The Geometry and Arithmetic of Sphere Packings

Alex Kontorovich

Rutgers

Apollonian Circle Packings

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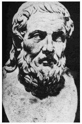

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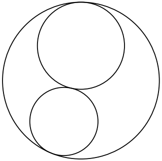

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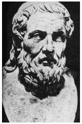

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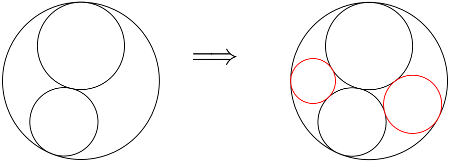

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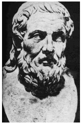

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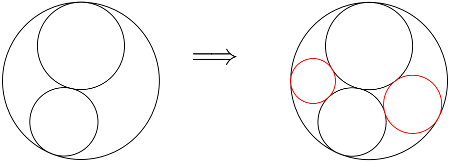

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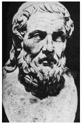

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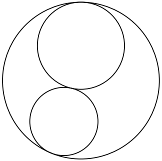

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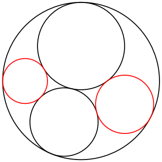

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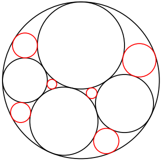

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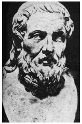

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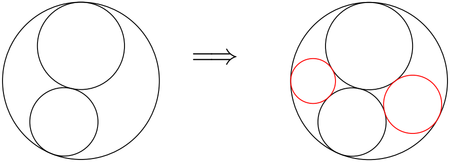

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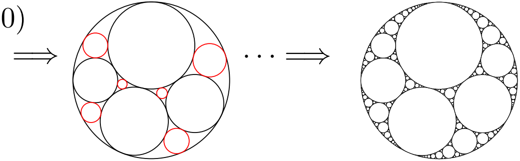

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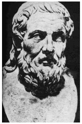

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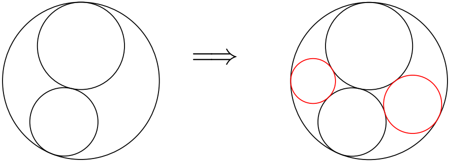

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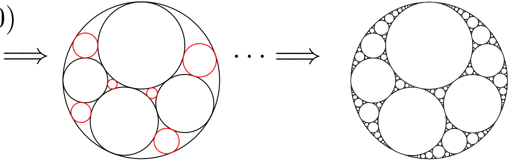

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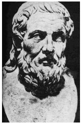

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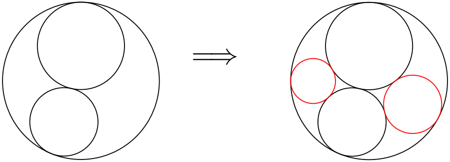

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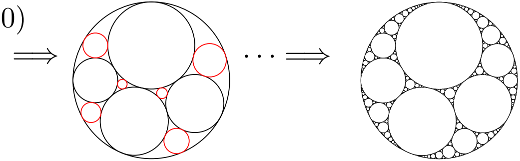

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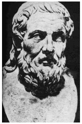

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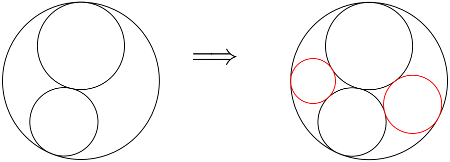

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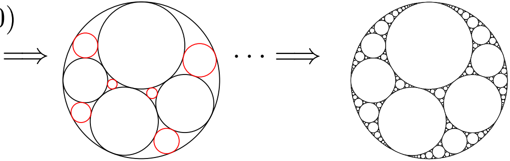

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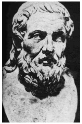

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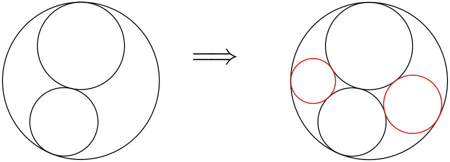

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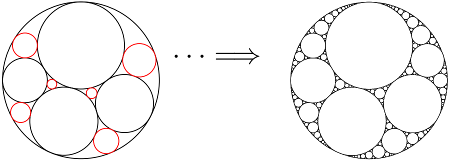

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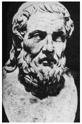

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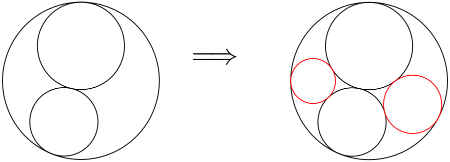

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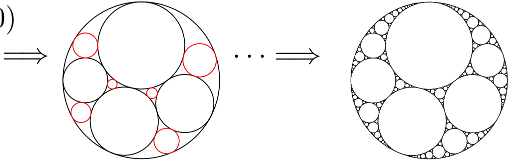

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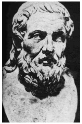

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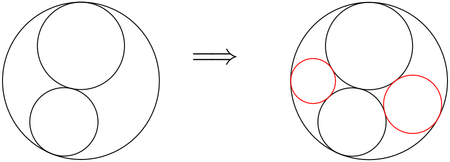

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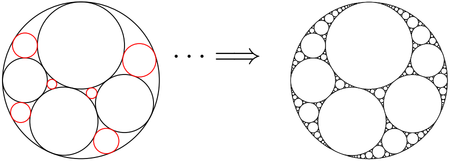

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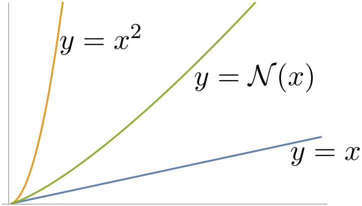

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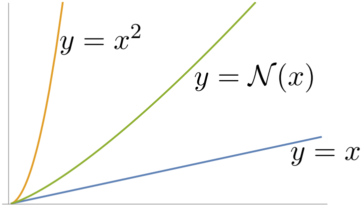

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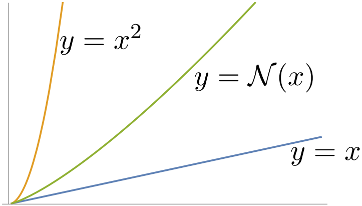

