Real normalized differentials and geometry of the moduli spaces of Riemann surfaces with points

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Vanishing properties of $\mathcal{M}_{g,k}$

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- Diaz' theorem (1986):
  There does not exist a complete (complex) cycle in $\mathcal{M}_g$ of dimension greater than $g-2$

Note, that is the upper bound. The know constructions give complete cycles of dimension of order $\log_3 g$, only.

- Looijenga theorem (1995):
  The tautological ring $R^*(\mathcal{M}_{g,k})$ vanishes in dimensions greater then $g-2+k$

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$$\psi_i = c_1(L_i), \quad \kappa_i = p_*(\psi_1^{i+1}) \in H^*(\mathcal{M}_g).$$

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Faber's conjecture

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Conjectural geometric explanations

Widely accepted by experts "geometric explanation" of vanishing properties of $\mathcal{M}_{g,k}$ is the existence of its stratification by certain number of affine strata or the existence of a cover of $\mathcal{M}_{g,k}$ by certain number of open affine sets.

Historically, Arbarello first realized that a stratification of $\mathcal{M}_g$ could be useful for a study of its geometrical properties. He studied the stratification (known already for Rauch)

$$\mathcal{W}_2 \subset \mathcal{W}_3 \subset \cdots \subset \mathcal{W}_{g-1} \subset \mathcal{W}_g = \mathcal{M}_g,$$

where $\mathcal{W}_n$ if the locus of curves having a Weierstrass point of order at most n, and then conjectured that $\mathcal{W}_n \setminus \mathcal{W}_{n-1}$ is affine.

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Alternative geometric explanation

Recently, the author jointly with S. Grushevsky proposed an alternative approach for geometrical explanation of the vanishing properties of $\mathcal{M}_{g,k}$ motivated by certain constructions of the Whitham perturbation theory of integrable systems. The key elements of the alternative geometrical explanation are:

- the moduli space $\mathcal{M}_{g,k}^{(n)}$, $n = (n_1, \ldots, n_k)$ of smooth genus g Riemann surfaces with the fixed n_α-jets of local coordinates in the neighborhoods of labeled points is the total space of a *real-analytic* foliation, whose leaves $\mathcal{L}$ are locally smooth *complex subvarieties* of real codimension $2g$;
- on $\mathcal{M}_{g,k}^{(n)}$ there is an ordered set of $(\dim_{\mathbb{R}} \mathcal{L})$ continuous functions, which restricted onto the leaves of the foliation are piecewise harmonic. Moreover, the first of these function restricted onto $\mathcal{L}$ is a subharmonic function.

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Results and conjectures

- Proof of Arbarello's conjecture

Theorem

Any compact complex cycle in $\mathcal{M}_g$ of dimension $g-n$ must intersect $\mathcal{W}_n$.

- New upper bound for dimensions of complete (complex) cycles in the moduli space $\mathcal{M}_g^{ct}$ of stable curves of compact type.

Conjecture

There do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ having non empty intersection with $\mathcal{M}_g$ of dimension greater than $g-1$.
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Previously known bounds

- Diaz:
  there is no compact cycle in $\mathcal{M}_g^{ct}$ of dimension greater that $2g - 3$.
- Keel and Sadun:
  for $g \geq 3$ there do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ of dimension greater than $2g - 4$.

The proof is by easy induction arguments starting from the base $g = 3$. The proof of the base statement is a corollary of remarkable vanishing result:

- *there do not exist a complete complex subvarieties of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of codimension g.*

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Real normalized differentials

The foliation structure arises through identification of $\mathcal{M}_{g,k}^{(n)}$ with the moduli space of curves with fixed *real-normalized* meromorphic differential. By definition a real normalized meromorphic differential is a differential whose periods over any cycle on the curve are real. The power of this notion is that:

Lemma

For any fixed singular parts of poles with pure imaginary residues, there exists a unique meromorphic differential Ψ, having prescribed singular part at p_α and such that all its periods on Γ are real, i.e.

$$\Im\left(\oint_c \Psi\right) = 0, \ \ \forall\, c \in H^1(\Gamma, \mathbb{Z}).$$

Foliation

Definition

A leaf $\mathcal{L}$ of the foliation on $\mathcal{M}_{g,k}^{(n)}$ defined to be the locus along which the periods of the corresponding differentials remain (covariantly) constant.

