left(\frac{\delta}{2}\right)^{s} \frac{\chi(M)}{6}.$$

Loop mass computations: proof overview

- **McKean and Singer (1967):** short time expansion of heat kernel trace:

$$\mathrm{tr}(e^{-t\Delta}) = \frac{\mathrm{Vol}_g(M)}{4\pi t} - \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi t}} + \frac{\chi(M)}{6} + O(t^{1/2}) \quad \text{as } t \to 0^+.$$

- Using this estimate, we have for $\Re(s) > 1$ that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_{\delta/2}^{\infty} t^{s-1}\, \mathrm{tr}(e^{-t\Delta}) dt$$

$$+\frac{1}{\Gamma(s)} \int_0^{\delta/2} t^{s-1} \left(\mathrm{tr}(e^{-t\Delta}) - \frac{\mathrm{Vol}_g(M)}{4\pi t} + \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi t}} - \frac{\chi(M)}{6} \right) dt$$

$$+\frac{1}{\Gamma(s)} \left(\frac{\mathrm{Vol}_g(M)}{4\pi(s-1)} \left(\frac{\delta}{2}\right)^{s-1} - \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi}(s-\frac{1}{2})} \left(\frac{\delta}{2}\right)^{s-\frac{1}{2}} \right) + \frac{1}{\Gamma(s+1)} \left(\frac{\delta}{2}\right)^{s} \frac{\chi(M)}{6}.$$

- Here we have used the identity $s\Gamma(s) = \Gamma(s+1)$. In the above form, it is clear that $\zeta(s)$ extends holomorphically to a neighborhood of $s = 0$.

Loop mass computations: proof overview

- Differentiate in s at $s=0$. Since $\lim_{s\to 0} s\Gamma(s) = 1$ and $\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s+1)} = -\gamma$, we have $\zeta'(0) =$

$$\int_{\delta/2}^{\infty} t^{-1}\,\mathrm{tr}(e^{-t\delta})\ dt + \int_0^{\delta/2} O(t^{-1/2})dt - \left(\frac{\delta}{2}\right)^{-1} \frac{\mathrm{Vol}_g(M)}{4\pi}$$

$$+ 2\left(\frac{\delta}{2}\right)^{-1/2} \frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi}} + (\log\delta - \log 2)\frac{\chi(M)}{6} - \gamma\frac{\chi(M)}{6}.$$

Loop mass computations: proof overview

- Differentiate in s at $s=0$. Since $\lim_{s\to 0} s\Gamma(s) = 1$ and $\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s+1)} = -\gamma$, we have $\zeta'(0) =$

$$\int_{\delta/2}^{\infty} t^{-1}\,\mathrm{tr}(e^{-t\delta})\ dt + \int_0^{\delta/2} O(t^{-1/2})dt - \left(\frac{\delta}{2}\right)^{-1}\frac{\mathrm{Vol}_g(M)}{4\pi}$$
$$+\,2\left(\frac{\delta}{2}\right)^{-1/2}\frac{\mathrm{Len}_g(\partial M)}{8\sqrt{\pi}} + (\log\delta - \log 2)\frac{\chi(M)}{6} - \gamma\frac{\chi(M)}{6}.$$

- Using the fact that $\zeta'(0) = -\log\det\Delta_{M,g}$ and $\int_{\delta/2}^{\infty} t^{-1}\,\mathrm{tr}(e^{-t\Delta})dt = \int_{\delta}^{\infty} u^{-1}\,\mathrm{tr}(e^{-u\Delta/2})du = \mu_{M,g}^{\mathrm{loop}}(\mathcal{L}(M,g,\delta))$ (since the generator of Brownian motion is $\frac{1}{2}\Delta$), we are done.

Loop mass computations

- The Polyakov-Alvarez conformal anomaly formula describes how the determinant of the Laplacian changes when one perturbs the conformal factor of the metric.