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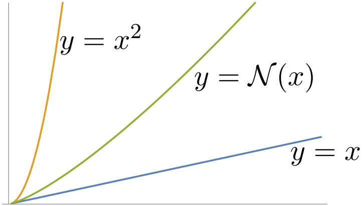

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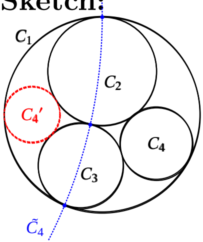

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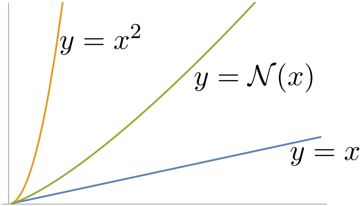

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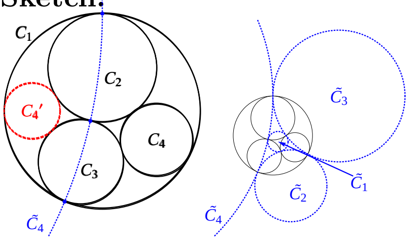

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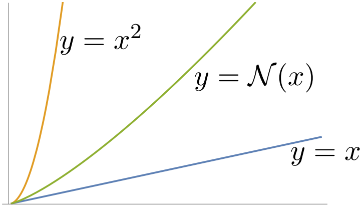

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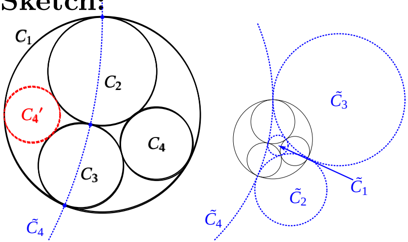

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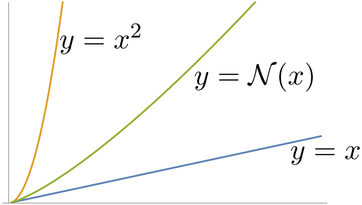

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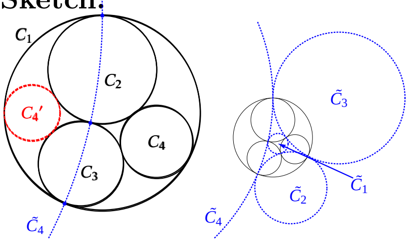

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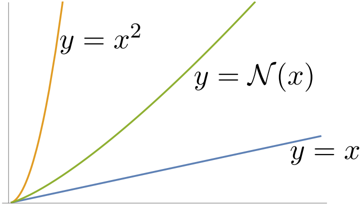

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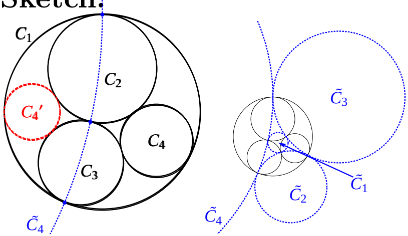

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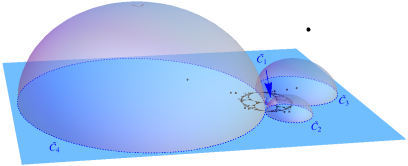

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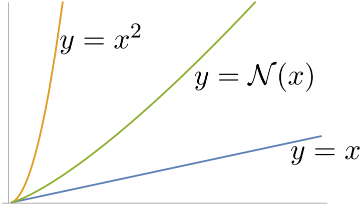

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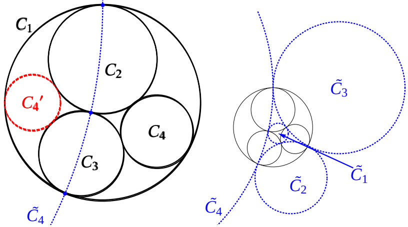

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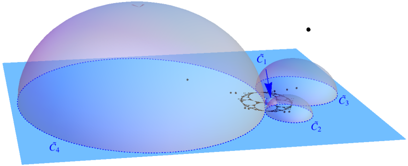

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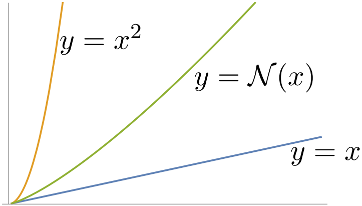

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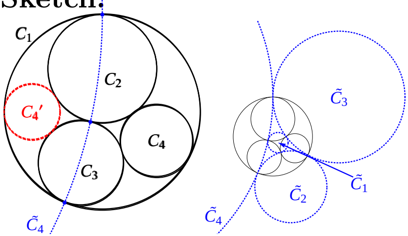

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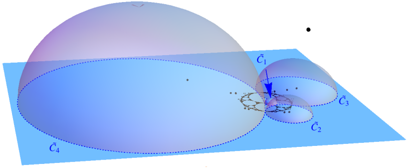

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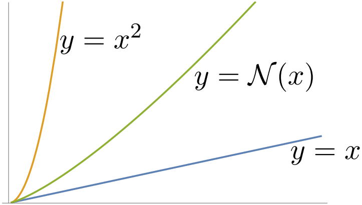

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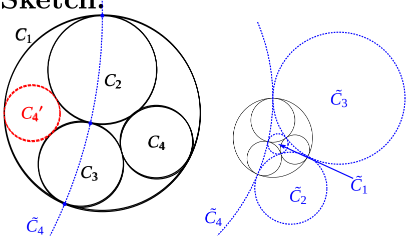

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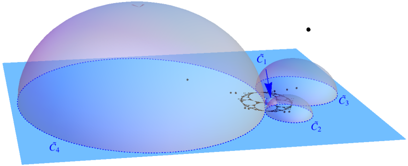
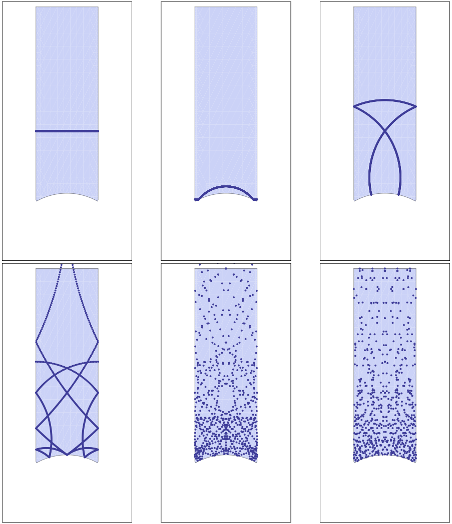


