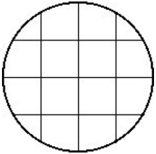
Igusa integrals and volume asymptotics in analytic and adelic geometry

joint work with A. Chambert-Loir

Counting lattice points



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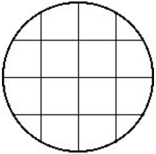

Basic observation

$\#$ of lattice points $\sim$ volume $+$ error term

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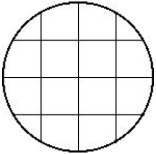

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- compute the volume

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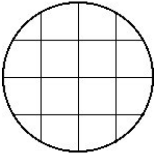

Basic observation

$\#$ of lattice points $\sim$ volume $+$ error term

Basic problems

- compute the volume
- prove that the error term is smaller than the main term

Rational points on $\mathbb{P}^1$

$$\mathbb{P}^1(\mathbb{Q}) = \{\mathbf{x} = (x_0, x_1) \in (\mathbb{Z}^2 \setminus 0)/\pm \mid \gcd(x_0, x_1) = 1\}$$

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Height function

$$\begin{array}{rccc} H\colon & \mathbb{P}^1(\mathbb{Q}) & \to & \mathbb{R}_{>0} \\ & \mathbf{x} & \mapsto & \sqrt{x_0^2 + x_1^2} \end{array}$$

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$$N(B) := \#\{\mathbf{x} \mid H(\mathbf{x}) \leq B\} \sim \frac{1}{2} \cdot \frac{1}{\zeta(2)} \cdot \pi \cdot B^2, \quad B \to \infty$$

Leading constant

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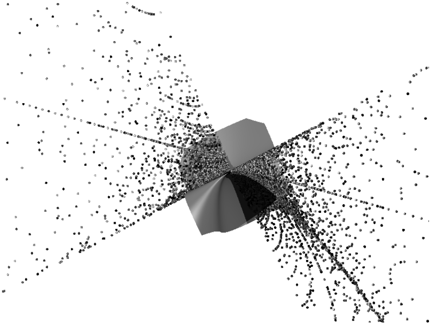
We will interpret this as a volume with respect to a natural regularized measure on the adelic space $\mathbb{P}^1(\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}})$.

Cubic forms



Points of height ≤ 1000 on the $\mathbf{E}_6$ singular cubic surface $X \subset \mathbb{P}^3$

$$x_1 x_2^2 + x_2 x_0^2 + x_3^3 = 0,$$

with $x_0, x_2 > 0$.

Counting points

Let $X^\circ := X \setminus \mathfrak{l}$, the unique line on X given by $x_2 = x_3 = 0$.

Derenthal (2005)

$$N(X^\circ(\mathbb{Q}), B) \sim c \cdot B \log(B)^6, \quad B \to \infty.$$

Leading constant

$$c = \alpha \cdot \beta \cdot \tau$$

where

- $\alpha = \frac{1}{6220800}$
- $\beta = 1$
- $\tau = \prod_p \tau_p \cdot \tau_\infty$ with

$$\tau_p = \frac{(p^2 + 7p + 1)}{p^2} \cdot (1 - \frac{1}{p})^7 = \frac{\#X(\mathbb{F}_p)}{p^2} \cdot (1 - \frac{1}{p})^7$$

$$\tau_\infty = 6 \int_{|tv^3| \leq 1, |t^2 + u^3| \leq 1, 0 \leq v \leq 1, |uv^4| \leq 1} \mathrm{d}t \mathrm{d}u \mathrm{d}v$$

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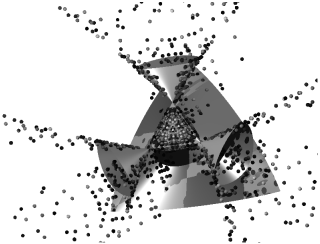

Points of height ≤ 50 on the Cayley cubic surface $(4\mathbf{A}_1)$ $X \subset \mathbb{P}^3$

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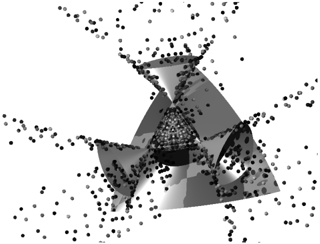

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Many recent results on asymptotics of points of bounded height on cubic surfaces and other Del Pezzo surfaces (Batyrev-Tschinkel, Browning, Derenthal, de la Breteche, Fouvry, Heath-Brown, Moroz, Salberger, Swinnerton-Dyer, ...)

