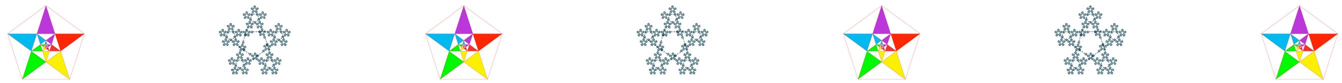
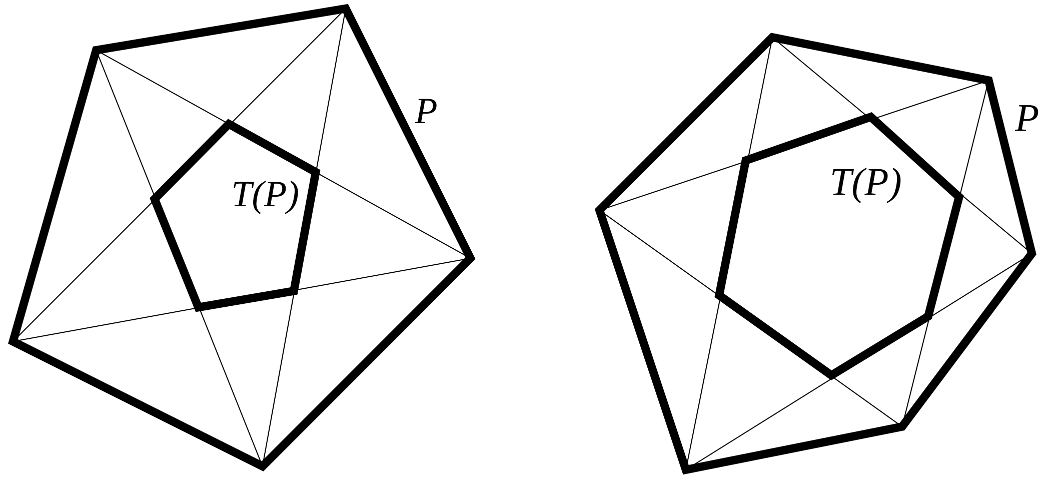
Pentagram map, twenty years after

Stony Brook, March 2014



Part 1. **A survey of older results**

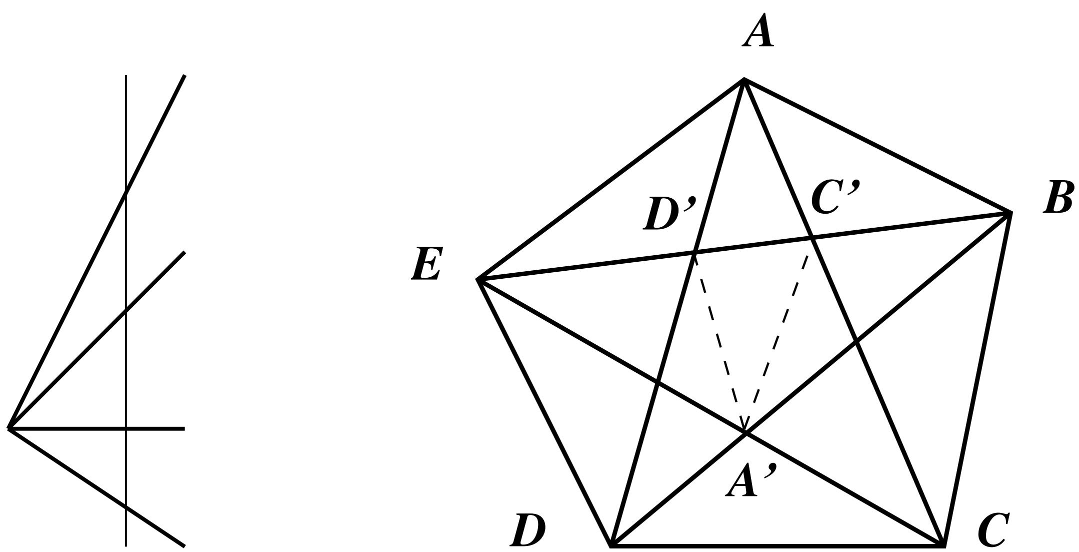
Pentagram Map of Richard Schwartz:



A good reference (among many others): `http://en.wikipedia.org/wiki/Pentagram_map`

Example: if n=5 then $T(P) = P$.

Cross-ratio: $[t_1, t_2, t_3, t_4] = \frac{(t_1-t_2)(t_3-t_4)}{(t_1-t_3)(t_2-t_4)}$.



Exercise: if $n = 6$ then $T^2(P) = P$.

Let's watch an animation

Relevant moduli spaces:

$\mathcal{C}_n$, space of projective equivalence classes of closed n-gons in $\mathbf{RP}^2$; dim $= 2n - 8$;

$\mathcal{P}_n$, space of projective equivalence classes of *twisted* n-gons in $\mathbf{RP}^2$; dim $= 2n$:

$$\phi : \mathbf{Z} \to \mathbf{RP}^2 \ \text{ s.t. } \ \phi(k+n) = M \circ \phi(k); \ \forall k.$$

M is the *monodromy*.

Theorem (OST 2010) The Pentagram Map is completely integrable on the space of twisted n-gons $\mathcal{P}_n$:
1). There are $2[n/2]+2$ *algebraically independent integrals;*
2). There is an invariant Poisson structure of corank 2 if n *is odd, and corank 4 if* n *is even, such that the integrals Poisson commute.*

In both cases,
dim $\mathcal{P}_n -$ corank $= 2$(number of integrals $-$ corank).