Key: Equidistribution of low-lying horospheres

Classical: $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



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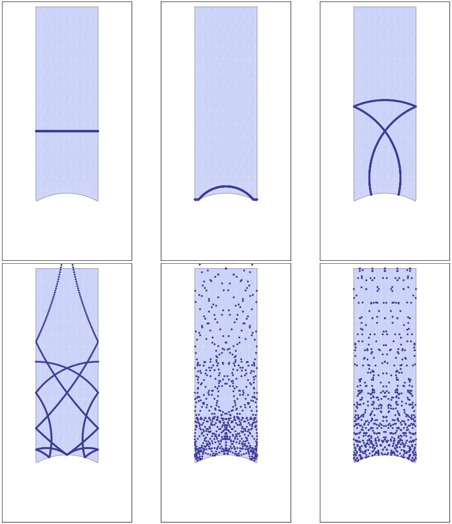

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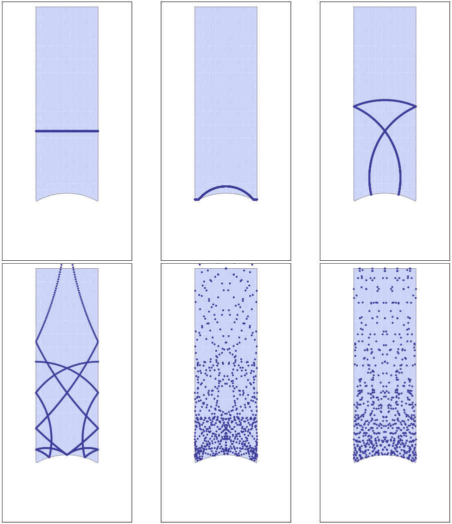

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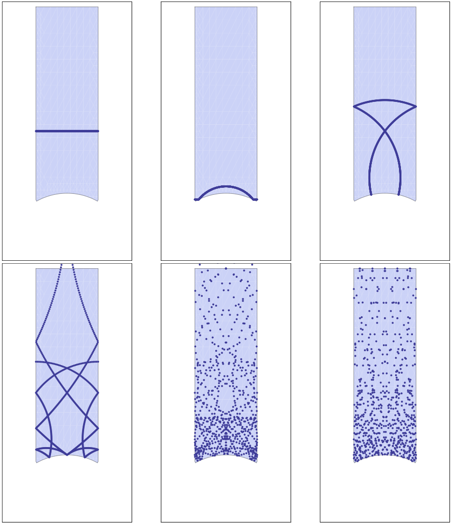
Note: $\ell(H_y) = 1/y \to \infty$.

Classical: $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



Let H_y=horocycle at height $y \to 0$.

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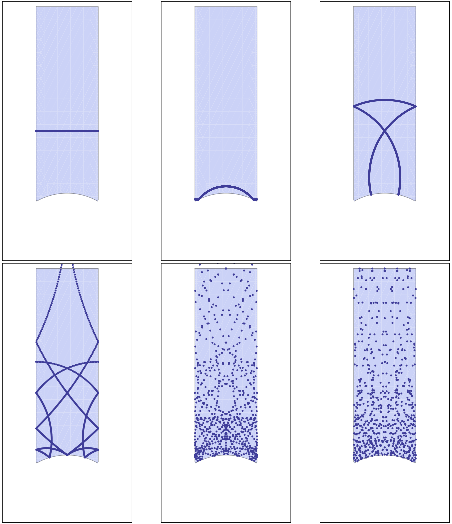

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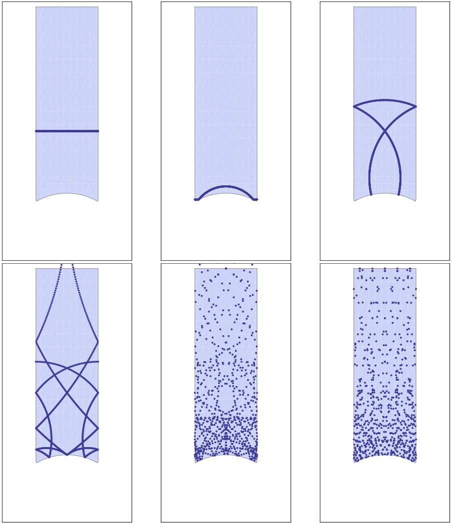

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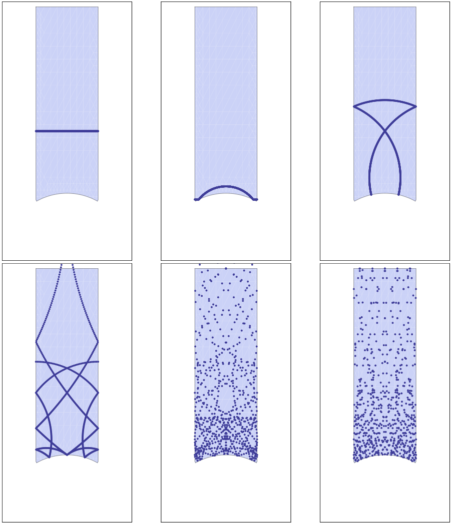

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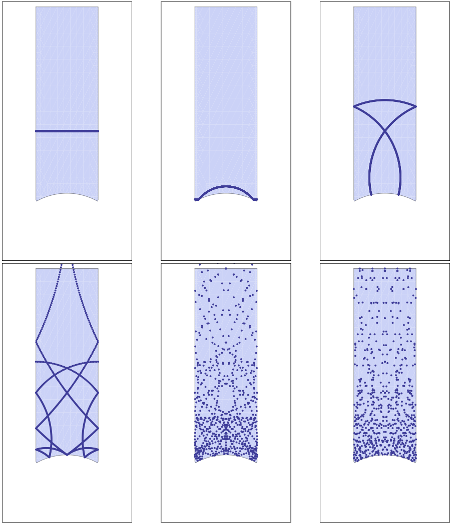