The framework: Manin's conjecture

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We do not know, in general, whether or not $X(F)$ is Zariski dense, even after a finite extension of F. Potential density of rational points has been proved for some families of Fano varieties, but is still open, e.g., for the quintic hypersurface $X_5 \subset \mathbb{P}^5$.

Algebraic groups

(G, ρ, V):

- G a (connected) linear algebraic group over F
- V a finite-dimensional F-vector space
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Example

There exists a Zariski open G-orbit in V, with complement $D \subset V$. Such triples (G, ρ, V) are called prehomogeneous vector spaces.

Algebraic flows

Data:

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- V a finite-dimensional vector space over F
- $\rho : \mathsf{G} \to \mathrm{End}(V)$ an algebraic representation
- fix $x \in V$ and consider the "flow" $\rho(\mathsf{G}) \cdot x$
- $H : V(F) \to \mathbb{R}_{>0}$ - height
- $\{\gamma \in \mathsf{G}(\mathfrak{o}_F) \mid H(\rho(\gamma) \cdot x) \leq B\}$

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Arithmetic problem:
Count $\mathfrak{o}_F$-integral (or F-rational points) on G/H, where H is the stabilizer of x.

Some results

Rational points: (Franke-Manin-T.) G/P; (Strauch) twisted products of G/P; (Batyrev-T.) $X \supset$ T; (Strauch-T.) $X \supset$ G/U; (Chambert-Loir-T.) $X \supset \mathbb{G}_a^n$; (Shalika-T.) $X \supset$ U (bi-equivariant); (Shalika-Takloo-Bighash-T.) $X \supset$ G, De Concini-Procesi varieties

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In all cases, Manin's conjecture, and its refinements by Batyrev-Manin, Peyre, Batyrev-T. hold.

Integral points on G/H: Duke-Rudnick-Sarnak; Eskin-McMullen; Eskin-Mozes-Shah; Borovoi-Rudnick; Gorodnik, Maucourant, Oh, Shah, Nevo, Weiss

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Nevertheless, one has to address the following

Problem

Compute these volumes.

Example

Consider the set $V_P(\mathbb{Z})$ of integral 2×2-matrices M with characteristic polynomial

$$P(X) := X^2 + 1.$$

Put

$$\|M\| = \| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

The volume of the "height ball" is given by $c \cdot B$, where

$$c = \zeta^*_{\mathbb{Q}(\sqrt{-1})}(1) \cdot \frac{\pi^{1/2}}{\Gamma(3/2)} \cdot \frac{\pi}{\Gamma(2/2)\zeta(2)}.$$

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The number of integral matrices in the ball of radius B converges to the volume.

Matrices with fixed characteristic polynomial

Eskin-Moses-Shah (1996), Shah (2000)

For general

$$V_P := \{M \in \mathrm{Mat}_n \mid \det(X \cdot Id - M) = P(X)\},$$

where P has n distinct roots, one has

$$\#\{M \in V_P(\mathbb{Z}) \mid \|\mathrm{M}\| \leq B\} \sim c_P \cdot B^m, \quad m = n(n-1)/2,$$

where

$$c_P = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{D}} \cdot \frac{\pi^{m/2}/\Gamma(1+(m/2))}{\prod_{j=2}^{n} \pi^{-j/2}\Gamma(j/2)\zeta(j)}$$

Volume asymptotics

Maucourant (2004)

Let G be a semi-simple (real) Lie group with trivial character, μ a Haar measure on G, V a finite-dimensional vector space over $\mathbb{R}$, and $\rho\colon G \to V$ a faithful representation. Let $\|\cdot\|$ be a norm on V. Then

$$\mathrm{vol}(B) = \mu(\{g \in G \mid \|\rho(g)\| \leq B\}) \sim c \cdot B^a \log(B)^{b-1}, \quad B \to \infty,$$

where a, b are defined in terms of the relative root system of G and the weights of ρ, and $1 \leq b \leq \mathrm{rank}_{\mathbb{R}}(G)$.

