Quadratic forms and cohomological operations

Alexander Vishik

SBU, 6 November 2014

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Moreover, all quadratic forms of a given dimension n are "forms" of a fixed one (say, a sum of squares $x_1^2 + \ldots + x_n^2$), and are described by the orthogonal group $O(n)$:

$$\text{The set of isom. classes of } n-\text{dim. forms} \quad = H^1_{et}(k, O(n)),$$

where the latter is the 1-st cohomology of the absolute Galois group G of k with coefficients in $O(n;\overline{k})$.

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4) $u(\mathbb{Q}_p) = 4$.

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Now we know that the world of quadratic forms is much more complex. But in 1953 even the class of the "best possible" forms was not discovered yet. These are "Pfister forms" introduced in the middle of 60-s.

Pfister forms

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An n-fold Pfister form is a 2^n-dimensional form

$$\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle\langle a_1 \rangle\rangle \otimes \ldots \otimes \langle\langle a_n \rangle\rangle,$$

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For $n = 1, 2, 3$ these are norms/reduced norms in the quadratic extension $k(\sqrt{a})$,
the (generalized) quaternion algebra

$$Quat(\{a, b\}; k) = k < u, v > /(u^2 = a, v^2 = b, uv = -vu),$$

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More explicitly, these forms are given by: $\langle 1, -a \rangle$, $\langle 1, -a, -b, ab \rangle$, and $\langle 1, -a, -b, -c, ab, bc, ca, -abc \rangle$.

In particular, on Pfister forms of foldness $0, 1, 2, 3$ we have a multiplicative structure: a bilinear map

$$V_q \times V_q \xrightarrow{\mu} V_q, \quad \text{such that} \quad q(\mu(u, v)) = q(u) \cdot q(v).$$

And no other forms have this property.

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This is the main property of Pfister forms, and (modulo scalar factors) no other forms posses it.

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$(\Rightarrow)$ Evident.

$(\Leftarrow)$ π_L-isotropic $\Rightarrow$ π_L-hyperbolic, that is, it has a totally isotropic subspace of dimension $= \frac{\dim(\pi)}{2}$. But such a subspace will intersect $V_p \subset V_\pi$. $\square$

The sum of squares $x_1^2 + \ldots + x_m^2$ is always a Pfister neighbor of some $x_1^2 + \ldots + x_{2^r}^2 = \langle\langle -1, \ldots, -1 \rangle\rangle$, and these two forms will be isotropic simultaneously.

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So, if the sum of 2^{l+1} squares represents zero non-trivially, then so does the sum of $2^l + 1$ of them. This proves the first part of the Conjecture of Kaplansky - due to Pfister ('60-s).

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$$q = (\underbrace{\mathbb{H} \perp \ldots \perp \mathbb{H}}_{i_W(q)}) \perp q_{an}, \qquad \mathbb{H} = x^2 - y^2$$

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r – fold Pfister : $SP(q) = (0, 2^{r-1})$;

"generic" of dimension n : $SP(q) = (0, 1, 2, \ldots, [n/2])$.

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Still, some important information is not detected by $SP(q)$.

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$q \ \longrightarrow \ Q = G(Q;0), G(Q;1), \ldots, G(Q,d)$

- the homogeneous varieties (quadratic Grassmannians) associated to our quadratic form. Here $d = [\frac{\dim(Q)}{2}]$.

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This invariant of geometric nature contains most of known discrete invariants.

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Elementary classes

For each i, the ring $\mathrm{CH}^*(G(Q;i)_{\overline{k}})$ is generated by the so-called "elementary classes" $y_{i,j}$, $j = 0, \ldots, d$ and the Chern classes of the tautological vector bundle. Here the Chern classes do not represent any interest for us, as they are always defined over k. In contrast, the rationality of "elementary classes" carries a very intersting information about q.

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And for an arbitrary Grassmannian these classes are obtained from those on Q. Namely, we have natural forgetful maps

$$G(Q;0) \xleftarrow{\alpha} F(Q;0,i) \xrightarrow{\beta} G(Q;i)$$

from the flag variety of pairs $(\pi_0 \subset \pi_i)$ to the quadric and the Grassmannian. Then $y_{i,j} := \beta_* \alpha^*(l_j)$. In other words, the class $y_{i,j}$ is given by the locus of those i-dimensional planes on $Q_{\overline{k}}$ which intersect a given j-dimensional plane.

