Cocenter of Hecke algebras

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Representation Theory 101

Let G be a finite group, e.g. $GL_n(\mathbb{F}_p)$.
Number of (ordinary) irr. repr. =number of conjugacy classes.

Reformulation:

- LHS=rank of $R(G)$, the Grothendieck group of fin. dim repr.
- RHS=dim of the cocenter $\overline{\mathbb{C}[G]} := \mathbb{C}[G]/[\mathbb{C}[G], \mathbb{C}[G]]$. Here the cocenter has a standard basis $\{\mathcal{O}\}$, where $\mathcal{O}$ runs over $Cl(G)$.

And a natural duality $Tr : \overline{\mathbb{C}[G]} \to R(G)^*, \quad g \mapsto (V \mapsto Tr(g, V))$.

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Twisted version

One may also consider the twisted version coming from twisted endoscopy.

Here θ is an automorphism of G and ω is a character of G.
We are interested in

- ω-representations of G, i.e. smooth admissible representations π of G such that $\pi^\theta = \pi \circ \theta$ is isomorphism $\omega \otimes \pi$.
- The twisted cocenter $\bar{H} = H/\langle f - {}^x f \rangle$, where $f \in H, x, g \in G$ and ${}^x f(g) = \omega(x) f(x^{-1} g \theta(x))$.
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Difficulties to understand cocenter

For the group algebra of G, we have

- For any conjugacy class $\mathcal{O}$ of G, and $g, g' \in \mathcal{O}$. The image of g and g' in the cocenter are the same.
- The cocenter has a standard basis $\{[g_{\mathcal{O}}]\}$. Here $\mathcal{O}$ runs over all the conjugacy classes of G and $g_{\mathcal{O}}$ is a representative of $\mathcal{O}$.

Such a simple and nice description does not apply to the Hecke algebra. The reason basically comes from the “locally constant” condition. Because of it, we are not able to separate a single conjugacy class from the others.

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Newton stratification

Solution: to separate nice (geometric) unions of conjugacy classes.

- F a nonarchimedean local field of arbitrary characteristic
- $\breve{F}$ the completion of its maximal unramified extension
- $\sigma \in Gal(\breve{F}/F)$ the Frobenius map
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- The dominance condition: $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$;
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Newton stratification (Cont')

We then have $G(\breve{F}) = \sqcup_\nu [b_\nu]$, the Newton stratification.
For split groups, we define $G(\nu) = G \cap [b_\nu]$. Then

$$G = \sqcup_\nu G(\nu).$$

☹ It also works for quasi-split groups under some modification, but not for non quasi-split groups as the special vertex of buildings over F and over $\breve{F}$ do not match.

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Newton decomposition

A key feature of the Newton strata is that they are all admissible.

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The Newton stratum $G(\nu)$ is open and for any compact subset X of G, there exists an open compact subgroup K of G such that $G(\nu) \cap X$ is stable under the left/right multiplication by K.

The admissibility of Newton strata guarantees that the Newton strata works well with the "locally constant" condition of Hecke algebra.

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Newton decomposition at a given level

Note that for a given open compact subgroup K, there is no Newton decomposition at the Hecke algebra level:

$$H(G, K) \neq \oplus_\nu H(G, K; \nu).$$

But quite amazingly, the cocenter of $H(G, K)$ (for "good" K) does have Newton decomposition.

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The proof is based on the establishment of the Iwahori-Matsumoto presentation of cocenter.

A quick review of history:

- Bruhat decomposition $G = \sqcup_{w \in \tilde{W}} IwI$, where $\tilde{W}$ is the Iwahori-Weyl group;
- The original Iwahori-Matsumoto presentation (IHES, 1965) is for the affine Hecke algebra $H(G, I)$: $H(G, I)$ has a basis $\{T_w\}$ for $w \in \tilde{W}$;
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Iwahori-Matsumoto presentation (Cont')

Let $\tilde{W}_{\min}$ be the set of elements in $\tilde{W}$ that are of minimal length in their conjugacy class. Now we have

Theorem

(1) For any n,

$$\bar{H}(G, I_n) = \sum_{w \in \tilde{W}_{\min}} \bar{H}(G, I_n)_w,$$

where $\bar{H}(G, I_n)_w$ is the image in the cocenter of I_n-biinvariant functions supported in IwI.
(2) For any n and Newton point ν, we have

$$\bar{H}(G, I_n; \nu) = \sum_{w \in \tilde{W}_{\min}, \nu_w = \nu} \bar{H}(G, I_n)_w.$$

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Application: Howe's conjecture

As we mentioned before, one major difficulty is that $\dim \bar{H} = \infty$. We need some finiteness results.

