How much of the Hilbert function do we really need to know?

János Kollár

Princeton University

April, 2015

Main question

- $(X, \mathcal{O}_X(1))$ projective scheme,
- F coherent sheaf on X.
- Basic numertical invariant: $\chi(X, F(t)) \in \mathbb{Q}[t]$.

Main question

- $(X, \mathcal{O}_X(1))$ projective scheme,
- F coherent sheaf on X.
- Basic numertical invariant: $\chi(X, F(t)) \in \mathbb{Q}[t]$.

Problem: We usually understand only a few of the coefficients of $\chi(X, F(t))$.
(Top one or two and the constant.)

Main question

– $(X, \mathcal{O}_X(1))$ projective scheme,
– F coherent sheaf on X.
– Basic numertical invariant: $\chi(X, F(t)) \in \mathbb{Q}[t]$.

Problem: We usually understand only a few of the coefficients of $\chi(X, F(t))$.
(Top one or two and the constant.)

Do we need the others?

Theme 1: Hilbert functions and volumes

– X normal variety, D a divisor or $\mathbb{R}$-divisor,

– $Hilb(X, D)(t) := h^0(X, \mathcal{O}_X(\lfloor tD \rfloor))$ for $t \geq 0$,

– $vol(X, D) = \lim_{t\to\infty} h^0(X, \mathcal{O}_X(\lfloor tD \rfloor)/(t^n/n!)$.

Theme 1: Hilbert functions and volumes

– X normal variety, D a divisor or $\mathbb{R}$-divisor,
– $Hilb(X, D)(t) := h^0(X, \mathcal{O}_X(\lfloor tD \rfloor))$ for $t \geq 0$,
– $vol(X, D) = \lim_{t \to \infty} h^0(X, \mathcal{O}_X(\lfloor tD \rfloor)/(t^n/n!)$.

Clear: E effective $\mathbb{R}$-divisor then

$$Hilb(X, D - E)(t) \leq Hilb(X, D)(t) \leq Hilb(X, D + E)(t).$$

Theme 1: Hilbert functions and volumes

- X normal variety, D a divisor or $\mathbb{R}$-divisor,
- $Hilb(X, D)(t) := h^0(X, \mathcal{O}_X(\lfloor tD \rfloor)$ for $t \geq 0$,
- $vol(X, D) = \lim_{t\to\infty} h^0(X, \mathcal{O}_X(\lfloor tD \rfloor)/(t^n/n!)$.

Clear: E effective $\mathbb{R}$-divisor then

$$Hilb(X, D - E)(t) \leq Hilb(X, D)(t) \leq Hilb(X, D + E)(t).$$

This implies that

$$vol(X, D - E) \leq vol(X, D) \leq vol(X, D + E).$$

Theme 1: Hilbert functions and volumes

– X normal variety, D a divisor or $\mathbb{R}$-divisor,
– $Hilb(X, D)(t) := h^0(X, \mathcal{O}_X(\lfloor tD \rfloor))$ for $t \geq 0$,
– $vol(X, D) = \lim_{t \to \infty} h^0(X, \mathcal{O}_X(\lfloor tD \rfloor))/(t^n/n!)$.

Clear: E effective $\mathbb{R}$-divisor then

$$Hilb(X, D - E)(t) \leq Hilb(X, D)(t) \leq Hilb(X, D + E)(t).$$

This implies that

$$vol(X, D - E) \leq vol(X, D) \leq vol(X, D + E).$$

Question: What if equality holds?

Theorem (Fulger-K.-Lehmann)

Assume D is big and E is effective. Then

$$vol(X, D - E) = vol(X, D) \quad \Leftrightarrow \quad Hilb(X, D - E)(t) \equiv Hilb(X, D)(t).$$

and hence also

$$vol(X, D + E) = vol(X, D) \quad \Leftrightarrow \quad Hilb(X, D + E)(t) \equiv Hilb(X, D)(t).$$

Theorem (Fulger-K.-Lehmann)

Assume D is big and E is effective. Then

$$vol(X, D - E) = vol(X, D) \quad \Leftrightarrow \quad Hilb(X, D - E)(t) \equiv Hilb(X, D)(t).$$

and hence also

$$vol(X, D + E) = vol(X, D) \quad \Leftrightarrow \quad Hilb(X, D + E)(t) \equiv Hilb(X, D)(t).$$

Non-Example: $D|_E \equiv 0$ and $E|_E \equiv 0$ can not happen.

Theme 2: Simultaneous canonical models

$f : X \to S$ with irreducible fibers of general type.

