From Laplacian growth to competitive erosion

Yuval Peres

September 8, 2016

Joint work with Lionel Levine
Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^d$. 

- To form $A + B$, let $C_0 = A \cup B$ and $C_j = C_{j-1} \cup \{y_j\}$ where $y_j$ is the endpoint of a random walk started at $x_j$ and stopped on exiting $C_{j-1}$.
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- $A \cap B = \{x_1, \ldots, x_k\}$. 

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  where $y_j$ is the endpoint of a random walk started at $x_j$ and stopped on exiting $C_{j-1}$.
- Define $A + B = C_k$.
- Abelian property: the law of $A + B$ does not depend on the ordering of $x_1, \ldots, x_k$. 
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From Laplacian growth to competitive erosion
Internal DLA

$A_1 = \{o\}$, $A_n = A_{n-1} + \{o\}$.

Lawler, Bramson and Griffeath (1992) proved that the limiting shape is a ball. More precisely, for any $\varepsilon > 0$, with probability one we have $B_{r(1-\varepsilon)} \subset A_{\lfloor \omega d r \rfloor} \subset B_{r(1+\varepsilon)}$ for all sufficiently large $r$. Here $B_r = \{x \in \mathbb{Z}^d : |x| < r\}$, and $\omega d$ is the volume of the unit ball in $\mathbb{R}^d$. Logarithmic error bounds recently proved by Assaleh-Gaudilierre and by Jerison-Levine-Sheffield. Yuval Peres (joint work with Lionel Levine)
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The Rotor-Router Model

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- Each site $x \in \mathbb{Z}^2$ has a **rotor** pointing North, South, East or West.
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- Deterministic analogue of random walk.
- Each site $x \in \mathbb{Z}^2$ has a **rotor** pointing North, South, East or West.
  (Start all rotors pointing North, say.)
- A particle starts at the origin. At each site it comes to, it
  1. Turns the rotor clockwise by 90 degrees;
  2. Takes a step in direction of the rotor.
Rotor-Router Aggregation (Proposed by Jim Propp)

- Sequence of lattice regions

\[ A_1 = \{ o \} \]

\[ A_n = A_{n-1} \cup \{ x_n \}, \]

- Choices of which particles to route in what order don't affect the final shape generated or the final rotor directions.
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where \( x_n \in \mathbb{Z}^2 \) is the site at which rotor walk first leaves the region \( A_{n-1} \).

- Makes sense in \( \mathbb{Z}^d \) for any \( d \).

- Choices of which particles to route in what order don't affect the final shape generated or the final rotor directions.
Yuval Peres (joint work with Lionel Levine)
From Laplacian growth to competitive erosion
Spherical Asymptotics

- **Theorem** (Levine-P.) Let $A_n$ be the region of $n$ particles formed by rotor-router aggregation in $\mathbb{Z}^d$.

  - $B_\rho$ is the ball of radius $\rho$ centered at the origin.
  - $n = \omega_d r^d$, where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$.
  - $c, c'$ depend only on $d$.

- **Corollary**: Inradius/Outradius $\to 1$ as $n \to \infty$. 

Yuval Peres (joint work with Lionel Levine) From Laplacian growth to competitive erosion
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▶ **Theorem** (Levine-P.) Let $A_n$ be the region of $n$ particles formed by rotor-router aggregation in $\mathbb{Z}^d$. Then

$$B_{r-c\log r} \subset A_n \subset B_r(1+c'r^{-1/d}\log r),$$

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From Laplacian growth to competitive erosion
Divisible Sandpile

- Start with mass $m$ at the origin.
Divisible Sandpile

- Start with mass $m$ at the origin.
- Each site keeps mass 1, divides excess mass equally among its neighbors.

Theorem (Levine-P.): There are constants $c$ and $c'$ depending only on $d$, such that

$$B_r - c \subset A_m \subset B_r + c'$$

where $m = \omega d r^d$. 

Yuval Peres (joint work with Lionel Levine)
Divisible Sandpile

- Start with mass $m$ at the origin.
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- As $t \to \infty$, get a limiting region $A_m$ of mass 1, fractional mass on $\partial A_m$, and zero outside.
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where $m = \omega_d r^d$. 

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Questions

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- As the lattice spacing goes to zero, is there a scaling limit?
- If so, can we describe the limiting shape?
- Is it the same for all three models?
- Not clear how to define dynamics in $\mathbb{R}^d$. 
Odometer Function

- $u(x) = \text{total mass emitted from } x$. 