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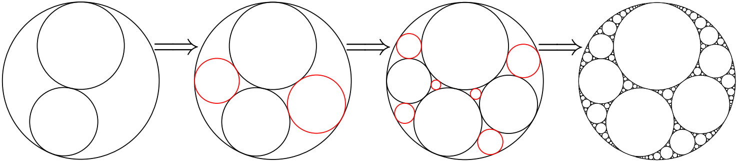
Analogue of this to our setting is used to prove

$$\mathcal{N}(X) = \#\{C \in \mathcal{P} : \kappa(C) < X\} \sim c \cdot X^{1.3057\ldots}$$

Leibniz:



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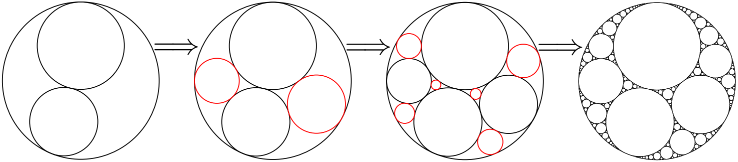

Soddy (1936):

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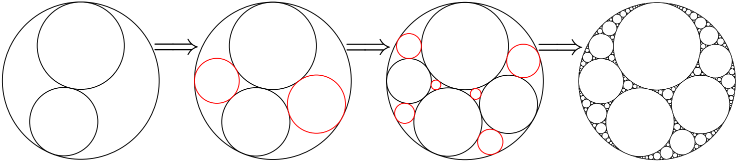

Soddy (1936):

Study the "bends" $\kappa = 1/r$!

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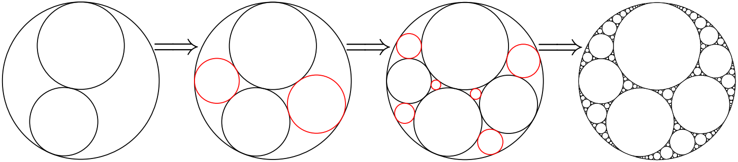

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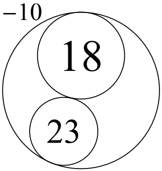

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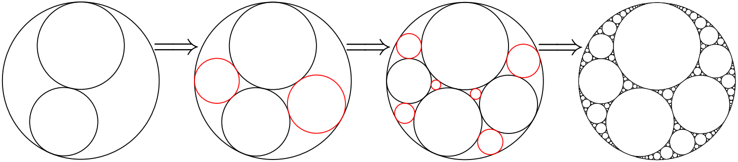

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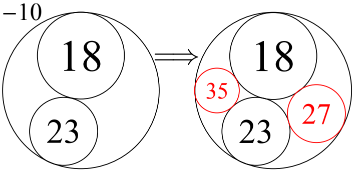

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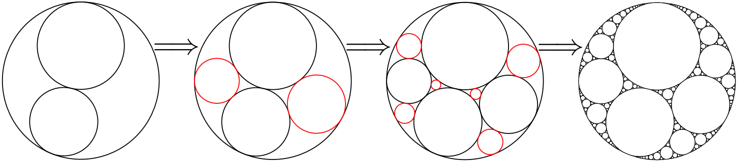

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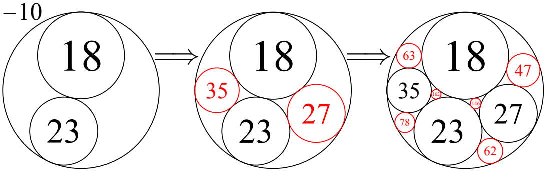

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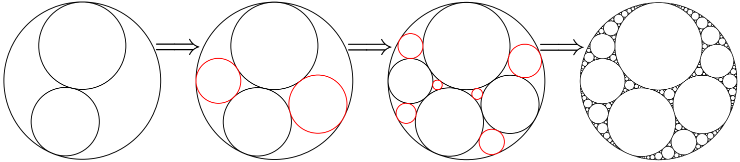

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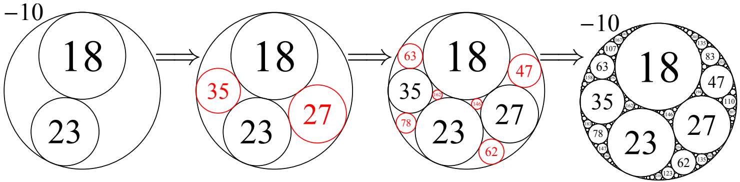

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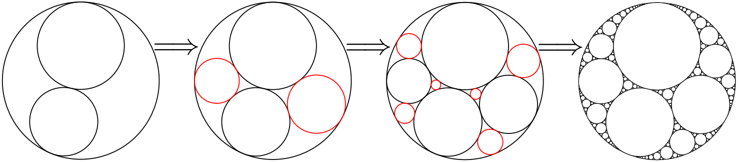

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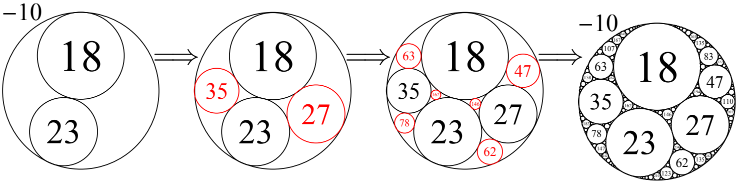

Integral Apollonian Circle Packings

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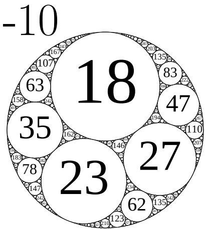

Integral Apollonian Circle Packings

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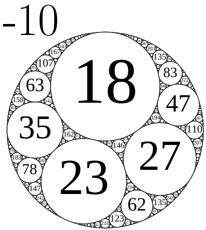