Simultaneous canonical model:
$f^{simcr} : X^{simcr} \to S$, flat, projective such that
$\left(X^{simcr}\right)_s = (X_s)^{cr}$ (:= can. model of resolution of X_s)

Theme 2: Simultaneous canonical models

$f : X \to S$ with irreducible fibers of general type.

Simultaneous canonical model:
$f^{simcr} : X^{simcr} \to S$, flat, projective such that
$\left(X^{simcr}\right)_s = (X_s)^{cr}$ (:= can. model of resolution of X_s)

Old (?) Theorem.

- $s \mapsto H^0\left(X_s^{cr}, \mathcal{O}(mK)\right)$ is **lower** semi-continuous,
- if $f^{simcr} : X^{simcr} \to S$ exists then $s \mapsto H^0\left(X_s^{cr}, \mathcal{O}(mK)\right)$
  is constant for every $m \geq 1$,
- converse also holds if S is reduced.

Simultaneous canonical model: strong form

Theorem

Assume that S is reduced. Equivalent:

- $f^{simcr} : X^{simcr} \to S$ *exists,*
- $s \mapsto H^0(X_s^{cr}, \mathcal{O}(mK_{X_s^{cr}}))$ *is constant* $\forall m \geq 1$,
- $s \mapsto vol(X_s^{cr}, K_{X_s^{cr}})$ *is constant.*
- $s \mapsto vol(X_s^{res}, K_{X_s^{res}})$ *is constant.*

Theme 3: Cartier divisors

Example: lines on families of quadric surfaces.

$Q := (x^2 - y^2 + z^2 - t^2w^2 = 0) \subset \mathbb{P}^3_{xyzw} \times \mathbb{A}^1_t,$

$L_t = (x - y = z - tw = 0)$ and $L'_t = (x + y = z - tw = 0)$.

Theme 3: Cartier divisors

Example: lines on families of quadric surfaces.
$Q := (x^2 - y^2 + z^2 - t^2 w^2 = 0) \subset \mathbb{P}^3_{xyzw} \times \mathbb{A}^1_t$,

$L_t = (x - y = z - tw = 0)$ and $L'_t = (x + y = z - tw = 0)$.

Compute self-intersections:
$(aL_0 + bL'_0)^2 = \frac{1}{2}(a+b)^2$ and $(aL_g + bL'_g)^2 = 2ab$.

Theme 3: Cartier divisors

Example: lines on families of quadric surfaces.

$Q := (x^2 - y^2 + z^2 - t^2 w^2 = 0) \subset \mathbb{P}^3_{xyzw} \times \mathbb{A}^1_t$,

$L_t = (x - y = z - tw = 0)$ and $L'_t = (x + y = z - tw = 0)$.

Compute self-intersections:

$(aL_0 + bL'_0)^2 = \frac{1}{2}(a+b)^2$ and $(aL_g + bL'_g)^2 = 2ab$. So

- $(aL_0 + bL'_0)^2 \geq (aL_g + bL'_g)^2$,
- $aL_t + bL'_t$ Cartier on every fiber iff $a + b$ is even,
- $aL + bL'$ is globally Cartier iff equality holds.

Theorem (Numerical Cartier condition; weak form)

– $f : X \to C$ *is flat, projective, relative dimension* n,

– *normal fibers (for simplicity)*

– D *divisor such that each* D_c *is Cartier and ample.*

Theorem (Numerical Cartier condition; weak form)

– $f : X \to C$ *is flat, projective, relative dimension* n,

– *normal fibers (for simplicity)*

– D *divisor such that each* D_c *is Cartier and ample. Then*

1. $c \mapsto (D_c^n)$ *is upper semi-continuous and*
2. D *is Cartier iff the above function is constant.*

Corollary (Numerical criterion of stability)

– $f : X \to C$ *flat, projective, relative dimension* n,
– *fibers are (semi) log canonical with*
– *ample canonical class* K_{X_c}.

Corollary (Numerical criterion of stability)

– $f : X \to C$ flat, projective, relative dimension n,
– fibers are (semi) log canonical with
– ample canonical class K_{X_c}. Then

1. $c \mapsto (K_{X_c}^n)$ *is upper semi-continuous and*
2. f *is stable iff* $(K_{X_c}^n)$ *is constant.*

Stable $:=$ $K_{X/C}$ is $\mathbb{Q}$-Cartier.

$(K_{X_c}^n)$= volume of X_c with Kähler–Einstein metric.