Arnold-Liouville integrability (Poisson set-up)

Let M^{p+2q} have independent Poisson commuting 'integrals' $f_1, \ldots, f_p, f_{p+1}, \ldots, f_{p+q}$ where $f_1, \ldots, f_p$ are Casimirs. One has the symplectic foliation given by

$$f_1 = const, f_2 = const, \ldots, f_p = const,$$

and the Lagrangian subfoliation $\mathcal{F}^q$ given by

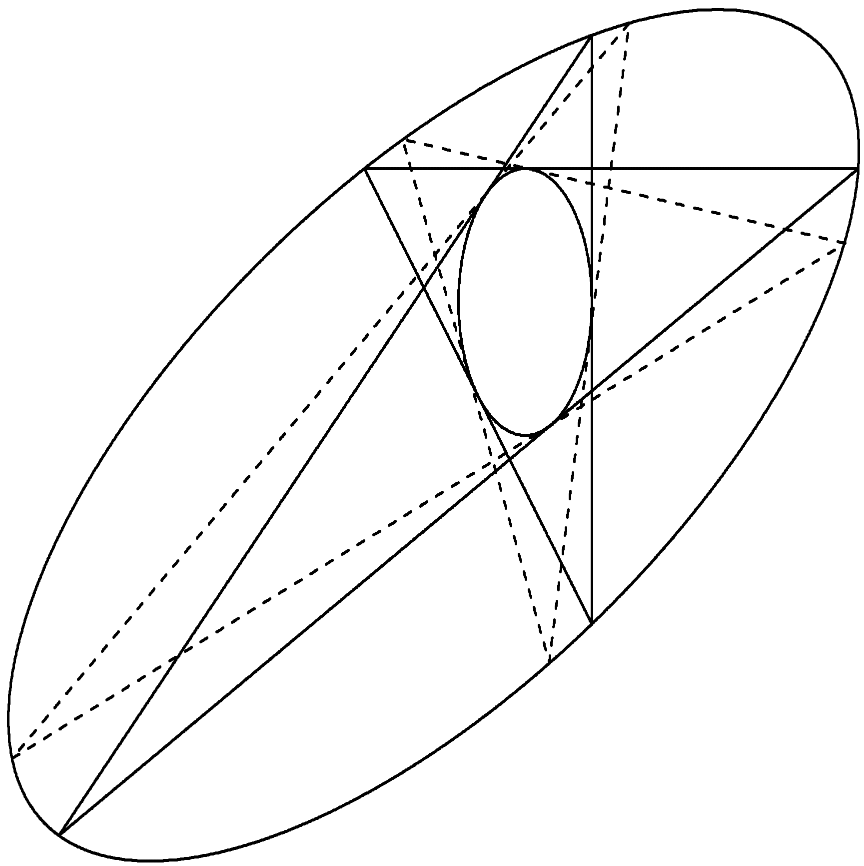
$$f_{p+1} = const, f_{p+2} = const, \ldots, f_{p+q} = const.$$

The leaves carry a *flat structure* given by the commuting fields

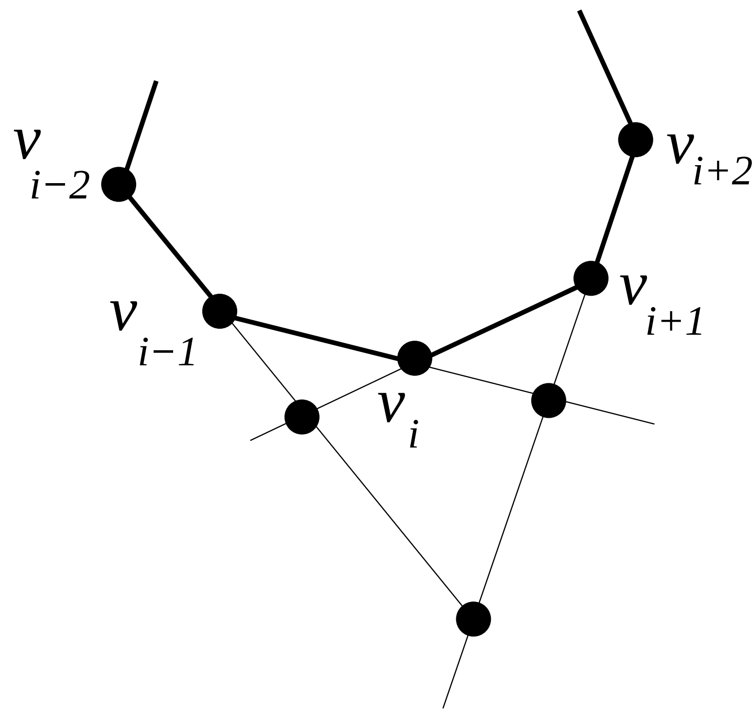
$$H_{f_{p+1}}, \ldots, H_{f_{p+q}}.$$

If a (discrete or continuous time) dynamical system on M preserves all this, then the motion on the leaves is a *parallel translation*. If the leaves are compact, they are tori, and the motion is quasi-periodic.

Poncelet-style Corollary. *If a point is T-periodic then all points on the same leaf are periodic with the same period.*



Corner coordinates: left and right cross-ratios $X_1, Y_1, \ldots, X_n, Y_n$.



The map is as follows:

$$X_i^* = X_i \, \frac{1 - X_{i-1}\, Y_{i-1}}{1 - X_{i+1}\, Y_{i+1}}, \qquad Y_i^* = Y_{i+1} \, \frac{1 - X_{i+2}\, Y_{i+2}}{1 - X_i \, Y_i}.$$

Two consequences:

1). Hidden *scaling symmetry*

$$(X_1, Y_1, ..., X_n, Y_n) \mapsto (tX_1, t^{-1}Y_1, ..., tX_n, t^{-1}Y_n)$$

commutes with the map.