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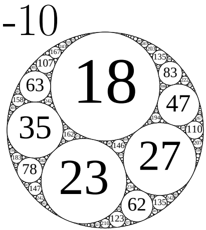
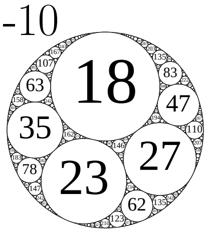


$\textcolor{blue}{\text{Integral}}$ Apollonian Circle Packings



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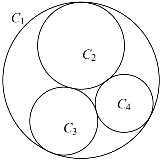
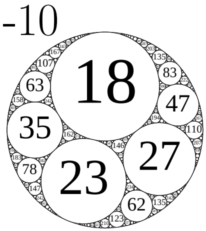


$\Longrightarrow$

Integral Apollonian Circle Packings



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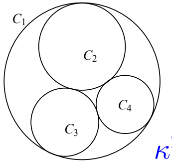

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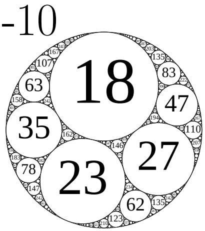
Integral Apollonian Circle Packings



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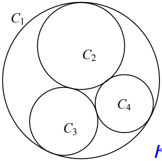

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Four circles to the kissing come.
The smaller are the bender.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

F. Soddy,
Nature (1936).

Integral Apollonian Circle Packings

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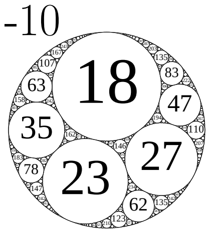

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Integral Apollonian Circle Packings

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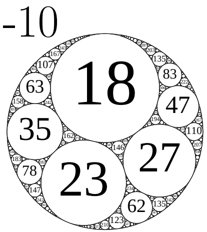

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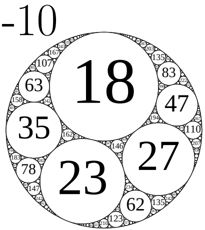

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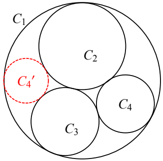

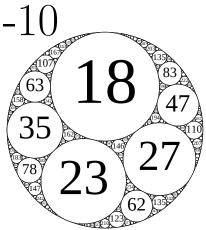

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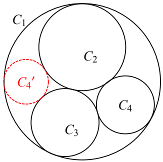

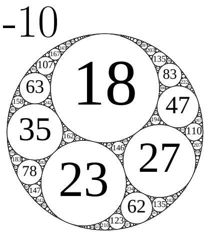

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Integral Apollonian Circle Packings

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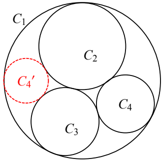

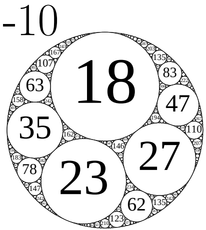

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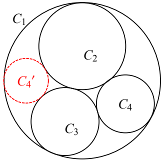

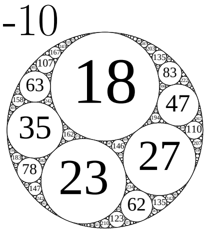

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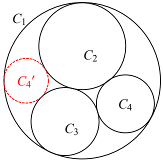

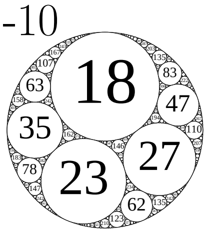

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But *every* circle $C \in \mathcal{P}$ is obtained from the previous ones by Viète moves.

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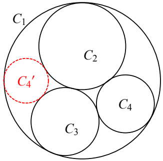

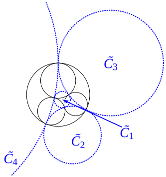

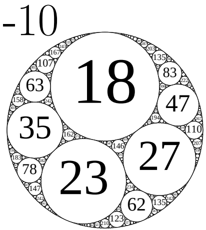

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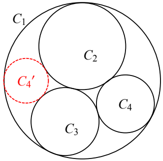

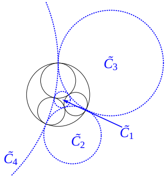

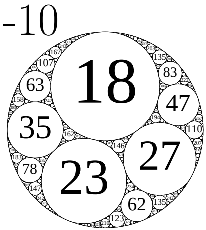

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But *every* circle $C \in \mathcal{P}$ is obtained from the previous ones by Viète moves.

$\Longrightarrow$ (Soddy) If $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ all integral,

Integral Apollonian Circle Packings

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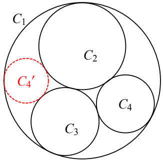

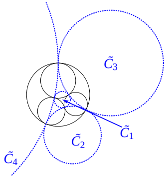

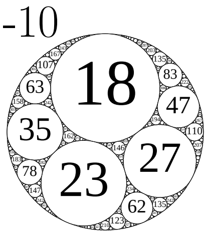

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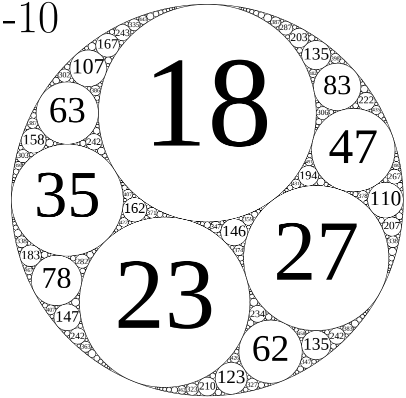
But *every* circle $C \in \mathcal{P}$ is obtained from the previous ones by Viète moves.

$\Longrightarrow$ (Soddy) If $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ all integral, then so are *all* curvatures!