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From Laplacian growth to competitive erosion
Odometer Function

- $u(x) =$ total mass emitted from $x$.
- Discrete Laplacian:

$$\Delta u(x) = \frac{1}{2d} \sum_{y \sim x} u(y) - u(x)$$
Odometer Function

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= mass received – mass emitted
Odometer Function

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= mass received − mass emitted

\[
\begin{cases} 
-1 & x \in A \cap B \\
0 & x \in A \cup B - A \cap B \\
1 & x \in A \oplus B - A \cup B.
\end{cases}
\]
Least Superharmonic Majorant

Let

\[ \gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y), \]

where \( g \) is the Green's function for SRW in \( \mathbb{Z}^d \), \( d \geq 3 \).
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- In dimension two, we use the negative of the potential kernel in place of $g$.
- Let $s(x) = \inf\{\phi(x) \mid \phi \text{ superharmonic, } \phi \geq \gamma\}$. 
Least Superharmonic Majorant

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- In dimension two, we use the negative of the potential kernel in place of \( g \).
- Let \( s(x) = \inf \{ \phi(x) \mid \phi \text{ superharmonic, } \phi \geq \gamma \} \).
- **Claim:** odometer = \( s - \gamma \).
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From Laplacian growth to competitive erosion
Proof of the claim

Let $m(x) =$ amount of mass present at $x$ in the final state.
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Then

\[ \Delta u = m - 1_A - 1_B \]
Proof of the claim

Let $m(x) =$ amount of mass present at $x$ in the final state. Then

$$
\Delta u = m - 1_A - 1_B \\
\leq 1 - 1_A - 1_B.
$$
Proof of the claim

- Let $m(x) =$ amount of mass present at $x$ in the final state. Then

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\Delta u = m - 1_A - 1_B \leq 1 - 1_A - 1_B.
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- Since

\[
\Delta \gamma = 1_A + 1_B - 1
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the sum $u + \gamma$ is superharmonic, so $u + \gamma \geq s$. 
Proof of the claim

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Reverse inequality: \( s - \gamma - u \) is superharmonic on \( A \oplus B \) and nonnegative outside \( A \oplus B \), hence nonnegative inside as well.
Defining the Scaling Limit

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have measure zero
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- Let

$$D = A \cup B \cup \{s > \gamma\}$$
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$$\gamma(x) = -|x|^2 - \int_A g(x, y) dy - \int_B g(x, y) dy$$

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From Laplacian growth to competitive erosion
Defining the Scaling Limit

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$$s(x) = \inf\{\phi(x) | \phi \text{ is continuous, superharmonic, and } \phi \geq \gamma\}$$

is the least superharmonic majorant of $\gamma$. 

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Defining the Scaling Limit

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  \[ s(x) = \inf\{ \phi(x) | \phi \text{ is continuous, superharmonic, and } \phi \geq \gamma \} \]
  is the least superharmonic majorant of $\gamma$.
- Odometer: $u = s - \gamma$. 

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From Laplacian growth to competitive erosion
The domain $D = \{s > \gamma\}$ for two overlapping disks in $\mathbb{R}^2$. 
The domain $D = \{s > \gamma\}$ for two overlapping disks in $\mathbb{R}^2$.  

The boundary $\partial D$ is given by the algebraic curve

$$(x^2 + y^2)^2 - 2r^2 (x^2 + y^2) - 2(x^2 - y^2) = 0.$$
Main Result

Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.
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Let \( A, B \subset \mathbb{R}^d \) be bounded open sets with \( \partial A, \partial B \) having measure zero.

- Lattice spacing \( \delta_n \downarrow 0 \).
- Write \( A:: = A \cap \delta_n \mathbb{Z}^d \).
Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- Write $A^{\cdot} = A \cap \delta_n \mathbb{Z}^d$.
- **Theorem** (Levine-P.) For any $\varepsilon > 0$, with probability one

  $$D^{\cdot} \subset D_n, R_n, I_n \subset D^{\varepsilon^{\cdot}}$$

  for all sufficiently large $n$, 

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From Laplacian growth to competitive erosion
Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.

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**Theorem** (Levine-P.) For any $\varepsilon > 0$, with probability one

$$D^{\vdash}_\varepsilon \subset D_n, R_n, I_n \subset D^{\vdash}_\varepsilon$$

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the Diaconis-Fulton sums of $A^{\vdash}$ and $B^{\vdash}$ in the lattice $\delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.
Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A$, $\partial B$ having measure zero.
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- $D = A \cup B \cup \{s > \gamma\}$. 
Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- Write $A_{\ddagger} = A \cap \delta_n \mathbb{Z}^d$.
- **Theorem** (Levine-P.) For any $\varepsilon > 0$, with probability one

$$D_{\varepsilon_{\ddagger}} \subset D_n, R_n, I_n \subset D_{\varepsilon_{\ddagger}}$$

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the Diaconis-Fulton sums of $A_{\ddagger}$ and $B_{\ddagger}$ in the lattice $\delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.
- $D = A \cup B \cup \{s > \gamma\}$.
- $D_{\varepsilon}, D_{\varepsilon}$ are the inner and outer $\varepsilon$-neighborhoods of $D$.
Multiple Point Sources

Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$. 