Conjecture (Howe)

Let X be a compact subset of G and $J(X)$ be the set of invariant distributions supported in $G \cdot X$. Then for any open compact subgroup K of G,

$$\dim J(X) \mid_{H(G,K)} < \infty.$$

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It is conjectured by Howe in 1973, proved by Clozel (Ann. Math) in 1989 for $\mathrm{char}(F) = 0$ and by Barbasch-Moy (JAMS) in 2000.

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Application: Howe's conjecture (Cont')

Now we give a short proof of it, for both the original version and the twisted version, based on the Iwahori-Matsumoto presentation.

Proof.

- Any open compact subgroup contains I_n for some n.
- Any compact subset of G is in a finite union of Newton strata $G(\nu)$.
- By definition, $J(G_\nu) \mid_{H(G,I_n)} = \bar{H}(G, I_n; \nu)^*$.
- $\forall \nu$, there are only finitely many $w \in \tilde{W}_{\min}$ associated to it.
- $\forall w$, $\dim \bar{H}(G, I_n)_w \leqslant \dim H(G, I_n)_w = \sharp(I_n \backslash IwI / I_n)$ is finite.
- $\bar{H}(G, I_n; \nu) = \sum_{w \in \tilde{W}_{\min}, \nu_w = \nu} \bar{H}(G, I_n)_w$ is a finite sum of finite dimensional spaces. □

Induction and restriction functors

We consider the representations of G over an algebraically closed field k of characteristic $\neq p$. Let $R(G)$ be the Grothendieck group $(\otimes_{\mathbb{Z}} k)$.
How to understand it?

An important family of representations comes from inductions.

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We should have the induction and restriction functors on the cocenter side.

- $\bar{r}_M : \bar{H}(G) \to \bar{H}(M)$ is dual to i_M and can be written down explicitly.
- The functor $\bar{i}_M : \bar{H}(M) \to \bar{H}(G)$ is more problematic.

It exists for affine Hecke algebras since $H(M, I \cap M) \hookrightarrow H(G, I)$ via Bernstein-Lusztig presentation. No such presentation in general.

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Bernstein-Lusztig presentation for $\bar{H}$

Recall that each Newton point ν is dominant. Thus its centralizer defines a standard Levi of G. We then define

$$\bar{H}(M)_{+,rig} = \sum_{Z_G^0(\nu)=M} \bar{H}(M;\nu).$$

We DO have a canonical (and explicit) map

$$\bar{i}_{M(\nu)} : \bar{H}(M;\nu) \cong \bar{H}(G;\nu).$$

That is enough for us since

$$\bar{H}(G) = \bigoplus_M \bar{i}_M(\bar{H}(M)_{+,rig})$$

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$$Tr(\bar{i}_M(h), V) = Tr(h, r_M(V)) \quad \forall h \in \bar{H}(M)_{+,rig}, V \in R(G).$$

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Trace Paley-Wiener Theorem

Now we describe the image of the map $Tr : \bar{H} \to R(G)^*$.
[Bernstein-Deligne-Kazhdan]: $f \in R(G)^*$ is good if

1. $\forall$ M, $\sigma \in R(M)$, $\psi \mapsto f(i_M(\psi\sigma))$ is regular on unramified char ψ
2. $\exists$ open compact subgroup K s.t. $f(V) = 0$ if $V^K = \{0\}$.

Theorem (Trace Paley-Wiener Theorem)

The image of $Tr : \bar{H} \to R(G)^*$ is the space of good forms.

Trace Paley-Wiener Theorem

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[Bernstein-Deligne-Kazhdan]: $f \in R(G)^*$ is good if

1. $\forall\ M, \sigma \in R(M)$, $\psi \mapsto f(i_M(\psi\sigma))$ is regular on unramified char ψ
2. $\exists$ open compact subgroup K s.t. $f(V) = 0$ if $V^K = \{0\}$.

Theorem (Trace Paley-Wiener Theorem)

The image of $Tr : \bar{H} \to R(G)^*$ is the space of good forms.

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- Henniart-Lemaire (Asterisque) 2015: complex ω-representations
- (Joint with Ciubotaru in progress): ordinary/modular/twisted representations under a mild assumption on the char(k)

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Rigid cocenter

A crucial part of the trace Paley-Wiener theorem is to reduce to finite dimensional case:

- In [BDK] and [HL], this is obtained by using unitarity, tempered modules etc. to understand discrete series. Thus only works over $\mathbb{C}$.
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Moreover, we have the rigid trace Paley-Wiener theorem:

Theorem

Suppose that G is semisimple. The trace map induces a surjection

$$Tr : \bar{H}(G)_{+,rig} \to R(G)^*_{rig},$$

where $R(G)^*_{rig}$ is the set of good forms that are constant on $i_M(\psi\sigma)$ (w.r.t unramified char ψ).

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