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$$\mathrm{vol}(B)^{-1} \cdot \int_{\|\rho(g)\| \leq B} f(\rho(g)) \mathrm{d}\mu(g) \to \int_{\mathbb{P}\mathrm{End}(V)} f(\rho(g)) \mathrm{d}\mu_\infty(g),$$

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where the limit measure μ_∞ is supported on a G bi-invariant submanifold of $\mathbb{P}\mathrm{End}(V)$.

The proof uses the $K\mathfrak{a}^+K$-decomposition and integration formula.

Difficulties

The computation of asymptotics of volumes of adelic "height balls" was an open problem, in many cases.

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- applicable to cubic surfaces and algebraic groups,
- applicable in the study of rational and integral points.

Heights

- $F/\mathbb{Q}$ number field
- $X = X_F$ projective algebraic variety over F
- $X(F)$ its F-rational points
- $\mathcal{L} = (L, (\| \cdot \|_v))$ adelically metrized very ample line bundle
- $H_{\mathcal{L}} : X(F) \to \mathbb{R}_{>0}$ associated height,
  depends on the metrization (choice of norms)
- $H_{\mathcal{L}}$ is not invariant with respect to field extensions
- $H_{\mathcal{L}+\mathcal{L}'} = H_{\mathcal{L}} \cdot H_{\mathcal{L}'}$ (height formalism)

Tamagawa numbers / Peyre (1995)

Let X be a smooth projective Fano variety of dimension d over a number field F. Assume that $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood U_x. In U_x, a section of the canonical line bundle has the form $\mathsf{s} := \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. Put

$$\omega_{\mathcal{K}_X, v} := \|\mathsf{s}\|_v \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

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where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. This local measure globalizes to $X(F_v)$. For almost all v,

$$\int_{X(F_v)} \omega_{\mathcal{K}_X,v} = \frac{X(\mathbb{F}_q)}{q^d}.$$

Tamagawa numbers / Peyre

Choose a finite set of places S, and put

$$\omega_{\mathcal{K}_X} := L_S^*(1, \mathrm{Pic}(\bar{X})) \cdot |\mathrm{disc}(F)|^{-1} \cdot \prod_v \lambda_v \omega_{\mathcal{K}_X, v},$$

with $\lambda_v = L_v(1, \mathrm{Pic}(\bar{X}))^{-1}$ for $v \notin S$ and $\lambda_v = 1$, otherwise. Put

$$\tau(\mathcal{K}_X) := \int_{\overline{X(F)} \subset X(\mathbb{A}_F)} \omega_{\mathcal{K}_X}.$$

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This constant appears in the contant $c = c(-\mathcal{K}_X)$ in Manin's conjecture above.

Tamagawa numbers / local theory

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A metrization of $K_X(D)$ defines a measure on $U(F)$

$$\tau_{(X,D)} = |\omega|/\|\omega f_D\|$$

Example

When X is an equivariant compactification of an algebraic group G and ω a left-invariant differential form on G, we have $\mathrm{div}(\omega) = -D$, so that $K_X(D)$ is a trivial line bundle, equipped with a canonical metrization. We may assume that its section ωf_D has norm 1. Then

$$\tau_{(X,D)} = |\omega| / \|\omega f_D\| = |\omega|$$

is a Haar measure on $\mathrm{G}(F)$.

Height balls

Let L be an effective divisor with support $|D| = X \setminus U$, equipped with a metrization. Then

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is a height ball, i.e., it is compact of finite measure $\mathrm{vol}(B)$.

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$$Z(s) := \int_0^\infty t^{-s} \mathrm{dvol}(t) = \int_{U(F)} \|f_L\|^s \tau_{(X,D)},$$

combined with a Tauberian theorem.