Theorem (Levine-P.) For any $\varepsilon > 0$, with probability one $D_\varepsilon \subset D_n, R_n, I_n \subset D_{\varepsilon}$ for all sufficiently large $n$, where $D_n, R_n, I_n$ are the domains of occupied sites $\delta_n \mathbb{Z}^d$, if $\lfloor \lambda_i \delta_n \rfloor$ particles start at each site $x_i$, computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.

$D$ is the continuum Diaconis-Fulton sum of the balls $B(x_i, r_i)$, where $\lambda_i = \omega d r_i$. 

Follows from the main result and the case of a single point source.
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- **Theorem** (Levine-P.) For any $\varepsilon > 0$, with probability one

$$D_{\varepsilon} \subset D_n, R_n, l_n \subset D_{\varepsilon}$$

for all sufficiently large $n$, 
Multiple Point Sources

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- **Theorem** (Levine-P.) For any $\varepsilon > 0$, with probability one

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- Follows from the main result and the case of a single point source.
Steps of the Proof

convergence of densities

⇓

convergence of obstacles
Steps of the Proof

convergence of densities
\[\downarrow\]
convergence of obstacles
\[\downarrow\]
convergence of odometer functions
Steps of the Proof

convergence of densities

⇓

convergence of obstacles

⇓

convergence of odometer functions

⇓

convergence of domains.
Adapting the Proof for Rotors

- Rotor-router odometer:
  \[ u(x) = \text{total number of particles emitted from } x. \]

- Instead of \( \Delta u = 1 \), we only know \( -2 \leq \Delta u \leq 4 \).

- Repeating the argument only gives \( B_{cr} \subset A_n \subset B_{c'} \).
Smoothing

To do better, let

\[ v(x) = \frac{1}{4k^2} \sum_{y \in S_k(x)} u(y) \]

where \( S_k(x) \) is a box of side length 2\( k \) centered at \( x \).

Using \( \Delta = \text{div grad} \), we get

\[ \Delta v(x) = \frac{1}{4k^2} \sum_{(y,z) \in \partial S_k(x)} \frac{u(z) - u(y)}{4} \]

\[ = 1 + O \left( \frac{1}{k} \right) \]

if \( o \notin S_k(x) \) and all sites in \( S_k(x) \) are occupied.
A Quadrature Identity

If $h$ is harmonic on $\delta_n \mathbb{Z}^d$, then

$$M_t = \sum_j h(X_t^j)$$

is a martingale for internal DLA, where $(X_t^j)_{t\geq 0}$ is the random walk performed by the $j$-th particle.

Therefore if $I_n \to D$, we expect the limiting domain $D \subset \mathbb{R}^d$ to satisfy

$$\int_D h(x) \, dx = k \sum_{i=1}^\lambda x_i h(x_i)$$

for all harmonic functions $h$ on $D$. 

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A Quadrature Identity

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  is a martingale for internal DLA, where $(X_t^j)_{t \geq 0}$ is the random walk performed by the $j$-th particle.
- Optional stopping:
  \[ \mathbb{E} \sum_{x \in I_n} h(x) = \mathbb{E} M_T = M_0 = \sum_{i=1}^k \left\lfloor \lambda_i \delta_n^{-d} \right\rfloor h(x_i). \]
A Quadrature Identity

- If \( h \) is harmonic on \( \delta_n \mathbb{Z}^d \), then
  \[
  M_t = \sum_j h(X_t^j)
  \]
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- Optional stopping:
  \[
  \mathbb{E} \sum_{x \in I_n} h(x) = \mathbb{E} M_T = M_0 = \sum_{i=1}^k \lfloor \lambda_i \delta_n^{-d} \rfloor h(x_i).
  \]

- Therefore if \( I_n \to D \), we expect the limiting domain \( D \subset \mathbb{R}^d \) to satisfy
  \[
  \int_D h(x)dx = \sum_{i=1}^k \lambda_i h(x_i).
  \]
  for all harmonic functions \( h \) on \( D \).
Quadrature Domains

- Given $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.
- $D \subset \mathbb{R}^d$ is called a quadrature domain for the data $(x_i, \lambda_i)$ if

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The smash sum $B_1 \oplus \ldots \oplus B_k$ is such a domain, where $B_i$ is the ball of volume $\lambda_i$ centered at $x_i$.

The boundary of $B_1 \oplus \ldots \oplus B_k$ lies on an algebraic curve of degree $2k$. 
\[ \int \int_D h(x, y) \, dx \, dy = h(-1, 0) + h(1, 0) \]
Further Directions and Open Problems: Rotor-Router

- How fast does $R(n) = \max_{k \leq n} (\text{outrad}(A_k) - \text{inrad}(A_k))$ really grow?

- Is the occupied region simply connected?

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From Laplacian growth to competitive erosion
$z \mapsto \frac{1}{z^2}$
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From Laplacian growth to competitive erosion

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Conjecture: As $n \to \infty$, the limiting shape $S_{n,h}$ is well approximated by a $(12 - 4h)$-gon.

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$h = 2$

$h = 1$

$h = 0$