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The $EDI(Q)$ measures which classes $y_{i,j}$ are defined over k. It can be visualized as a $d \times d$-square, where integral nodes are marked if the respective elementary classes are k-rational.

$$
\begin{array}{ccccccccc}
 & \circ & \circ & \circ & \circ & \circ & \circ & & G(Q;d) \\
 & \circ & \circ & \circ & \circ & \circ & \circ & & . \\
i & \circ & \circ & \bullet & \circ & \circ & \circ & & . \\
 & \circ & \circ & \circ & \circ & \circ & \circ & & . \\
 & \circ & \circ & \circ & \circ & \circ & \circ & & G(Q;1) \\
\uparrow & \circ & \circ & \circ & \circ & \circ & \circ & & G(Q;0) = Q \\
 & \rightarrow & & j & & & & &
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i & \circ & \circ & \bullet & \circ & \circ & \circ & . \\
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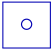

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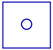

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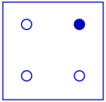

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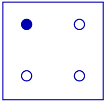

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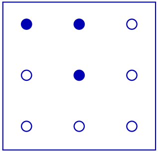

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$$\boxed{\begin{array}{ccc} \circ & \bullet & \bullet \\ \circ & \circ & \bullet \\ \circ & \circ & \circ \end{array}}$$

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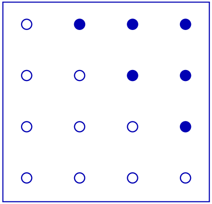

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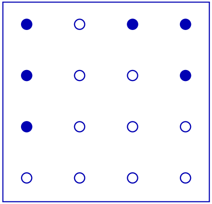

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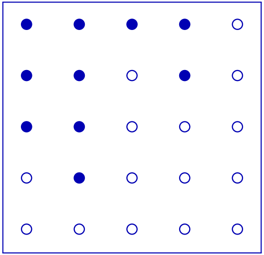

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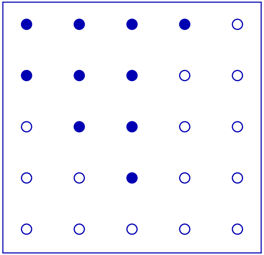

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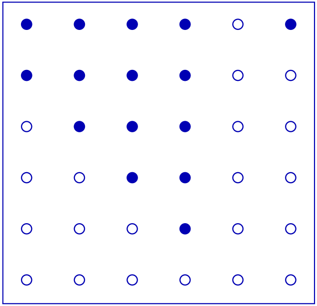

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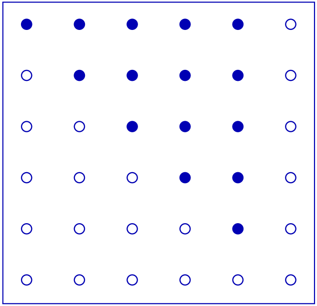
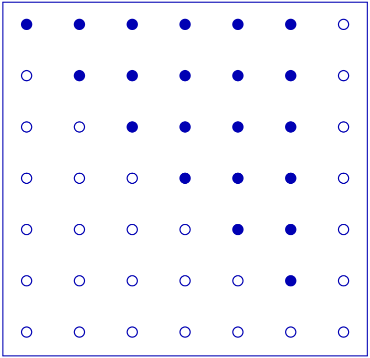
$n = 13$



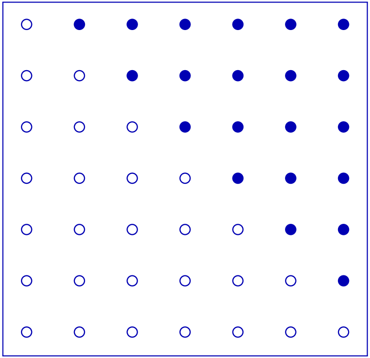
Examples: $k = \mathbb{R}$, $q = x_1^2 + \ldots + x_n^2$

$n = 14$



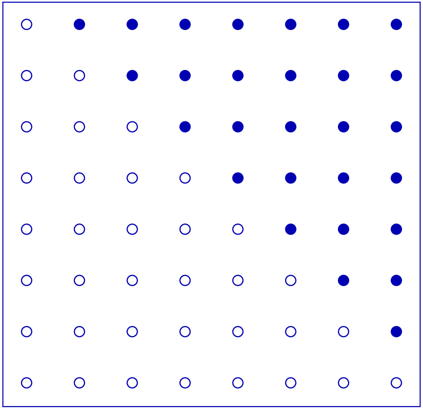
Examples: $k = \mathbb{R}$, $q = x_1^2 + \ldots + x_n^2$