Numerical Cartier condition (strong form)

- S reduced scheme over a field k,
- $f : X \to S$ flat, proper, pure relative dimension n,
- S_2 fibers,
- $Z \subset X$ such that $Z \cap X_s$ has codim ≥ 2,
- L^0 line bundle on $X \setminus Z$ such that
- $L^0|_{X_s \setminus Z}$ extends to an ample line bundle L_s on X_s.

Numerical Cartier condition (strong form)

- S reduced scheme over a field k,
- $f : X \to S$ flat, proper, pure relative dimension n,
- S_2 fibers,
- $Z \subset X$ such that $Z \cap X_s$ has codim ≥ 2,
- L^0 line bundle on $X \setminus Z$ such that
- $L^0|_{X_s \setminus Z}$ extends to an ample line bundle L_s on X_s. Then

1. $s \mapsto (L_s^n)$ is upper semi-continuous and
2. L^0 extends to a line bundle L on X iff (L_s^n) is constant.

Numerical Cartier condition (strong local form)

- S reduced scheme over a field k,
- $f : X \to S$ flat, projective, pure relative dimension n,
- S_2 fibers.
- $Z \subset X$ such that $Z \cap X_s$ has codim ≥ 2,
- L^0 line bundle on $X \setminus Z$ such that
- $L^0|_{X_s \setminus Z}$ extends to an arbitrary line bundle L_s on X_s.
- H relatively ample on X/S.

Numerical Cartier condition (strong local form)

- S reduced scheme over a field k,
- $f : X \to S$ flat, projective, pure relative dimension n,
- S_2 fibers.
- $Z \subset X$ such that $Z \cap X_s$ has codim ≥ 2,
- L^0 line bundle on $X \setminus Z$ such that
- $L^0|_{X_s \setminus Z}$ extends to an arbitrary line bundle L_s on X_s.
- H relatively ample on X/S. Then

1. $s \mapsto (H_s^{n-2} \cdot L_s^2)$ is upper semi-continuous and
2. L^0 extends to a line bundle L on X iff $(H_s^{n-2} \cdot L_s^2)$ is constant.

Reminder: what general theory says

Old (?) Theorem. Equivalent:

1. L^0 extends to a line bundle L on X.
2. Hilbert pol. $\chi\big(X_s, L_s(m)\big)$ is constant.
3. all the $\big(L_s^i \cdot H_s^j \cdot Td_{n-i-j}(X_s)\big)$ are constant.

Reminder: what general theory says

Old (?) Theorem. Equivalent:

1. L^0 extends to a line bundle L on X.
2. Hilbert pol. $\chi\big(X_s, L_s(m)\big)$ is constant.
3. all the $\big(L_s^i \cdot H_s^j \cdot Td_{n-i-j}(X_s)\big)$ are constant.

New Theorem.
$\big(L_s^2 \cdot H_s^{n-2}\big)$ constant $\quad \Rightarrow$

all the $\big(L_s^i \cdot H_s^j \cdot Td_{n-i-j}(X_s)\big)$ are constant.

Upper semi-continuity over a smooth curve

L^0 extends to a reflexive sheaf L^* and we have

$r_0 : L^*|_{X_0} \hookrightarrow L_0,$ called restriction map.

Upper semi-continuity over a smooth curve

L^0 extends to a reflexive sheaf L^* and we have
$r_0 : L^*|_{X_0} \hookrightarrow L_0$, called restriction map.
By semicontinuity

$$h^0(X_0, L_0) \geq h^0(X_0, L^*|_{X_0}) \geq h^0(X_g, L^*|_{X_g}) = h^0(X_g, L_g).$$

Upper semi-continuity over a smooth curve

L^0 extends to a reflexive sheaf L^* and we have
$r_0 : L^*|_{X_0} \hookrightarrow L_0,$ called restriction map.
By semicontinuity

$$h^0(X_0, L_0) \geq h^0(X_0, L^*|_{X_0}) \geq h^0(X_g, L^*|_{X_g}) = h^0(X_g, L_g).$$

If L_0 and L_g are ample, then
applying it to $(L^0)^{\otimes m}$ and using Riemann–Roch:

$$(L_0)^n = \lim \frac{h^0(X_0, L_0^{\otimes m})}{m^n/n!} \geq \lim \frac{h^0(X_g, L_g^{\otimes m})}{m^n/n!} = (L_g^n).$$

Proof in dimension 2 (L_t need not be ample)

Set $\chi(X_t, L_t^{\otimes m}) = a_t m^2 + b_t m + c_t$.
Cokernel of r_0^m is Artinian, so
$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g$ for every m.