2). "Easy" *integrals*:

$$O_n = \prod_{i=1}^{n} X_i, \quad E_n = \prod_{i=1}^{n} Y_i,$$

and, for even n,

$$O_{n/2} = \prod_{i \text{ even}} X_i + \prod_{i \text{ odd}} X_i, \quad E_{n/2} = \prod_{i \text{ even}} Y_i + \prod_{i \text{ odd}} Y_i.$$

These are the Casimirs.

Monodromy invariants:

$$\frac{O_n^{2/3} E_n^{1/3} (\text{Tr } M)}{(\det M)^{1/3}} = \sum_{k=1}^{[n/2]} O_k$$

are polynomials in (X_i, Y_i), decomposed into homogeneous components; likewise, for E_k with M^{-1} replacing M.

They are algebraically independent. There is a combinatorial description, and a description in terms of 4-diagonal determinants.

Poisson bracket:

$$\{X_i, X_{i+1}\} = -X_i\, X_{i+1}, \quad \{Y_i, Y_{i+1}\} = Y_i\, Y_{i+1},$$

and the rest $= 0$.

Complete integrability on the space of closed polygons $\mathcal{C}_n$ holds as well:

V. Ovsienko, R. Schwartz, S. T. *Liouville-Arnold integrability of the pentagram map on closed polygons*, Duke Math. J. 162 (2013), 2149–2196;
F. Soloviev. *Integrability of the Pentagram Map*, Duke Math. J., 162 (2013), 2815–2996.

Continuous limit of the pentagram map

Object of study: the space $\mathcal{P}$ of non-degenerate twisted parameterized curves in $\mathbf{RP}^2$ modulo projective equivalence:

$$\gamma(x+1) = M(\gamma(x)).$$

Lift so that

$$\left|\Gamma(x)\,\Gamma'(x)\,\Gamma''(x)\right| = 1.$$

Then

$$\Gamma'''(x) + u(x)\,\Gamma'(x) + v(x)\,\Gamma(x) = 0.$$

Thus $\mathcal{P} \equiv$ space of linear differential operators on $\mathbf{R}$:

$$A = \left(\frac{d}{dx}\right)^3 + u(x)\,\frac{d}{dx} + v(x),$$

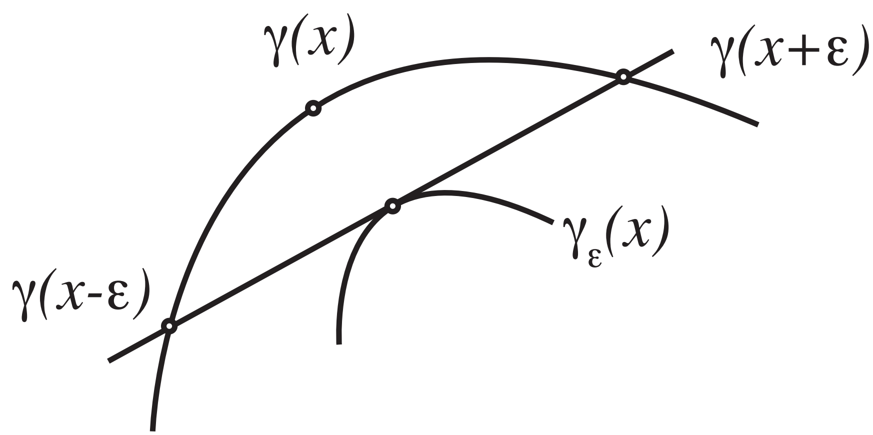
with u and v smooth 1-periodic functions.

Rewrite as

$$A = \left(\frac{d}{dx}\right)^3 + \frac{1}{2}\left(u(x)\frac{d}{dx} + \frac{d}{dx}u(x)\right) + w(x)$$

where $w(x) = v(x) - \frac{u'(x)}{2}$ (sum of a skew-symmetric and zero-order symmetric operators). These u and w are the "projective curvature" and "projective length element".

Construction:



It turns out that

$$u_\varepsilon = u + \varepsilon^2 \widetilde{u} + (\varepsilon^3), \qquad w_\varepsilon = w + \varepsilon^2 \widetilde{w} + (\varepsilon^3),$$

giving the flow: $\dot{u} = \widetilde{u}$, $\dot{w} = \widetilde{w}$.
A computation reveals :

$$\dot{u} = w', \qquad \dot{w} = -\frac{u\,u'}{3} - \frac{u'''}{12},$$

or

$$\ddot{u} + \frac{\left(u^2\right)''}{6} + \frac{u^{(IV)}}{12} = 0,$$

the Boussinesq equation!

Other interesting things (not for this talk):

- New configuration theorems of elementary projective geometry (Schwartz & S.T.)
- Combinatorics of monodromy invariants on inscribed polygons (Schwartz & S.T.)
- Algebraic combinatorics of frieze and 2-frieze patterns, and cluster algebras (Mourier-Genoud, Ovsienko & S.T.)
- Pentagram Map, cluster algebras, Y- and T-patterns (M. Glick; R. Kedem)
- Singularity confinement (M. Glick)
- Milti-dimensional pentagram maps (B. Khesin & F. Soloviev; G. Mari-Beffa)
- Loop groups and cluster algebras (V. Fock & A. Marshakov)
- Combinatorial Gale transform and commuting difference operators ((Mourier-Genoud, Ovsienko & S.T.; I. Krichever)

Part 2. **Higher- (and lower-) dimensional Pentagram Maps**

Joint work (still in progress) with M. Gekhtman, M. Shapiro and A. Vainshtein, ERA 19 (2012), 1–17.