$n = 15$



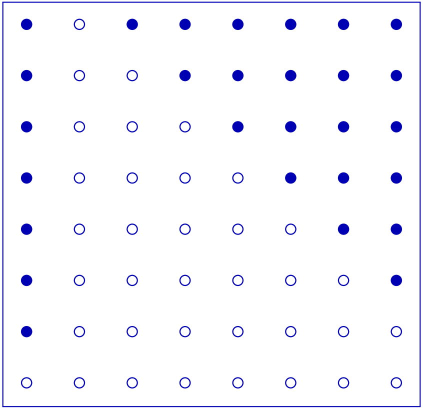
Examples: $k = \mathbb{R}$, $q = x_1^2 + \ldots + x_n^2$

$n = 16$



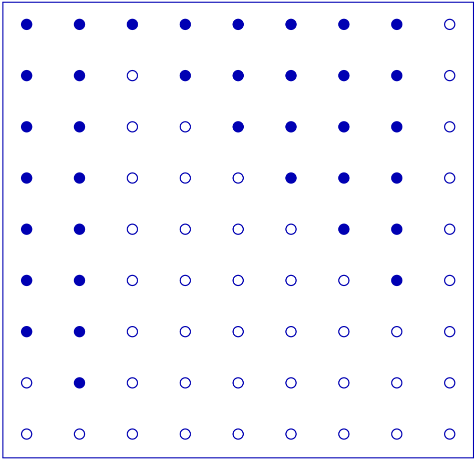
Examples: $k = \mathbb{R}$, $q = x_1^2 + \ldots + x_n^2$

$n = 17$



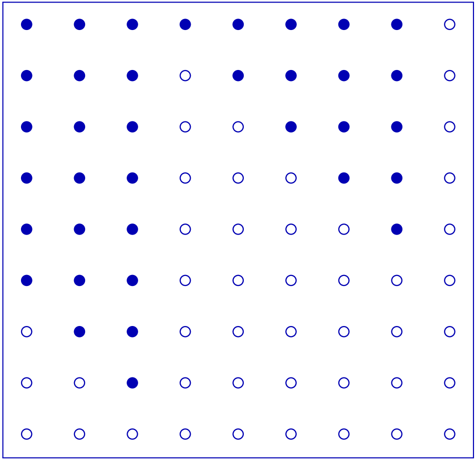
Examples: $k = \mathbb{R}$, $q = x_1^2 + \ldots + x_n^2$

$n = 18$



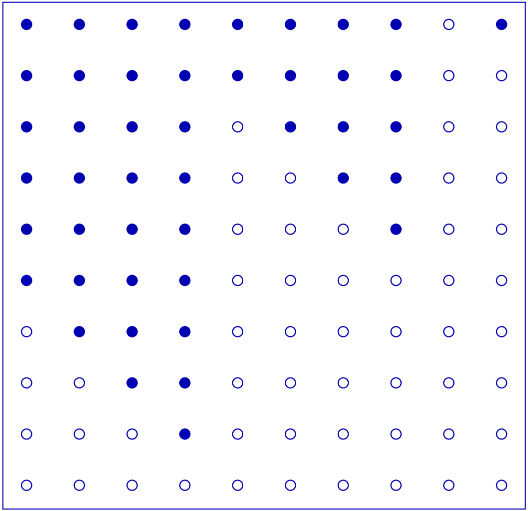
Examples: $k = \mathbb{R}$, $q = x_1^2 + \ldots + x_n^2$

$n = 19$



Examples: $k = \mathbb{R}$, $q = x_1^2 + \ldots + x_n^2$

$n = 20$



The Elementary Discrete Invariant can be applied to the u-invariant of fields.

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The second part of the Conjecture of Kaplansky was disproven by Merkurjev (1986) who showed that $\forall n = 2m$ there exists a field with such u-invariant.

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Using our methods we can show:

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For any $r \geqslant 3$ there exists a field of u-invariant $2^r + 1$.

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The case $u = 9$ was done by Izhboldin (1999).
It is easy to see that $u \neq 3, 5, 7$.
Nothing else is known.

Aside from the Elementary Discrete Invariant, another major ingredient of the proof of Theorem A is the following result concerning the ”field of definition” of the cohomology element.

Aside from the Elementary Discrete Invariant, another major ingredient of the proof of Theorem A is the following result concerning the ”field of definition” of the cohomology element.