Loop mass computations

- The Polyakov-Alvarez conformal anomaly formula describes how the determinant of the Laplacian changes when one perturbs the conformal factor of the metric.
- **Proposition:** Let $D \subset \mathbf{C}$ be a simply connected domain, and consider two conformally equivalent metrics g, g_0 on it with $g = e^{2\sigma} g_0$ for some smooth function σ. Then writing K_0 and k_0 for the Gauss and geodesic curvatures with respect to g_0, we have

$$\log \det \Delta_g = -\frac{1}{12\pi} \int_D |\nabla_{g_0} \sigma|^2 \ d\mathrm{Vol}_{g_0} + \frac{1}{6\pi} \int_D K_0 \sigma \ d\mathrm{Vol}_{g_0} + \frac{1}{6\pi} \int_{\partial D} k_0 \sigma \ d\mathrm{Len}_{g_0} - \frac{1}{4\pi} \int_{\partial D} \partial_n \sigma \ d\mathrm{Len}_{g_0} + \log \det \Delta_{g_0}.$$

Loop mass computations

- The Polyakov-Alvarez conformal anomaly formula describes how the determinant of the Laplacian changes when one perturbs the conformal factor of the metric.
- **Proposition:** Let $D \subset \mathbf{C}$ be a simply connected domain, and consider two conformally equivalent metrics g, g_0 on it with $g = e^{2\sigma} g_0$ for some smooth function σ. Then writing K_0 and k_0 for the Gauss and geodesic curvatures with respect to g_0, we have

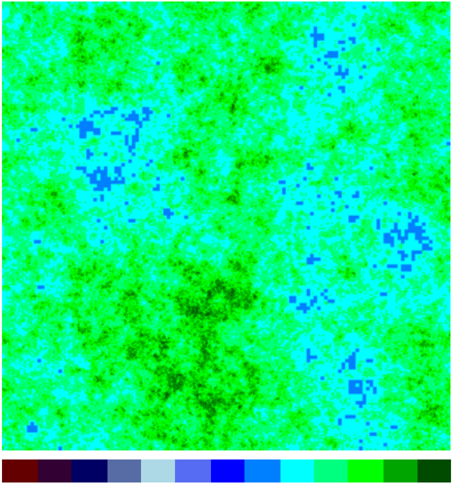
$$\begin{aligned}\log \det \Delta_g = &-\frac{1}{12\pi}\int_D |\nabla_{g_0}\sigma|^2 \; d\mathrm{Vol}_{g_0} + \frac{1}{6\pi}\int_D K_0\sigma \; d\mathrm{Vol}_{g_0} \\ &+ \frac{1}{6\pi}\int_{\partial D} k_0\sigma \; d\mathrm{Len}_{g_0} - \frac{1}{4\pi}\int_{\partial D} \partial_n\sigma \; d\mathrm{Len}_{g_0} + \log\det \Delta_{g_0}.\end{aligned}$$

- Similar statement for Polyakov-Alvarez without boundary.

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0, 2)$

$\gamma = 0.5$

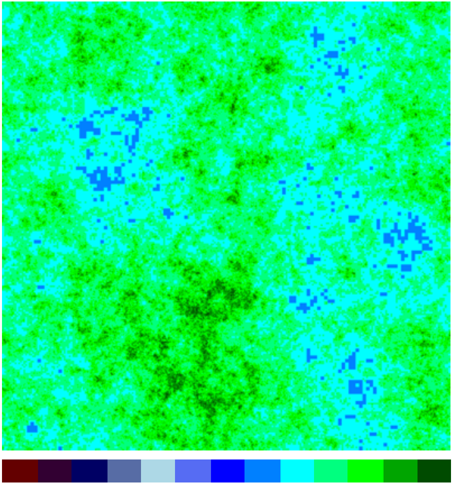


(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0, 2)$
- Random surface model: Polyakov, 1980. Motivated by string theory.
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0, \sqrt{2})$. Kahane, 1985, $\gamma \in [0, 2)$.