Proof in dimension 2 (L_t need not be ample)

Set $\chi(X_t, L_t^{\otimes m}) = a_t m^2 + b_t m + c_t$.
Cokernel of r_0^m is Artinian, so
$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g$ for every m.

RR: $a_t = \frac{1}{2}(L_t \cdot L_t)$ and $c_t = \chi(X_t, \mathcal{O}_{X_t})$.

If $(L_0 \cdot L_0) = (L_g \cdot L_g)$. Then $a_0 = a_g$ thus
$b_0 m + c_0 \geq b_g m + c_g$ for every m.

Proof in dimension 2 (L_t need not be ample)

Set $\chi\left(X_t, L_t^{\otimes m}\right) = a_t m^2 + b_t m + c_t$.
Cokernel of r_0^m is Artinian, so
$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g$ for every m.

RR: $a_t = \frac{1}{2}\left(L_t \cdot L_t\right)$ and $c_t = \chi\left(X_t, \mathcal{O}_{X_t}\right)$.

If $(L_0 \cdot L_0) = (L_g \cdot L_g)$. Then $a_0 = a_g$ thus
$b_0 m + c_0 \geq b_g m + c_g$ for every m.

$m \gg 1$ gives $b_0 \geq b_g$ and $m \ll -1$ gives $-b_0 \geq -b_g$.
So $b_0 = b_g$ and $c_0 = c_g$ since f is flat.

Strong local form; second look

- $f : X \to S$ flat, projective, pure dim n with S_2 fibers
- L^0 line bundle on $X \setminus Z$
- every L_s line bundle

1. $s \mapsto (H_s^{n-2} \cdot L_s^2)$ is upper semi-continuous and
2. L^0 extends to line bundle L iff constant.

Strong local form; second look

– $f : X \to S$ flat, projective, pure dim n with S_2 fibers
– L^0 line bundle on $X \setminus Z$
– every L_s line bundle

1. $s \mapsto (H_s^{n-2} \cdot L_s^2)$ is upper semi-continuous and
2. L^0 extends to line bundle L iff constant.

Note: $(H_s^{n-2} \cdot$ takes general surface section so

1. upper semi-continuity follows from 2-dim case,

Strong local form; second look

– $f : X \to S$ flat, projective, pure dim n with S_2 fibers
– L^0 line bundle on $X \setminus Z$
– every L_s line bundle

1. $s \mapsto (H_s^{n-2} \cdot L_s^2)$ is upper semi-continuous and
2. L^0 extends to line bundle L iff constant.

Note: $(H_s^{n-2} \cdot$ takes general surface section so

1. upper semi-continuity follows from 2-dim case,
2. codim ≥ 3 singularities do not matter!

Grothendieck–Lefschetz in SGA2

- $(x \in X)$ local scheme, $x \in D \subset X$ Cartier divisor,
- $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$,
- L line bundle on U such that
- $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$.
- Assume that $\operatorname{depth}_x \mathcal{O}_D \geq 3$.

$\Rightarrow L \cong \mathcal{O}_U$.

Reminder on local cohomology

– X affine, $x \in X$, $U := X \setminus \{x\}$
– F coherent sheaf that is S_2.

Then:

1. $H^2_x(X, F) = H^1(U, F|_U)$,
2. finite if X has pure dimension ≥ 3,
3. vanishes iff $\operatorname{depth}_x F \geq 3$.

Proof. $0 \to L \xrightarrow{t} L \xrightarrow{r} L_D \cong \mathcal{O}_{U_D} \to 0$ gives

$$\begin{array}{ccccccc} H^0(U,L) & \xrightarrow{t} & H^0(U,L) & \xrightarrow{r} & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) & \to \\ H^1(U,L) & \xrightarrow{t} & H^1(U,L) & \to & H^1(U_D, L_D \cong \mathcal{O}_{U_D}). & \end{array}$$

Proof. $0 \to L \xrightarrow{t} L \xrightarrow{r} L_D \cong \mathcal{O}_{U_D} \to 0$ gives

$$\begin{array}{ccccccc} H^0(U,L) & \xrightarrow{t} & H^0(U,L) & \xrightarrow{r} & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) & \to \\ H^1(U,L) & \xrightarrow{t} & H^1(U,L) & \to & H^1(U_D, L_D \cong \mathcal{O}_{U_D}). & \end{array}$$

$\operatorname{depth}_x \mathcal{O}_D \geq 3 \Rightarrow H^1(U_D, \mathcal{O}_{U_D}) = 0$ and so
$t : H^1(U,L) \to H^1(U,L)$ is surjective.