Theorem B

Let Y be a smooth variety, $\bar{y} \in \mathrm{CH}^m(Y_{\bar{k}})/2$, and P be a smooth quadric of $\dim(P) > 2m$. Then

$$\bar{y} \text{ is defined over } k \quad \Leftrightarrow \quad \bar{y} \text{ is defined over } k(P).$$

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$$\overline{y} \text{ is defined over } k \;\Leftrightarrow\; \overline{y} \text{ is defined over } k(P).$$

Thus, studying the rationality of $\overline{y}$ we can assume that all sufficiently large quadratic forms are isotropic. This helps to compute $EDI(Q)$.

The proof of Theorem B is based on the use of ”Symmetric Operations” in the Algebraic Cobordism Ω^* of Levine-Morel (by the way, these operations were discovered in the study of $GDI(Q)$).

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Ω^* – is an algebro-geometric analogue of MU^*.

- generators: classes $[V \xrightarrow{v} X]$ of projective maps from smooth varieties + some relations
- $\Omega^*(\mathrm{Spec}(k)) = MU^*(pt) = \mathbb{L} \cong \mathbb{Z}[x_1, x_2, \ldots], \deg(x_i) = i$

- the Lazard ring (the coefficient ring of the universal FGL).
- Chow groups (as well as K_0) can be reconstructed out of Ω^* by the change of coefficients:

$$\mathrm{CH}^*(X) = \Omega^*(X)_{\otimes_{\mathbb{L}}}\mathbb{Z}, \quad \text{where} \quad x_i \mapsto 0.$$

Idea of the proof of Theorem B:

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$$\begin{tikzcd} \bar{y} \\ \mathrm{CH}^*(Y_{k(P)}) \\ \mathrm{CH}^*(Y) \arrow[u, two heads] \end{tikzcd}$$

Idea of the proof of Theorem B:

$$\begin{tikzcd}
\bar{y} & \\
\mathrm{CH}^*(Y_{k(P)}) & \mathrm{CH}^*(Y \times P) \arrow[l, two heads] \\
\mathrm{CH}^*(Y) \arrow[u] &
\end{tikzcd}$$

Idea of the proof of Theorem B:

$$\bar{y}$$

$$\begin{tikzcd}
\mathrm{CH}^*(Y_{k(P)}) & \mathrm{CH}^*(Y \times P) \arrow[l, two heads] & \Omega^*(Y \times P) \arrow[l, two heads] \arrow[d, "\pi_*"] \\
\mathrm{CH}^*(Y) \arrow[u] & & \Omega^*(Y) \arrow[ll, two heads]
\end{tikzcd}$$

Idea of the proof of Theorem B:

$$
\begin{tikzcd}
\bar{y} \arrow[r, dotted, "\mathit{lift}"] & {} \arrow[r, dotted, "\mathit{lift}"] & {} \\
\mathrm{CH}^*(Y_{k(P)}) & \mathrm{CH}^*(Y \times P) \arrow[l, two heads] & \Omega^*(Y \times P) \arrow[l, two heads] \arrow[d, "\pi_*"] \\
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\end{tikzcd}
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\begin{tikzcd}
\bar{y} \arrow[r, dotted, "\mathit{lift}"] & {} \arrow[r, dotted, "\mathit{lift}"] & {} \\
\mathrm{CH}^*(Y_{k(P)}) & \mathrm{CH}^*(Y \times P) \arrow[l, two heads] & \Omega^*(Y \times P) \arrow[l, two heads] \arrow[loop above, "\cdot h^r"] \arrow[d, "\pi_*"] \\
\mathrm{CH}^*(Y) \arrow[u] & & \Omega^*(Y) \arrow[ll, two heads]
\end{tikzcd}
$$

Idea of the proof of Theorem B:

$$\begin{tikzcd}
\overline{y} \arrow[r, dotted, "lift"] & {} \arrow[r, dotted, "lift"] & {} \\
\mathrm{CH}^*(Y_{k(P)}) & \mathrm{CH}^*(Y \times P) \arrow[l, two heads] & \Omega^*(Y \times P) \arrow[l, two heads] \arrow[loop above, "\cdot h^r"] \arrow[loop right, "coh.op{-}s"] \arrow[d, "\pi_*"] \\
\mathrm{CH}^*(Y) \arrow[u] & & \Omega^*(Y) \arrow[ll, two heads] \arrow[loop below, "coh.op{-}s"]
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\mathrm{CH}^*(Y) \arrow[u] & & \Omega^*(Y) \arrow[ll, two heads] \arrow[loop below, "coh.op{-}s"]
\end{tikzcd}
$$

If $\dim(P)$ is sufficiently large w.r. to the $\operatorname{codim}(\bar{y})$, we can choose appropriate coefficients so that the result will not depend on any choices we made, and will give us the original element, but now defined over k instead of $k(P)$!

Thank you!