Igusa zeta functions / local theory

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$$D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha.$$

Local computations

The Mellin transform $Z(s)$ can be computed in charts, via partition of unity. In a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_\alpha \|\mathsf{f}_{D_\alpha}\|(x)^{\lambda_\alpha s - \rho_\alpha} \,\mathrm{d}\tau_X(x) = \int \prod_{\alpha \in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \phi(x; y; s) \prod_\alpha \mathrm{d}x_\alpha \,\mathrm{d}y.$$

Local computations

The Mellin transform $Z(s)$ can be computed in charts, via partition of unity. In a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_\alpha \|\mathsf{f}_{D_\alpha}\|(x)^{\lambda_\alpha s - \rho_\alpha}\, \mathrm{d}\tau_X(x) = \int \prod_{\alpha \in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \phi(x; y; s) \prod_\alpha \mathrm{d}x_\alpha\, \mathrm{d}y.$$

Essentially, this is a product of integrals of the form

$$\int_{|x| \leq 1} |x|^{s-1} \mathrm{d}x.$$

Analytic properties of $Z(s)$ are encoded in the combinatorics of the stratification (D_A).

Igusa zeta functions / local theory

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Leading coefficient = sum of integrals over all D_A of minimal dimension where A consists only of such αs.

Global theory

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$$\mathrm{H}^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0.$$

Let

$$\mathrm{EP}(U) = \Gamma(U_{\bar{\mathbb{F}}}, \mathscr{O}_X^*)/\bar{\mathbb{F}}^* - \mathrm{Pic}(U_{\bar{\mathbb{F}}})/\textbf{torsion}$$

be the virtual Galois module.

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be the virtual Galois module. Put

$$\lambda_v = L_v(1, \mathrm{EP}(U)), \quad v \nmid \infty, \quad \lambda_v = 1, \quad v \mid \infty.$$

We have a global measure on $U(\mathbb{A}_F)$ given by

$$\tau_{(X,D)} = L^*(1, \mathrm{EP}(U))^{-1} \cdot \prod_v \lambda_v \tau_{(X,D),v}$$

Height on the adelic space $U(\mathbb{A}_F)$

Let $\mathcal{L} = (L, (\|\cdot\|_v))$ be an adelically metrized effective divisor supported on $|D|$. This defines a height function on $U(\mathbb{A}_F)$

$$H_{\mathcal{L}}((x_v)) = \prod_v \|\mathsf{f}_L(x_v)\|_v^{-1}.$$

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To compute the volume of the height ball

$$\mathrm{vol}(B) := \{x \in U(\mathbb{A}_F) \mid H_{\mathcal{L}}(x) \leq B\},$$

for $\mathcal{L}$ and $\tau_{(X,D)}$, we use the adelic Mellin transform:

$$Z(s) = \int_0^\infty t^{-s} \mathrm{dvol}(t) = \int_{U(\mathbb{A}_F)} H_{\mathcal{L}}(x)^{-s} \, \mathrm{d}\tau_{(X,D)}(x) = \prod_v \int_{U(F_v)} \cdots.$$

Denef's formula

Recall that

$$D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha.$$

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For almost all v and $\Re(s) > (\rho_\alpha - 1)/\lambda_\alpha$, one has

$$Z_v(s) = \sum_A \frac{\#D_A^\circ(\mathbb{F}_q)}{q^{\dim X}} \prod_{\alpha \in A} \frac{q-1}{q^{s\lambda_\alpha - \rho_\alpha + 1} - 1}.$$

Analyzing the Euler product

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$$\lim_{s \to a} Z(s)(s-a)^b \prod_{\alpha \in A(L,D)} \lambda_\alpha = \int_{X(\mathbb{A}_F)} H_E(x)^{-1} \, \mathrm{d}\tau_X(x).$$

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A Tauberian theorem implies the volume asymptotics with respect to $\mathcal{L}$ and $\tau_{(X,D)}$, for $B \to \infty$, of the form

$$B^a \log(B)^{b-1} \left(a(b-1)! \prod_{\alpha\in A(L,D)} \lambda_\alpha \right)^{-1} \int_{X(\mathbb{A}_F)} H_E(x)^{-1}\,\mathrm{d}\tau_X(x).$$

Integral points

- F number field, $\mathfrak{O}_F$ ring of integers
- S finite set of places of F, $S \supset S_\infty$
- X smooth projective variety over F, $D \subset X$ subvariety
- $\mathcal{D} \subset \mathcal{X}$ models over $\mathrm{Spec}(\mathfrak{O}_F)$

A rational point $x \in X(F)$ gives rise to a section

$$\sigma_x : \mathrm{Spec}(\mathfrak{O}_F) \to \mathcal{X}.$$

A $(\mathcal{D}, S)$-integral point on X is a rational point $x \in X(F)$ such that $\sigma_{x,v} \notin \mathcal{D}_v$ for all $v \notin S$.