Proof. $0 \to L \xrightarrow{t} L \xrightarrow{r} L_D \cong \mathcal{O}_{U_D} \to 0$ gives

$$\begin{array}{ccccccc} H^0(U,L) & \xrightarrow{t} & H^0(U,L) & \xrightarrow{r} & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) & \to \\ H^1(U,L) & \xrightarrow{t} & H^1(U,L) & \to & H^1(U_D, L_D \cong \mathcal{O}_{U_D}). \end{array}$$

$\operatorname{depth}_x \mathcal{O}_D \geq 3 \Rightarrow H^1(U_D, \mathcal{O}_{U_D}) = 0$ and so
$t : H^1(U,L) \to H^1(U,L)$ is surjective.

$\dim U \geq 4$ implies $H^1(U,L)$ has finite length, so
$t : H^1(U,L) \to H^1(U,L)$ isomorphism.

Proof. $0 \to L \xrightarrow{t} L \xrightarrow{r} L_D \cong \mathcal{O}_{U_D} \to 0$ gives

$$\begin{array}{ccccccc} H^0(U,L) & \xrightarrow{t} & H^0(U,L) & \xrightarrow{r} & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) & \to \\ H^1(U,L) & \xrightarrow{t} & H^1(U,L) & \to & H^1(U_D, L_D \cong \mathcal{O}_{U_D}). \end{array}$$

$\operatorname{depth}_x \mathcal{O}_D \geq 3 \Rightarrow H^1(U_D, \mathcal{O}_{U_D}) = 0$ and so
$t : H^1(U,L) \to H^1(U,L)$ is surjective.

$\dim U \geq 4$ implies $H^1(U,L)$ has finite length, so
$t : H^1(U,L) \to H^1(U,L)$ isomorphism.

Thus $r : H^0(U,L) \to H^0(U_D, L_D)$ is surjective.

Lift back the constant 1 section to L. □

Stronger Grothendieck–Lefschetz

- $(x \in X)$ local scheme, $x \in D \subset X$ Cartier divisor,
- $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$,
- L line bundle on U such that
- $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$.
- Remove assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 3$.
- New assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 2$ and $\dim D \geq 3$.

Stronger Grothendieck–Lefschetz

- $(x \in X)$ local scheme, $x \in D \subset X$ Cartier divisor,
- $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$,
- L line bundle on U such that
- $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$.
- Remove assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 3$.
- New assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 2$ and $\dim D \geq 3$.

$\Rightarrow L \cong \mathcal{O}_U$.

Stronger Grothendieck–Lefschetz

- $(x \in X)$ local scheme, $x \in D \subset X$ Cartier divisor,
- $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$,
- L line bundle on U such that
- $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$.
- Remove assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 3$.
- New assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 2$ and $\dim D \geq 3$.

$\Rightarrow L \cong \mathcal{O}_U$.

- Conjectured around 2010
- Proved for semi-log-canonical (arXiv:1211.0317)
- Bhatt – de Jong: X normal over field (arXiv:1302.3189)
- General case (over a field) (arXiv:1407.5108)

Normal case in characteristic p

$\pi : X^+ \to X$ normalization in algebraic closure of $k(X)$.

Normal case in characteristic p

$\pi : X^+ \to X$ normalization in algebraic closure of $k(X)$.

– Hochster–Huneke: X^+ is CM.

– previous proof runs on X^+ (almost).

Normal case in characteristic p

$\pi : X^+ \to X$ normalization in algebraic closure of $k(X)$.

– Hochster–Huneke: X^+ is CM.

– previous proof runs on X^+ (almost).

– L becomes trivial on X^+, so

– L becomes trivial on some finite degree cover.

Normal case in characteristic p

$\pi : X^+ \to X$ normalization in algebraic closure of $k(X)$.

– Hochster–Huneke: X^+ is CM.

– previous proof runs on X^+ (almost).

– L becomes trivial on X^+, so

– L becomes trivial on some finite degree cover.

– use norm map to show that $L^m \cong \mathcal{O}_U$ for some $m > 0$,

Normal case in characteristic p

$\pi : X^+ \to X$ normalization in algebraic closure of $k(X)$.

– Hochster–Huneke: X^+ is CM.

– previous proof runs on X^+ (almost).

– L becomes trivial on X^+, so

– L becomes trivial on some finite degree cover.

– use norm map to show that $L^m \cong \mathcal{O}_U$ for some $m > 0$,

– work a little more ...