$\gamma = 0.5$

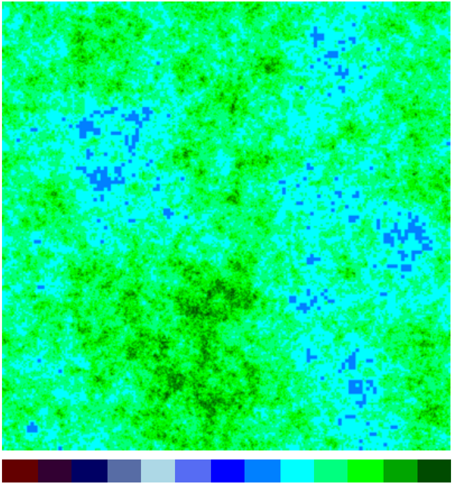


(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0, 2)$
- Random surface model: Polyakov, 1980. Motivated by string theory.
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0, \sqrt{2})$. Kahane, 1985, $\gamma \in [0, 2)$.
- Does not make literal sense since h takes values in the space of distributions.

$\gamma = 0.5$

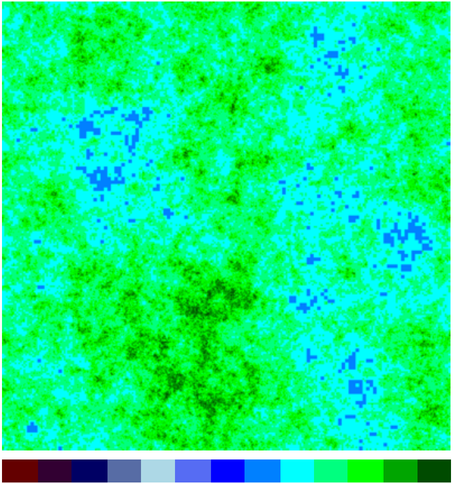


(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0, 2)$
- Random surface model: Polyakov, 1980. Motivated by string theory.
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0, \sqrt{2})$. Kahane, 1985, $\gamma \in [0, 2)$.
- Does not make literal sense since h takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.

$\gamma = 0.5$

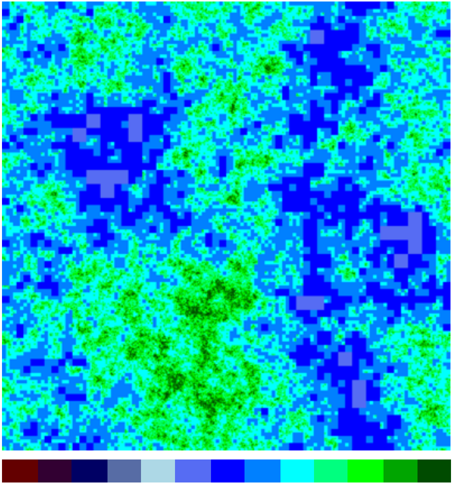


(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0,2)$
- Random surface model: Polyakov, 1980. Motivated by string theory.
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0,\sqrt{2})$. Kahane, 1985, $\gamma \in [0,2)$.
- Does not make literal sense since h takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.
- Areas of regions and lengths of curves are well defined.

$\gamma = 1.0$

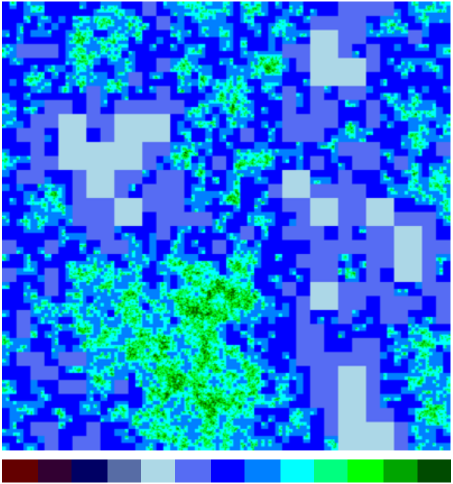


(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)}dz$ where h is a GFF and $\gamma \in [0,2)$
- Random surface model: Polyakov, 1980. Motivated by string theory.
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0,\sqrt{2})$. Kahane, 1985, $\gamma \in [0,2)$.
- Does not make literal sense since h takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.
- Areas of regions and lengths of curves are well defined.