The change of variables

$$x_i = Y_i, \quad y_i = -Y_i X_{i+1} Y_{i+1}$$

yields the map T_3:

$$x_i^* = x_{i-2} \frac{x_i + y_i}{x_{i-2} + y_{i-2}}, \quad y_i^* = y_{i-1} \frac{x_{i+1} + y_{i+1}}{x_{i-1} + y_{i-1}}.$$

Generalize to the family of map $T_k,\ k = 2, 3, \ldots$:

$$x_i^* = x_{i-r-1}\frac{x_{i+r} + y_{i+r}}{x_{i-r-1} + y_{i-r-1}}, \qquad y_i^* = y_{i-r}\frac{x_{i+r+1} + y_{i+r+1}}{x_{i-r} + y_{i-r}}, \qquad k \text{ even},$$

$$x_i^* = x_{i-r-2}\frac{x_{i+r} + y_{i+r}}{x_{i-r-2} + y_{i-r-2}}, \qquad y_i^* = y_{i-r-1}\frac{x_{i+r+1} + y_{i+r+1}}{x_{i-r-1} + y_{i-r-1}}, \qquad k \text{ odd},$$

where $r = \left[\frac{k}{2}\right] - 1$.

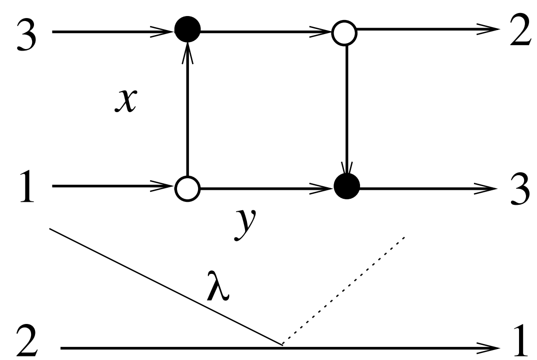
As we shall see, $k - 1$ is the dimension of the ambient projective space.

Weighted directed networks on the cylinder and the torus

(A. Postnikov math/0609764, for networks in a disc;

GSV, *Cluster algebras and Poisson geometry, AMS, 2010*).

Example:

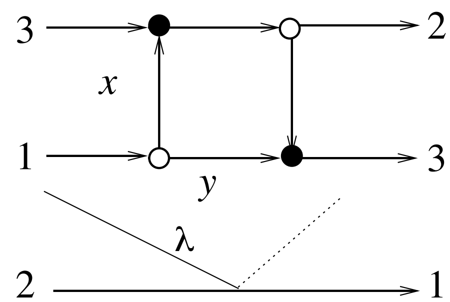


No cycles. Two kind of vertices, white and black.

Convention: an edge weight is 1, if not specified.

The *cut* is used to introduce a *spectral parameter* λ.

Boundary measurements: the network



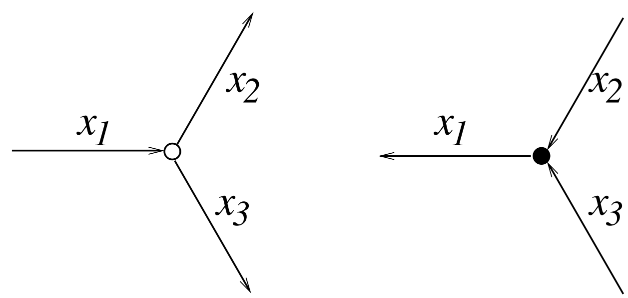
corresponds to the matrix

$$\begin{pmatrix} 0 & x & x+y \\ \lambda & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Concatenation of networks $\mapsto$ product of matrices.

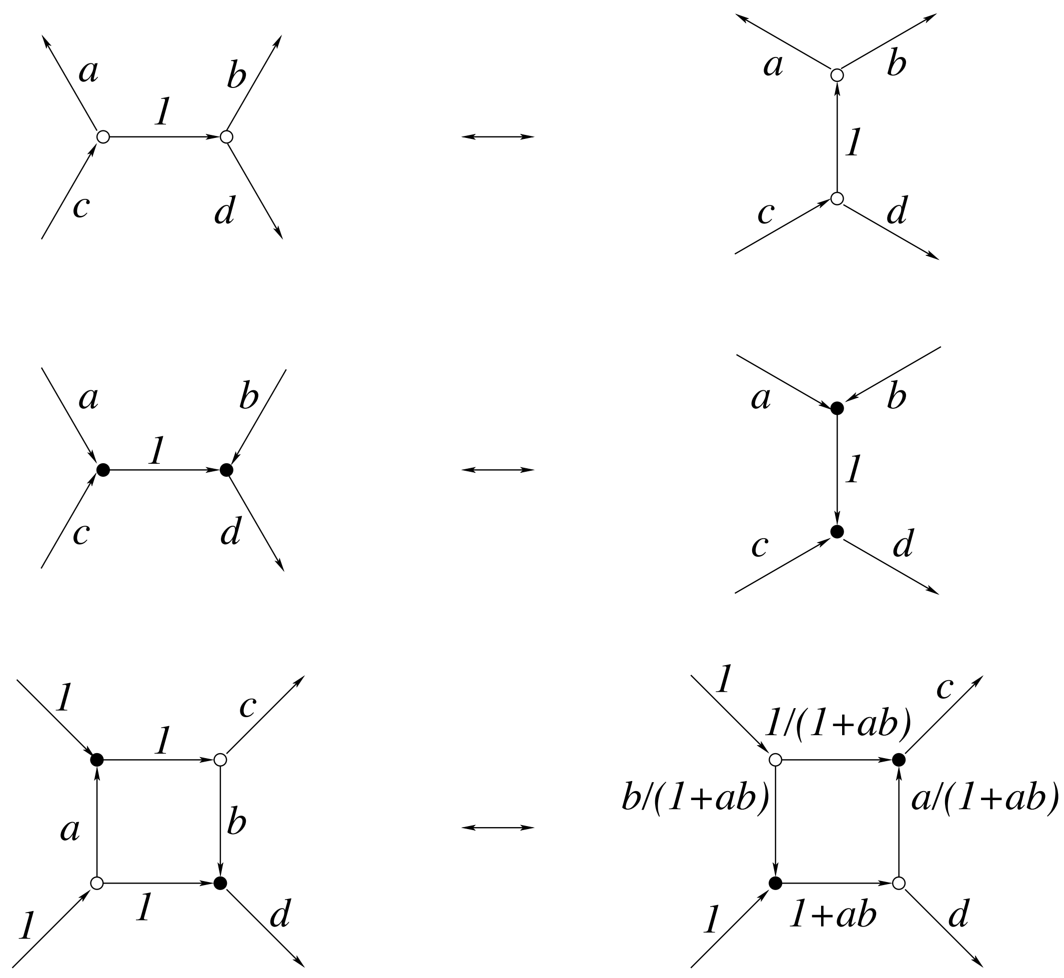
Gauge group: at a vertex, multiply the weights of the incoming edges and divide the weights of the outgoing ones by the same monomial in the weights. Preserves the boundary measurements.