Integral Apollonian Circle Packings



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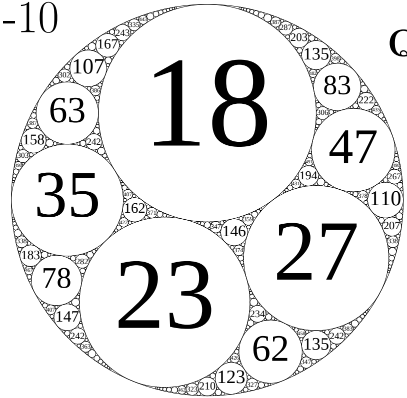

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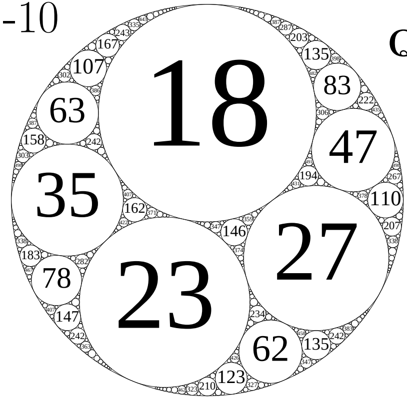

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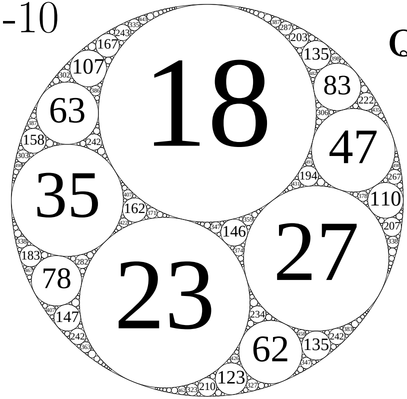

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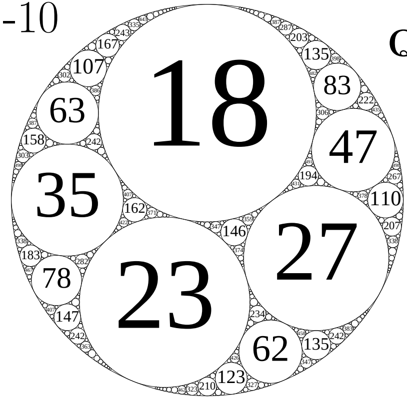

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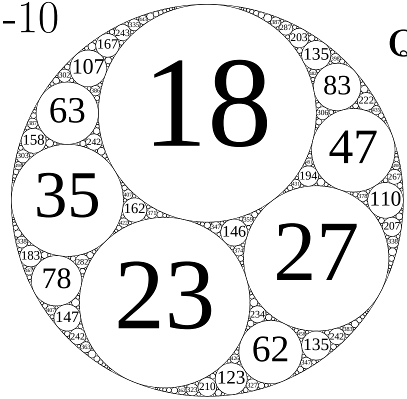

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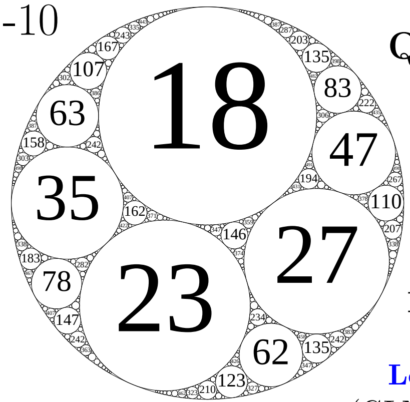

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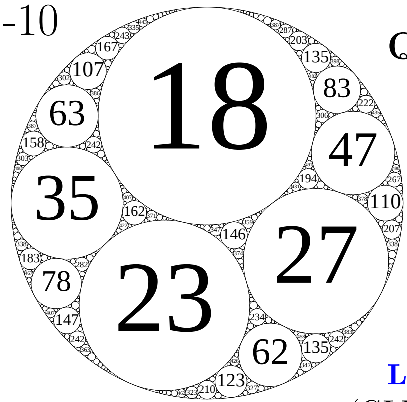

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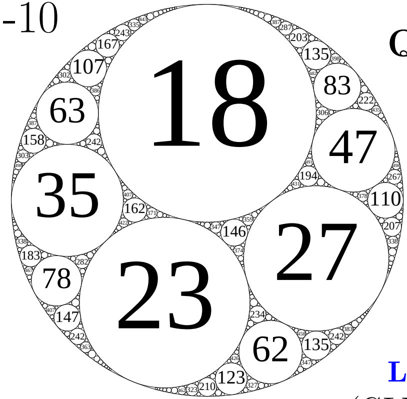

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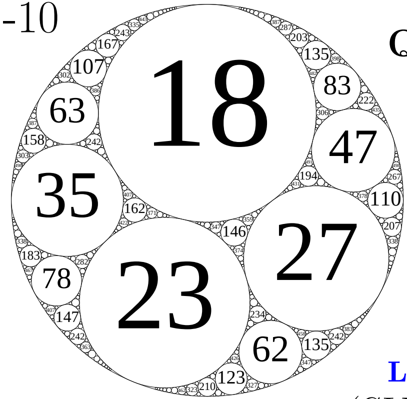

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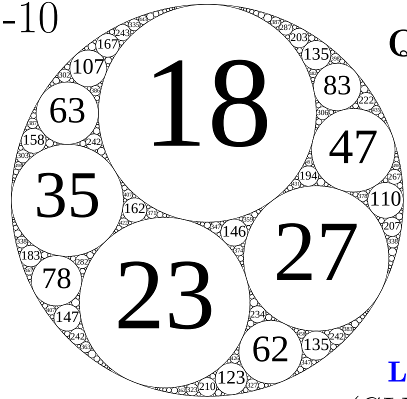

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Builds on GLMWY, Sarnak '07, Fuchs '10, Bourgain-Fuchs '11

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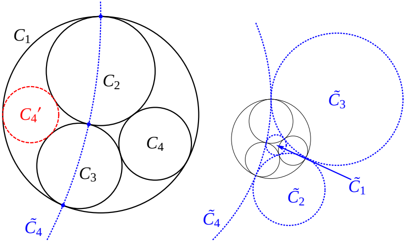

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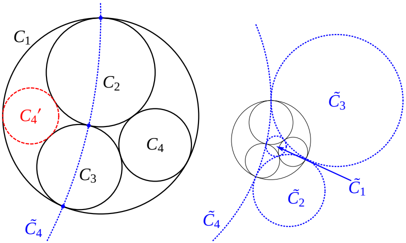