$\gamma = 1.5$

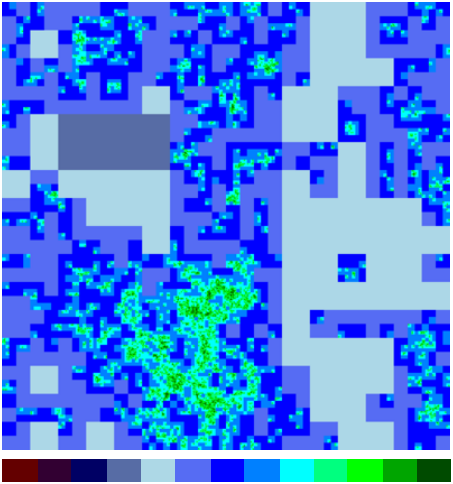


(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)}dz$ where h is a GFF and $\gamma \in [0,2)$
- Random surface model: Polyakov, 1980. Motivated by string theory.
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0,\sqrt{2})$. Kahane, 1985, $\gamma \in [0,2)$.
- Does not make literal sense since h takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.
- Areas of regions and lengths of curves are well defined.

$\gamma = 2.0$



(Number of subdivisions)

Smoothed planar maps

- One can do square subdivions but observe mean of GFF on each box before deciding whether to divide it.

Smoothed planar maps

- One can do square subdivions but observe mean of GFF on each box before deciding whether to divide it.
- Then *given* what one has learned after subdividing, the *conditional law* has a particular form.

Smoothed planar maps

- One can do square subdivions but observe mean of GFF on each box before deciding whether to divide it.
- Then *given* what one has learned after subdividing, the *conditional law* has a particular form.
- Weighting by Dirichlet energy changes variance with same stopping time rule, corresponds precisely to changing c in the way that we expect.

Smoothed planar maps

- One can do square subdivions but observe mean of GFF on each box before deciding whether to divide it.
- Then *given* what one has learned after subdividing, the *conditional law* has a particular form.
- Weighting by Dirichlet energy changes variance with same stopping time rule, corresponds precisely to changing c in the way that we expect.
- Allows us to construct some credible approximations to LQG for which the loop soup weighting (in small cutoff limit) *exactly* corresponds to changing c in the way we expect.

Smoothed planar maps

- One can do square subdivions but observe mean of GFF on each box before deciding whether to divide it.
- Then *given* what one has learned after subdividing, the *conditional law* has a particular form.
- Weighting by Dirichlet energy changes variance with same stopping time rule, corresponds precisely to changing c in the way that we expect.
- Allows us to construct some credible approximations to LQG for which the loop soup weighting (in small cutoff limit) *exactly* corresponds to changing c in the way we expect.
- Unlike ordinary loop-soup-weighted planar maps, the "smoothed planar maps" are equally well understood for each c.

Mixing different κ values and/or mismatched γ

- Start with a quantum wedge or another random surface, possibly decorated by curves.

Mixing different κ values and/or mismatched γ

- Start with a quantum wedge or another random surface, possibly decorated by curves.
- One can weight loop soup measure, but one has choice of whether to counting loops hitting curves or not hitting curves.

Mixing different κ values and/or mismatched γ

- Start with a quantum wedge or another random surface, possibly decorated by curves.
- One can weight loop soup measure, but one has choice of whether to counting loops hitting curves or not hitting curves.
- This suggests a whole zoo of variants of the quantum zipper, involving loops and a wild mixture of κ values and γ values.

Note

- We considered three quantities associated to a compact manifold
  1. A certain term in the truncated loop soup measure expansion
  2. The zeta function Laplacian determinant
  3. The Dirichlet energy of h

Note

- We considered three quantities associated to a compact manifold
  1. A certain term in the truncated loop soup measure expansion
  2. The zeta function Laplacian determinant
  3. The Dirichlet energy of h
- We can then say that
  A. McKean/Singer/Osgood/Philipps/Sarnak plus work connects 1 and 2.
  B. Polyakov-Alvarez connects 2 and 3.
  C. Another approach connects 1 and 3 directly.

Happy birthday Chris!