A sample problem

Let X be a projective equivariant compatification of $G = \mathbb{G}_a^n$, and

$$\cup_{\alpha \in \mathcal{A}} D_\alpha = X \setminus G$$

the boundary divisor, whose irreducible components D_α are smooth and intersect transversally. Choose a subset $\mathcal{A}_D \subseteq \mathcal{A}$ and put $U = X \setminus \cup_{\alpha \in \mathcal{A}_D} D_\alpha$.

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$$X \setminus U = D = \sum_{\alpha \in \mathcal{A}_D} D_\alpha.$$

Let $\mathcal{L}$ be an adelically metrized line bundle on X.

Problem

Establish an asymptotic formula for

$$N(B) := \#\{\gamma \in G(F) \cap U(\mathfrak{O}_{F,S}) \,|\, H_{\mathcal{L}}(\gamma) \leq B\}.$$

Techniques

Height pairing

$$G(\mathbb{A}_F) \quad \times \quad \oplus_\alpha \mathbb{C} D_\alpha \to \mathbb{C}$$

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Height zeta function

$$Z(g, \mathbf{s}) = \sum_{\gamma \in G(F) \cap U(\mathfrak{O}_{F,S})} H(\gamma g, \mathbf{s})^{-1},$$

is holomorphic for $\Re(\mathbf{s}) \gg 0$ and all g.

Techniques

"Fourier" expansion - "Poisson formula"

$$Z(g, \mathbf{s}) = \sum_{\psi} \hat{H}(\mathsf{s}, \psi),$$

a sum over all (automorphic) characters of $G(\mathbb{A}_F)/G(F)$.

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a volume integral computed above.

Asymptotics

For $L = -(K_X + D)$ we obtain

Chambert-Loir–T. (2009)

$$N(B) \sim c \cdot B \log(B)^{b-1},$$

$$b := \mathrm{rk}(\mathrm{Pic}(U)) + \sum_{v \in S} (1 + \dim \mathcal{C}^{\mathrm{an}}_{F_v}(D)),$$

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the analytic Clemens complex of the stratification of D, and

$$c = \alpha\beta\tau,$$

- $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$;
- $\tau = \tau^S_{(X,D)}(U(\mathcal{O}_S)) \cdot \prod_{v \in S} \left(\sum_{\sigma \in \mathcal{C}^{\mathrm{an}}_{\max, F_v}(D_v)} \tau_v(\sigma) \right)$
- $\tau_v(\sigma)$ Tamagawa volume of σ, (adjunction!).

Contributions from nontrivial characters

$$\hat{H}(\mathbf{s}, \psi) = \int_{G(\mathbb{A}_F)} H(g, \mathbf{s})^{-1} \psi(g) \mathrm{d}g.$$

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Only unramified ψ appear. Uniform bounds needed for summation over the lattice of these ψ are (relatively) easy to obtain.

Complications for integral points

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$$\int_{\sigma} \prod_{\alpha} |x_{\alpha}|^{s_{\alpha}} \psi(u(\mathbf{x})\mathbf{x}^{\lambda}) \phi(\mathbf{x}, \mathbf{s}, \psi) \mathrm{d}x,$$

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Similar integrals appeared in the work of Cluckers (2010) on *Analytic van der Corput Lemma*....

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- The spectral method to establish asymptotics for the number of integral points of bounded height leads to interesting v-adic oscillatory integrals. This should allow to establish asymptotics for $\mathfrak{O}_{F,S}$-integral points on general quasi-projective embeddings of algebraic groups.
- A framework to generalize Manin's conjectures to integral points.