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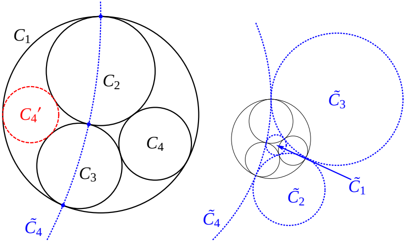

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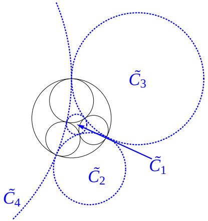

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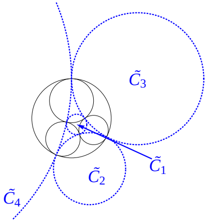

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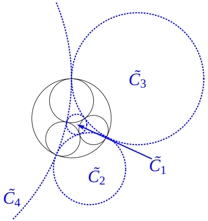

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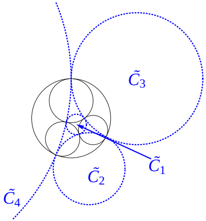

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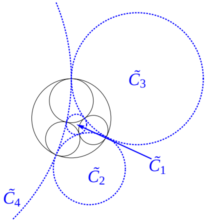

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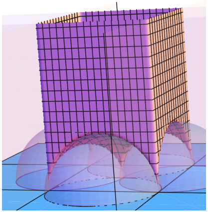

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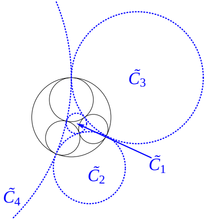

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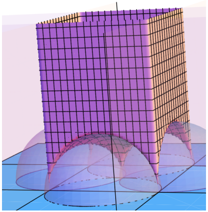

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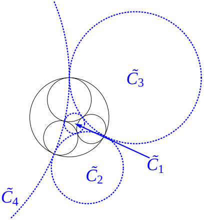

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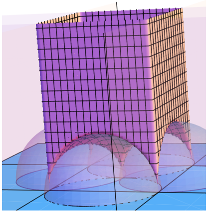

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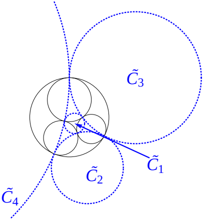

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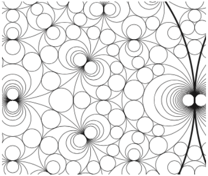

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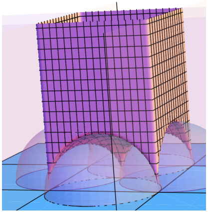

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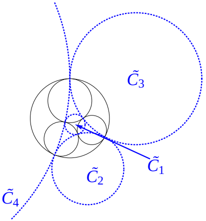

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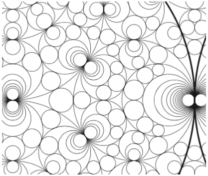

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Corollary: SuperPAC $\Longrightarrow$ essentially only finitely many super-integral Γ-packings.

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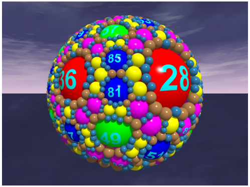

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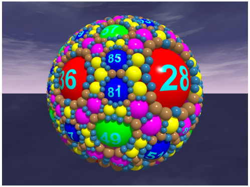

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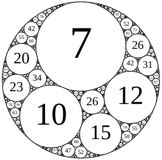

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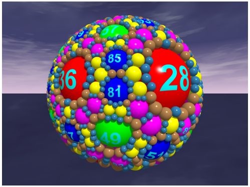

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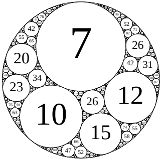

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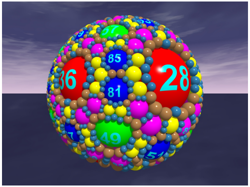

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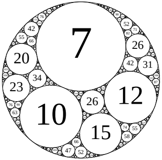

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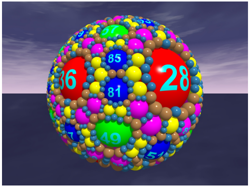

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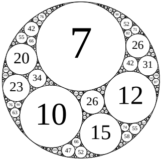

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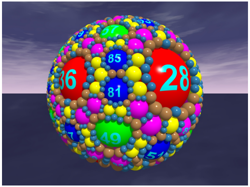

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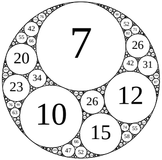

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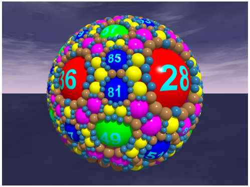

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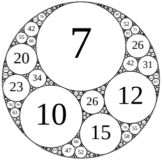

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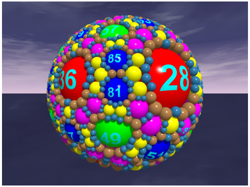

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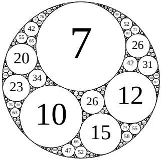

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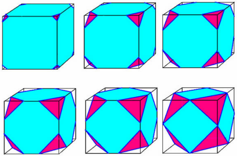

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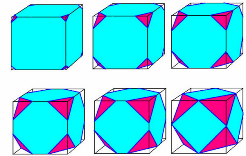

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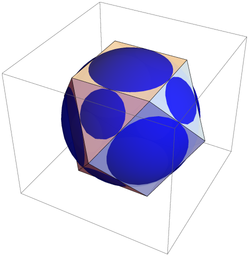

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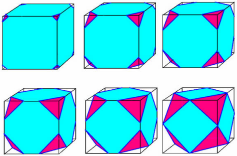

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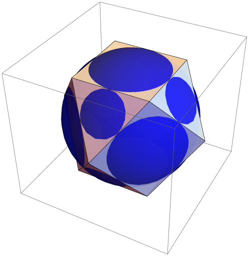

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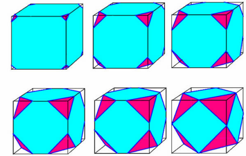

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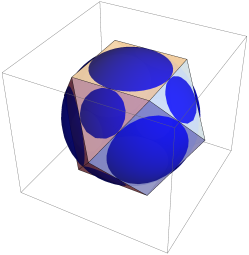