Poisson bracket on the edge weight space: $\{x_i, x_j\} = c_{ij} x_i x_j,\ i \neq j \in \{1, 2, 3\}$

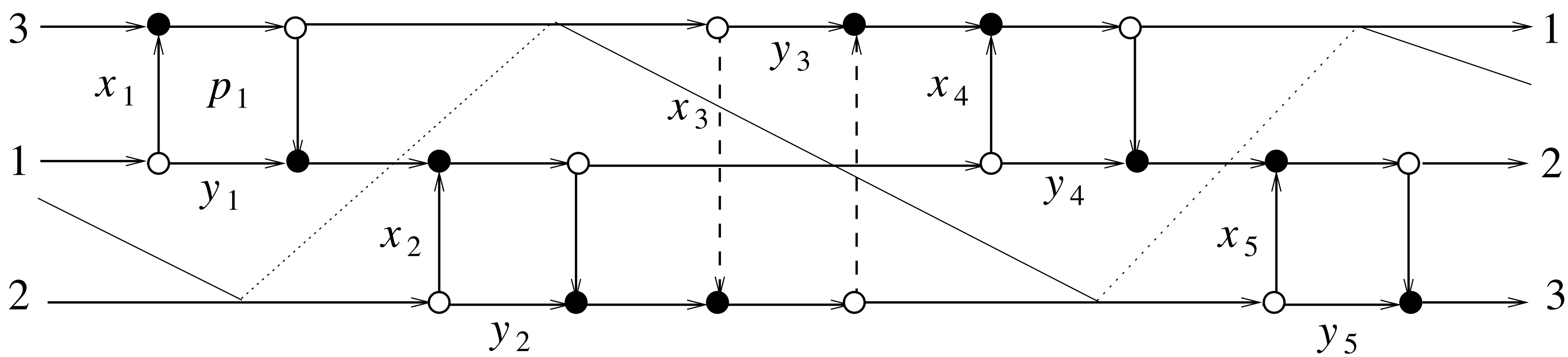


Descends to the quotient of the edge weight space by the gage group. One chooses the *standard* Poisson structure (described in terms of the dual graph).

Postnikov moves (preserve the boundary measurements):

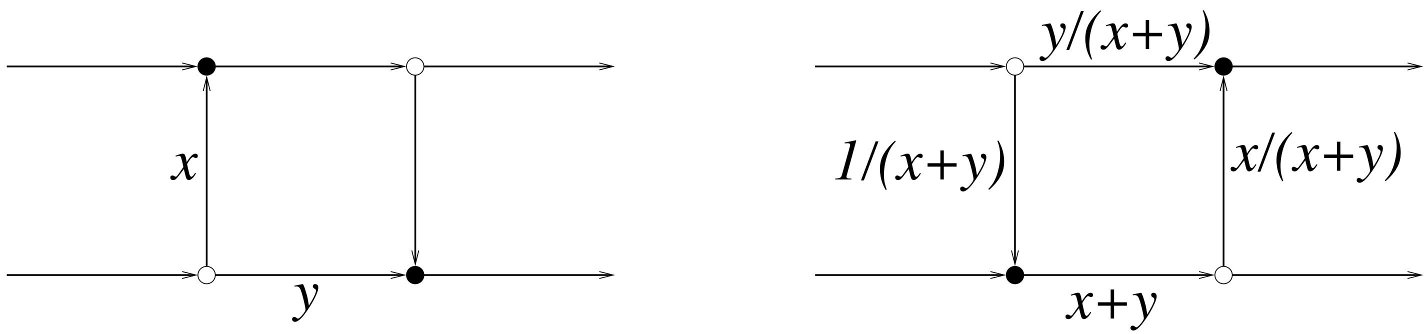


Consider a network drawn on the torus. Example, $k = 3, n = 5$:



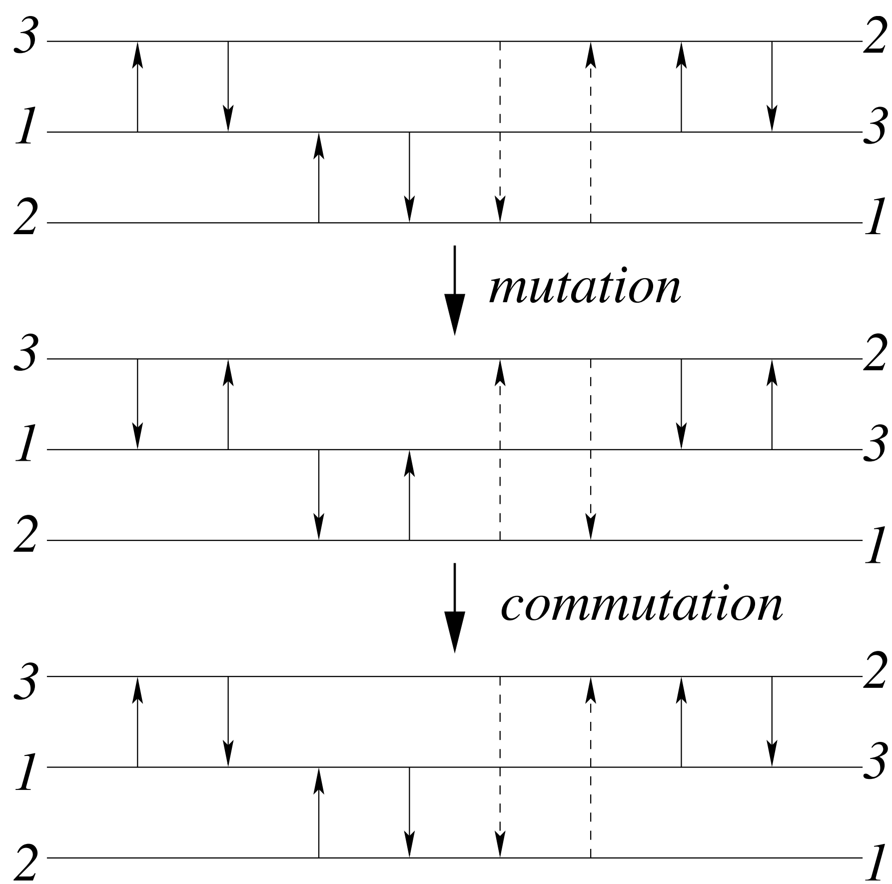
The faces are squares and octagons.

The dynamics: mutation (Postinov type 3 move on squares),



followed by the Postnikov type 1 and 2 moves on the white-white and black-black edges, including moving across the vertical cut, and finally, re-calibration to restore 1s at the appropriate places. These moves preserve the graph and the conjugacy class of the boundary measurement matrix.

Schematically:



Complete integrability of the maps T_k

The ingredients are suggested by the combinatorics of the network.

Invariant Poisson bracket (in the "stable range" $n \geq 2k-1$):

$\{x_i, x_{i+l}\} = -x_i x_{i+l}, 1 \leq l \leq k-2$; $\{y_i, y_{i+l}\} = -y_i y_{i+l}, 1 \leq l \leq k-1$;
$\{y_i, x_{i+l}\} = -y_i x_{i+l}, 1 \leq l \leq k-1$; $\{y_i, x_{i-l}\} = y_i x_{i-l}, 0 \leq l \leq k-2$;
the indices are cyclic.

The functions $\prod x_i$ and $\prod y_i$ are Casimirs. If n is even and k is odd, one has four Casimir functions:

$$\prod_{i \text{ even}} x_i, \quad \prod_{i \text{ odd}} x_i, \quad \prod_{i \text{ even}} y_i, \quad \prod_{i \text{ odd}} y_i.$$

Lax matrices, monodromy, integrals: for $k \geq 3$,

$$L_i = \begin{pmatrix} 0 & 0 & 0 & \dots & x_i & x_i + y_i \\ \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix},$$

and for $k = 2$,

$$L_i = \begin{pmatrix} \lambda x_i & x_i + y_i \\ \lambda & 1 \end{pmatrix}.$$

The boundary measurement matrix is $M(\lambda) = L_1 \cdots L_n$. The characteristic polynomial

$$\det(M(\lambda) - z) = \sum I_{ij}(x, y) z^i \lambda^j.$$

is T_k-invariant: the integrals I_{ij} are in involution.

Geometric interpretations

Twisted corrugated polygons in $\mathbf{RP}^{k-1}$ and $k-1$-diagonal maps

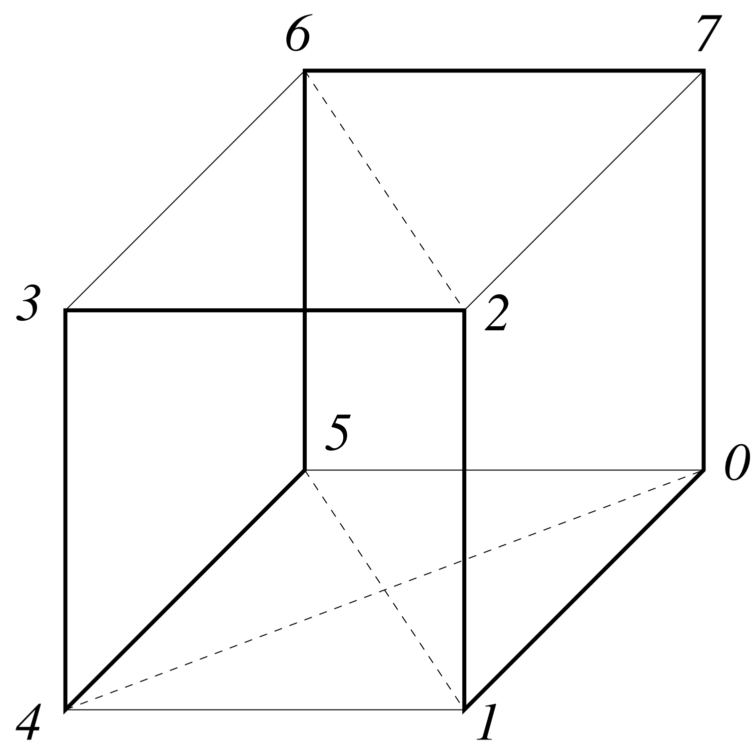
Let $k \geq 3$. $\mathcal{P}_{k,n}$ = projective equivalence classes of generic twisted n-gons in $\mathbf{RP}^{k-1}$; dim $\mathcal{P}_{k,n} = n(k-1)$.

$\mathcal{P}^c_{k,n} \subset \mathcal{P}_{k,n}$ consist of *corrugated* polygons: for every i, the vertices V_i, V_{i+1}, V_{i+k-1} and V_{i+k} are coplanar.