The leaves $\mathcal{L}$ of the foliation can be regarded as a generalization of the Hurwitz spaces of $\mathbb{P}^1$ covers.

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Coordinates along a leaf

A set of holomorphic coordinates on $\mathcal{M}_{g,k}^{(n)}$ are "critical" values of the corresponding abelian integral $F(p) = c + \int^p \Psi$, $p \in \Gamma$:

At the generic point, where zeros q_s of Ψ are distinct, the coordinates on $\mathcal{L}$ are the evaluation of F at these critical points:

$$\varphi_s = F(q_s), \quad \Psi(q_s) = 0, \quad s = 0, \ldots, d = \dim \mathcal{L}, \tag{1}$$

normalized by the condition $\sum_s \varphi_s = 0$.

A direct corollary of the real normalization is the statement that:

- *imaginary parts $f_s = \Im \varphi_s$ of the critical values depend only on labeling of the critical points*

They can be arranged into decreasing order

$$f_0 \geq f_1 \geq \cdots \geq f_{d-1} \geq f_d.$$

After that f_j can be seen as a well-defined continuous function on $\mathcal{M}_{g,k}^{(n)}$, which restricted onto $\mathcal{L}$ is a piecewise harmonic function. Moreover, f_0 restricted onto $\mathcal{L}$ is a *subharmonic function*, i.e, f_0 *has no local maximum on unless it is constant*.

Diaz' theorem revisited

Let X be a complete cycle in $\mathcal{M}_g$ and Z be its preimage under the forgetfull map: $\mathcal{M}_{g,2} \subset \mathcal{C}_g^2 \longmapsto \mathcal{M}_g$.

$\rightarrow$ On Z the function f_0 (defined by critical values of real-normalized differential with two simple poles) must achieve its maximum at some point.

$\rightarrow$ At this point the function f_0 achieves its maximum on $Z \cap \mathcal{L}$.

$\rightarrow$ Hence, it is a constant on $Z \cap \mathcal{L}$.

$\rightarrow$ If f_0 is a constant then (inductively) all the other functions f_j are constants.

$\rightarrow$ Then, $Z \cap \mathcal{L}$ is at most zero-dimensional, i.e. Z intersects $\mathcal{L}$ transversally.

$\rightarrow$ $\dim X \leq g - 2$

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- Additional difficulty: the space of singular parts of real-normalized differentials is non-compact.
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Cusps of plane curves.

- Classical problem: What is the maximum number $s(d)$ of cusps on degree d plane curve ?

Plane curves of degree d are defined by the equation

$$\sum_{i+j\leq d} \alpha_{ij} w^i z^j = 0$$

- Expected answer: $s(d)_{exp} = d(d+1)/4$

Hirano and Kulikov constructed a families of curves with large number of cusps that give

$$\sup \lim_{d\to\infty} \frac{s(d)}{d^2} \geq \frac{9}{32},$$

$$\sup \lim_{d\to\infty} \frac{s(d)}{d^2} \geq \frac{283}{960} \simeq 0.2948$$

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$$s(d) \leq \frac{5}{16}d^2 - \frac{3}{8}d \simeq 0.3125d^2 + O(d)$$

In 2004 Lander using generalization of Bogomolov-Miyaoka-Yau inequality proved

$$s(d) \leq \frac{125+\sqrt{73}}{432}d^2 \simeq 0.309d^2$$

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- *What is the maximal number of common zeros of two real normalized differentials having fixed orders of poles?*

Conjecture (Theorem ? (Grushevsky-Kr))

Two real normalized meromorphic differentials with $d > 1$ poles of order 2 on a smooth genus g algebraic curve can not have more that $\frac{3}{2}(g + d - 1)$ common zeros.

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