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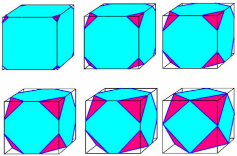

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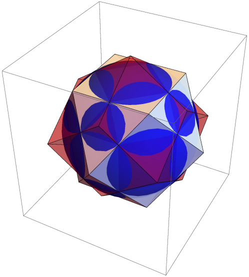

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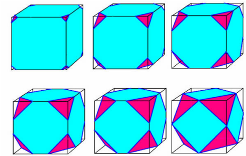

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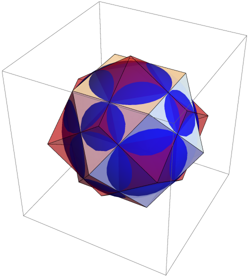

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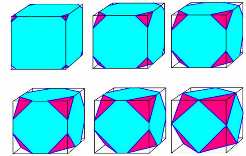

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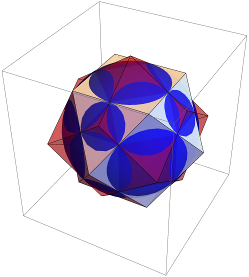

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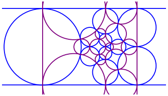

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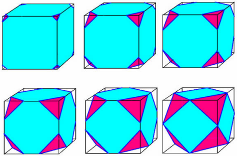

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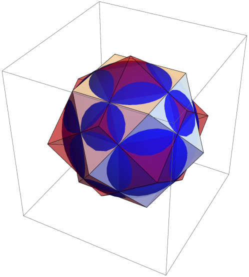

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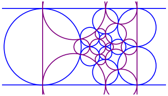

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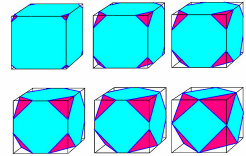

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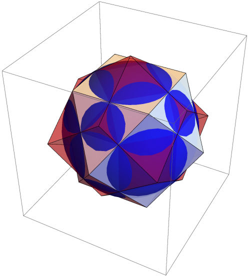

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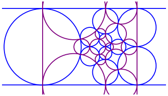

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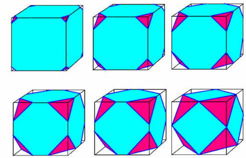

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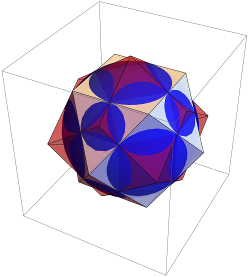

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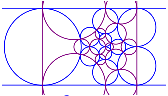

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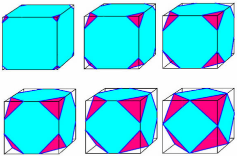

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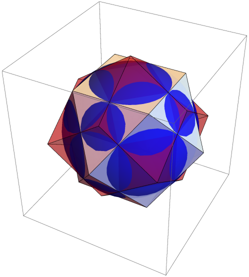

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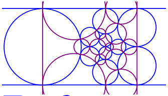

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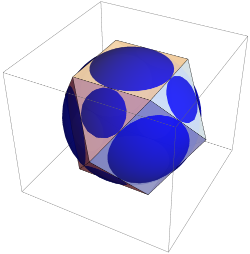

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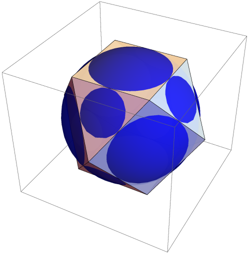
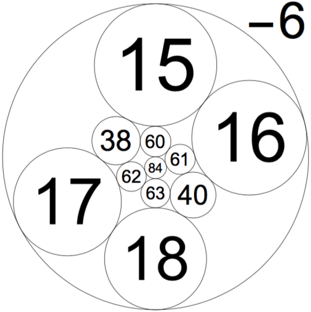
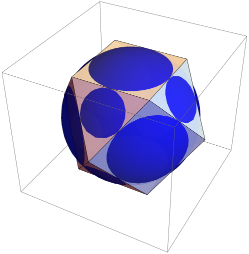


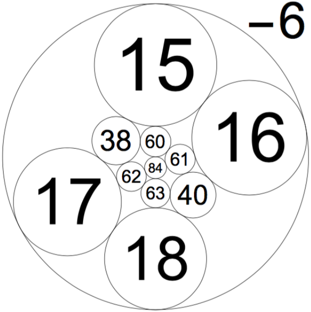
Π-cluster:



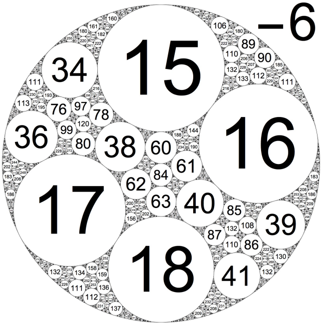
E.g.: Π=Cuboctahedron



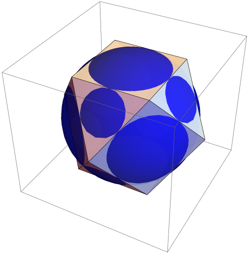
Π-cluster:



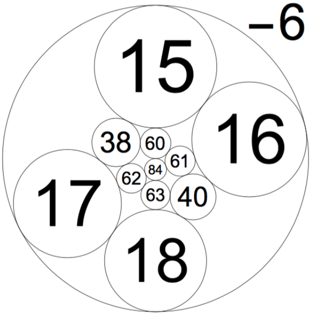
$\Longrightarrow$



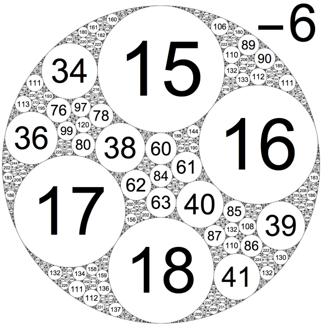
E.g.: Π=Cuboctahedron



Π-cluster:



$\Longrightarrow$



thm: For all known integral $\mathcal{P}(\Pi)$, $\dfrac{\#\{\text{bends}<X\}}{\#\{\text{admissible}<X\}} \to 1$