A polygon, projectively dual to a corrugated polygon, is corrugated.

The consecutive $k-1$-diagonals of a corrugated polygon intersect. The resulting polygon is again corrugated. One gets a pentagram-like $k-1$-diagonal map on $\mathcal{P}^c_{k,n}$. For $k=3$, this is the pentagram map.

Example: a 5-corrugated octagon

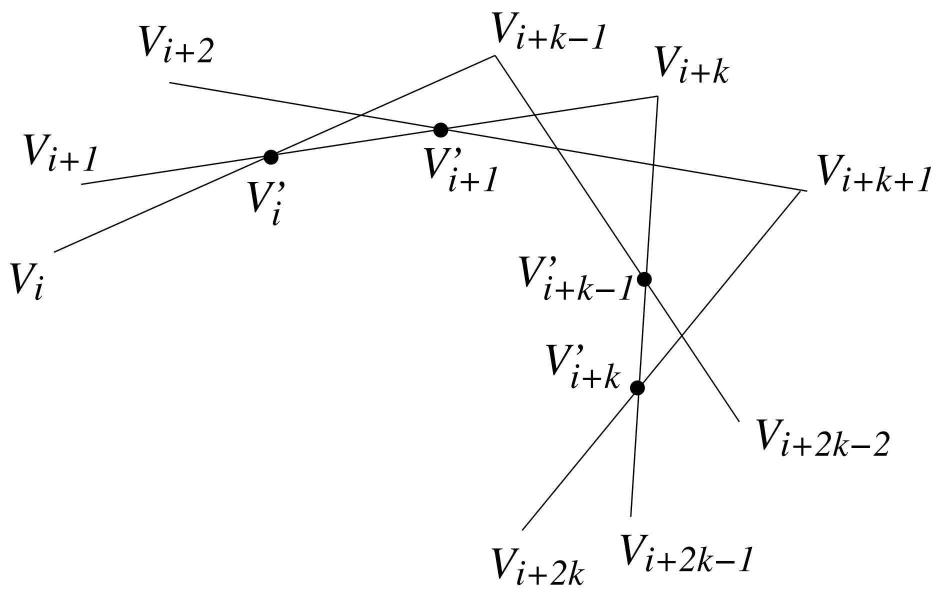


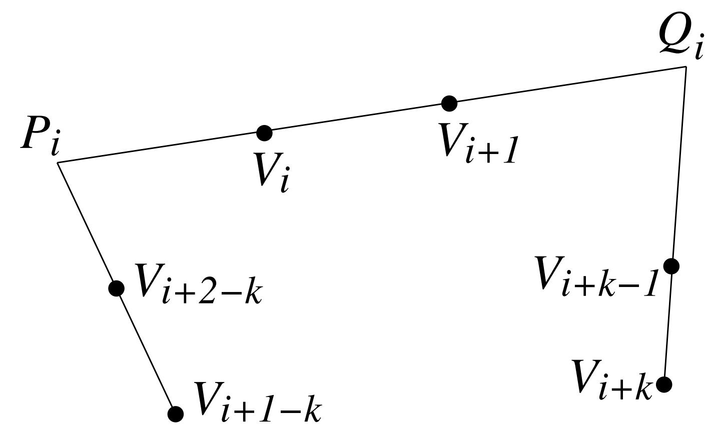
Coordinates: lift the vertices V_i of a corrugated polygon to vectors $\widetilde{V}_i$ in $\mathbf{R}^k$ so that the linear recurrence holds

$$\widetilde{V}_{i+k} = y_{i-1}\widetilde{V}_i + x_i\widetilde{V}_{i+1} + \widetilde{V}_{i+k-1},$$

where x_i and y_i are n-periodic sequences. These are coordinates in $\mathcal{P}^c_{k,n}$. In these coordinates, the map is identified with T_k.

Projective interpretation. The coordinates x, y are determined by two cross-ratios:





$$[V_{i+1}, V_i', V_{i+1}', V_{i+k}] \quad \text{and} \quad [P_i, V_i, V_{i+1}, Q_i].$$

Case $k = 2$

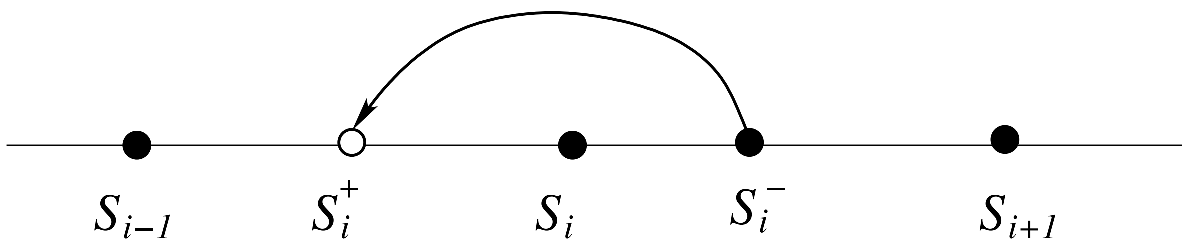
$\mathcal{S}_n$ is the space of pairs of twisted n-gons (S^-, S) in $\mathbf{RP}^1$ with the same monodromy. Consider the (multiple) cross-ratios:

$$x_i = \frac{(S_{i+1} - S^-_{i+2})(S^-_i - S^-_{i+1})}{(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}$$

$$y_i = \frac{(S^-_{i+1} - S_{i+1})(S^-_{i+2} - S_{i+2})(S^-_i - S^-_{i+1})}{(S^-_{i+1} - S_{i+2})(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}.$$

Then x_i, y_i are coordinates in $\mathcal{S}_n/PGL(2, \mathbf{R})$.

