The Geometry of Ricci Curvature

Aaron Naber

Talk Outline

- Background
- Lower Ricci Curvature
- Bounded Ricci Curvature
- Examples and Degeneracies
- Extension of Ideas to Other Areas

BACKGROUND

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- For a point $x \in M$ and $r > 0$, we denote by $B_r(x)$ the ball of radius r centered at x. We denote the volume of this ball by $Vol(B_r(x))$.

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- Definition: We say a metric space X_x is a tangent cone of X at x if there exists a sequence $r_i \to 0$ such that $X_{(x,r_i)} \to X_x$.

Background: Stratification

- Given a limit space X one decomposes X into pieces based on the conical behavior of the tangent cones. Rigorously Speaking:
- Definition: We say a metric space X is 0-conical if X=C(Y) is the cone of a metric space Y. We say X is k-conical if $X=R^k\times C(Y)$.
- - For a limit space X every tangent cone X_x is 0-conical.
- Definition: For a limit space X we define the strata $S^k(X) \equiv$ {x \in X: no tangent cone at x is k+1-conical}.
- Its known that dim $S^k \leq k$, where the dimension is the Hausdorff dimension.

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- 4. If $\sec_i > -K$ and $\mathrm{Vol}(B_1(p_i)) \to 0$, then X is homeomorphic to a stratified Riemannian manifold (Perelman). The Isometry Group is a Lie Group (Fukaya-Yamaguchi).

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- If collapsed, $Vol(B_1(p_i)) \to 0$, then there exists a set of full measure $\mathcal{R} \subseteq X$ such that every tangent cone at $x \in \mathcal{R}$ is *some* Euclidean space $R^{k(x)}$, where $0<k(x)<n$.

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- Similar results hold in higher dimensions if $\chi(M_i) < K$ is replaced with $\int |Rm|^{n/2} < K$ (Anderson).
- Under only $|Rc_i| \leq K$ and $Vol(M_i) > v > 0$, it can be said X is a Riemannian manifold away from a set of Hausdorff codim 2. If in addition one assumes $\int |Rm|^{p/2} < K$, then one can say X is a Riemannian manifold away from a set of Hausdorff codim p.

LOWER RICCI CURVATURE

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- Theorem (Petrunin): Let X be an Alexandroff space (e.g. $\sec_i > -K$) and $\gamma:[0,1]\to X$ a minimizing geodesic on X, then if X_0 and X_1 are tangent cones at $\gamma(t_0)$ and $\gamma(t_1)$ with $t_0,t_1 \in (0,1)$, then $X_0 \equiv X_1$ are isometric.

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- \- Tangent cones change at most at a Holder rate. In particular they change in a continuous fashion.
  \- For all intensive purposes α is $1/2$. -
  Holder rate is sharp. Tangent cones do not need to change at a Lipschitz rate, or even a C^{α} rate for $\alpha > 1/2$. -
  Effective version which says $\forall\ r > 0$ that $r^{-1} d_{GH}(B_r(\gamma(s)), B_r(\gamma(t))) < C|t-s|^{\alpha}$

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- New Estimates: Flow d by the heat equation to get smooth approximation h_t .
  - Previously known: For harmonic approximation h one has $\mathrm{Vol}(B_r)^{-1}\int_{B_r(\gamma(t))} |\nabla^2 h|^2 < C r^{2-\varepsilon}$. At best => $r^{-1} d_{GH}(B_r(\gamma(s)), B_r(\gamma(t))) < C|t-s|^{\alpha} r^{-1+\varepsilon/2}$. - Need $\varepsilon = 2$. Can (essentially) be proved for parabolic approximation h_{r^2}, but wrong for harmonic approximation h.

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- - Curvature blows up at most polynomially along a minimizing geodesic.

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- Question 2. Is an assumption on L^p control on the curvature necessary?

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- - Previously theorems do not prove *any* effective lower bounds for r(x).

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- \- We will prove stronger Minkowski estimates on this effective set, as opposed to the weaker Hausdorff estimates proved for the ineffective standard singular set.

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- - Controls *tubes* around the singular set.
- - Proof requires new ideas besides those of the standard dimension reduction for the estimate $\dim S^k \leq k$. In fact, the above gives a new and distinct proof of this estimate.

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- - Can prove the theorem for each E_{T_α} . Apriori there may seem to be 2^{α} such sets, but previous bound says there are only α^N such sets. Hence can add error up for all such sets to prove the theorem.

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  \- 4) for Kahler Einstein with bounded chern classes $\mathrm{Vol}(B_r(\mathrm{Sing}) \cap B_1) \le C\, r^4$
  \- Improves Hausdorff dimension estimates to effective Minkowski dimension estimates
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- Conjecture(Na): There exists $K(n,\nu)>0$ such that $\int_{B_1(p)} |Rm|^2 \le K$, and hence $\mathrm{Vol}(\{r(x)\le r\} \cap B_1) \le C\, r^4$.

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Examples and Degeneracies

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- 3. A tangent cone $X_p = C(Y)$ is a metric cone over a compact metric space Y.

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- Question: If these are necessary conditions, are they sufficient?

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- Primary application of the above is to construct new examples of limit spaces with various degenerate behaviors.

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- 1) It is the first example of a three dimensional limit space with nonunique tangent cones at some point $p \in X$.
- 2) It is the first example of a limit space where at a point $p \in X$, some tangent cones are smooth, and others are not.

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- Similarly let Ω be a smooth family on S^{n-1} which vary from a small sphere to the spaces $Sus^k(S^{n-k-1})$,where Sus is the suspension of a space (so $Sus^k(S^{n-k-1})$ is the k-th suspension of a n-k-1 sphere).

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- 1) At $p \in X$ there are tangent cones of the form $R^k \times C(Y)$ for all $0 \leq k \leq n-2$, where Y is a smooth space.
- - In particular the dimension of the singular set of a tangent cone is not an invariant of the point in question.
  - Hence, one cannot stratify (in the sense of a stratified space) limit spaces which are limits with only lower Ricci bounds.

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- Conjecture: Given a noncollapsed limit space X^n, the set of points *NH* where the tangent cones are not homeomorphic satisfies dim $NH \leq$ n-5.

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