Define a transformation $(S^-, S) \mapsto (S, S^+)$, where S^+ is given by the following local leapfrog rule: given points $S_{i-1}, S_i^-, S_i, S_{i+1}$, the point S_i^+ is obtained by the reflection of S_i^- in S_i in the projective metric on the segment $[S_{i-1}, S_{i+1}]$:



The projective distance between points S_i and S_i^- on a segment $S_{i-1}S_{i+1}$ is as given by the formula

$$d(S_i, S_i^-) = \frac{1}{2} \ln \frac{(S_i^- - S_{i-1})(S_{i+1} - S_i)}{(S_i - S_{i-1})(S_{i+1} - S_i^-)}.$$

The projection to x, y conjugates the leapfrog map and T_2.

In formulas (equivalently):

$$\frac{1}{S_i^+ - S_i} + \frac{1}{S_i^- - S_i} = \frac{1}{S_{i+1} - S_i} + \frac{1}{S_{i-1} - S_i},$$

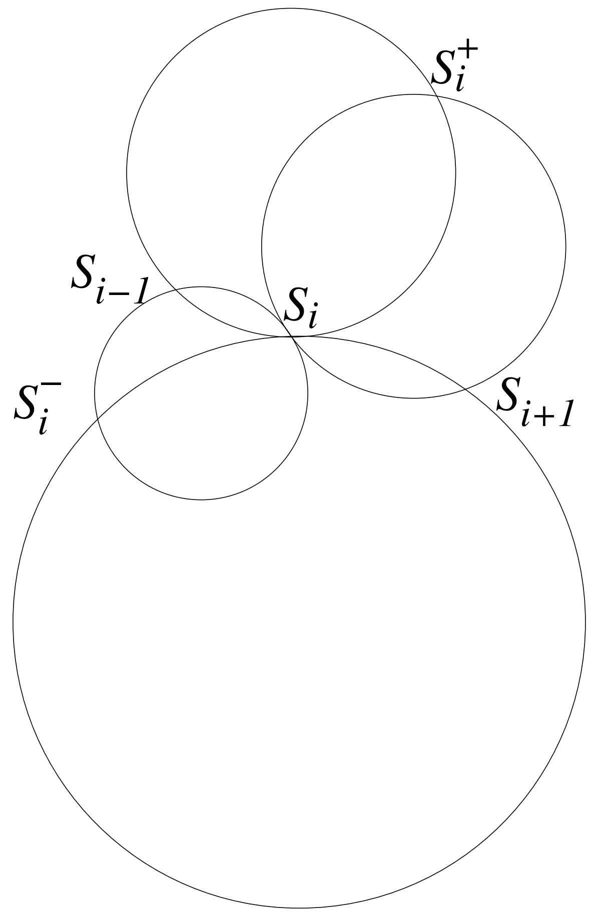
or

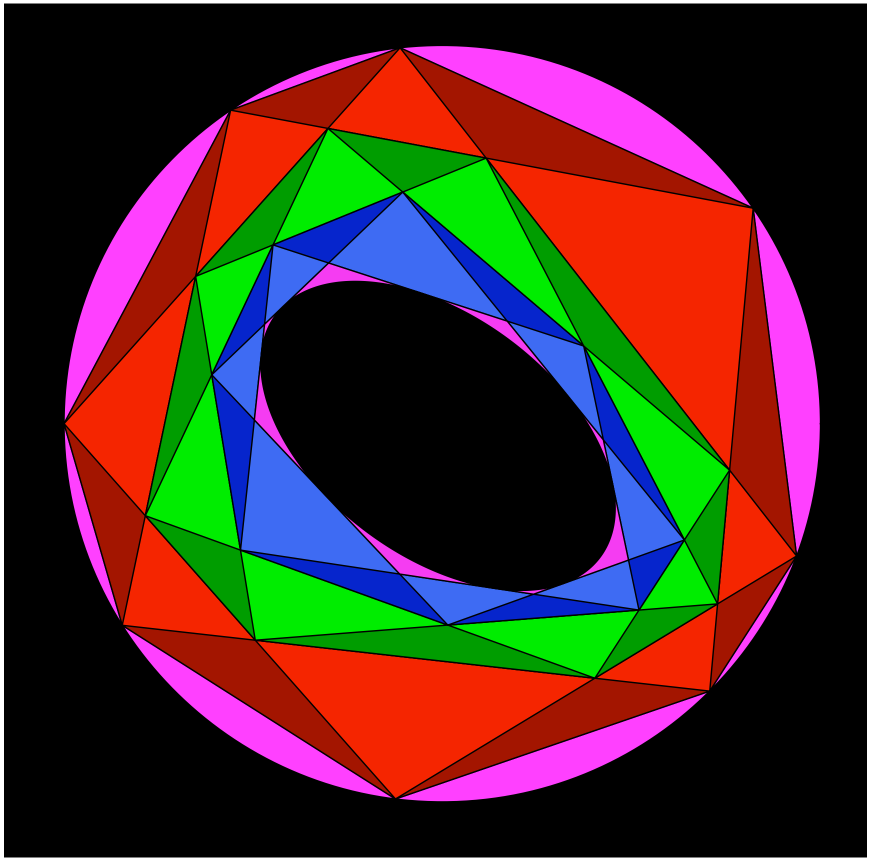
$$\frac{(S_i^+ - S_{i+1})(S_i - S_i^-)(S_i - S_{i-1})}{(S_i^+ - S_i)(S_{i+1} - S_i)(S_i^- - S_{i-1})} = -1,$$

or

$$\frac{(S_i^+ - S_{i-1})(S_i - S_i^-)(S_{i+1} - S_i)}{(S_i^+ - S_i)(S_i - S_{i-1})(S_i^- - S_{i+1})} = -1.$$

In $\mathbf{CP}^1$, a circle pattern interpretation:





Thank you!