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Local entropy along the Ricci flow

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Geometry Festival

April 25h, 2021

(M^m, g) Riemannian manifold, $p \in M$, $u, v, w \in T_pM$. Then curvature tensor

$$Rm(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis of T_pM. Then

Sectional curvature : $K(e_i, e_j) = -\langle Rm(e_i, e_j)e_i, e_j\rangle$,

Ricci curvature : $Rc(e_i, e_j) = \sum_{k=1}^{m} \langle Rm(e_k, e_i)e_j, e_k\rangle$,

Scalar curvature : $R = \sum_{k=1}^{m} R_{kk}$.

Rm can be regarded as an operator $\mathrm{Rm} : \Lambda^2 \mathrm{T_p M} \to \Lambda^2 \mathrm{T_p M}$:

$$\langle \mathrm{Rm}(u \wedge v), w \wedge z \rangle := -\langle \mathrm{Rm}(u, v)w, z \rangle.$$

Quarter-Pinch:

$$\frac{1}{4}\sigma^2 < K_{uv} \leq \sigma^2$$

for some $\sigma > 0$.

PIC condition: For any orthonormal four frame $\{e_1, e_2, e_3, e_4\}$ we have the inequality $R_{1331} + R_{1441} + R_{3223} + R_{4224} + 2R_{1234} > 0$.

Quater-pinch implies PIC_2, which means that $M \times \mathbb{R}^2$ has positive isotropic curvature.

- On a closed manifold M of dimension m, a smooth family of metrics $g(t)$ is called a Ricci flow solution if it satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

where R_{ij} is the Ricci curvature.

- The normalized Ricci flow solution.

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{r}{n} g_{ij}$$

where r is the average of scalar curvature.

Ricci flow solution on standard sphere: the sphere shrinks to a point in finite time.

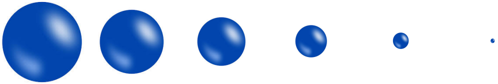

图: shrinking spheres

Normalized Ricci flow on standard sphere: the sphere stays unchanged.

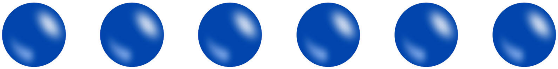

图: steady spheres

1. Einstein manifold: $\mathrm{Rc} - \lambda g = 0$.
Example: $(\mathbb{CP}^n, g_{FS})$, the Fubini-Study metric.
(X, g_{CY}), the Calabi-Yau metric.

2. Ricci soliton: $\mathrm{Rc} - \lambda g = \mathrm{Hess}_f$.
Example: $(\mathbb{CP}^2 \sharp \bar{\mathbb{CP}}^2, g_{CK})$, the Cao-Koiso metric.

The flow may develop finite time singularity, i.e., maximal curvature norm blows up.

- Starting from every dumb bell metric on S^2, the normalized flow has global existence and converge to round sphere(Hamilton, 1988; Chow, 1991).
- Starting from some dumb bell metric on $S^m(m \geq 3)$, the normalized flow may develop neck pinch singularities.

Normalized Ricci flow on topological S^2.

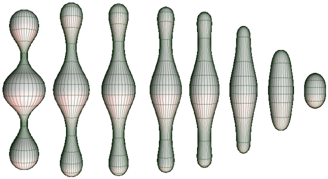

图: No singularity on S^2

Normalized Ricci flow on topological S^3.

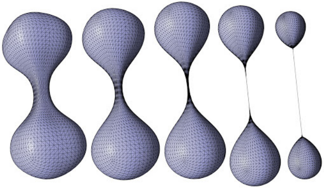

图: Formation of neckpinch singularity on S^3

Theorem[Hamilton, 1982]: Suppose (M^3, g) is a Riemannian manifold with positive Ricci curvature. Then the normalized Ricci flow solution starting from (M^3, g) exists immortally and converges exponentially fast to g_∞ with constant sectional curvature.

Theorem[Böhm-Wilking, 2006]: Suppose (M^m, g) is a Riemannian manifold with positive curvature operator. Then the normalized Ricci flow solution starting from (M^m, g) exists immortally and converges exponentially fast to g_∞ with constant sectional curvature.

Theorem[Brendle-Schoen, 2009]: Suppose (M^m, g) is a Riemannian manifold whose sectional curvature satisfies the quater-pinch condition. Then the normalized Ricci flow solution starting from (M^m, g) exists immortally and converges exponentially fast to g_∞ with constant sectional curvature.

These can be deduced from the preservation and improvement of some curvature condition:

(1) Positivity of curvature operator is preserved under the Ricci flow.

(2) $\mathrm{PIC}_2 > 0$ is preserved under the Ricci flow.

If the initial curvature condition is NOT preserved by the Ricci flow, then in general it is not clear whether the flow exists immortally and converges. Note that $\mathrm{Rc} > 0$ is NOT preserved by the Ricci flow if $m > 3$.

However, we still have some convergence results even if the initial curvature condition is not preserved.

Theorem(W., 2020): Suppose (M^m, g) is a Riemannian manifold satisfying

$$\begin{cases} Rc \geq (m-1)(1-\delta), \\ |M|_{dv_g} = (m+1)\omega_{m+1}, \end{cases} \tag{1}$$

where ω_{m+1} is the volume of unit ball in $\mathbb{R}^{m+1}$, and $\delta \in (0, \delta_0(m))$. Then the normalized Ricci flow initiated from (M^m, g) exists immortally and converges to a standard round metric g_∞ exponentially fast.

Furthermore, for each pair of points $x, y \in M$ satisfying $d_g(x,y) \le 1$, the following distance bi-Hölder estimate holds:

$$e^{-\psi} d_g^{1+\psi}(x,y) \le d_{g_\infty}(x,y) \le e^{\psi} d_g^{1-\psi}(x,y), \tag{2}$$

where $\psi = \psi(\delta|m)$ such that $\lim_{\delta \to 0} \psi(\delta|m) = 0$.

Remark: The diffeomorphism and the existence of distance bi-Hölder estimate was due to Cheeger-Colding 1997, based on the homeomorphism theorem of Perelman 1994.
Based on the development in algebraic geometry by Liu-Zhang 2020, we obtain a Kähler version of the above theorem, which is also proved independently by Datar-Seshadri-Song 2020.

Theorem(W., 2020, Datar-Seshadri-Song 2020): Suppose (M^n, g, J) is a Fano manifold satisfying

$$\begin{cases} Rc \geq 2(n+1)(1-\delta), \\ |M|_{dv_g} = \Omega_n, \end{cases} \tag{3}$$

where Ω_n is the volume of $(\mathbb{CP}^n, g_{FS})$, and $\delta \in (0, \delta_0(n))$. Then the normalized Ricci flow initiated from (M, g) exists immortally and converges smoothly to a metric g_∞ such that (M, g_∞, J) is biholomorphic-isometric to $(\mathbb{CP}^n, g_{FS}, J_{FS})$.

Namely, there exists a diffeomorphism $\Phi : \mathbb{CP}^n \to M$ such that $\Phi^* g_\infty = g_{FS}$ and $\Phi^*(J) = J_{FS}$. Furthermore, for each pair of points $x, y \in M$ satisfying $d_g(x, y) \leq 1$, the distance bi-Hölder estimate (2) holds.

Theorem(W., 2020): Suppose (N^n, h, J) is a closed Kähler manifold of complex dimension n satisfying $c_1(N, J) = 0$. There exists a small constant $\epsilon = \epsilon(N, h, J)$ with the following properties.
If (M, g) is a Riemannian manifold satisfying

$$Rc_g > -2(m-1), \quad d_{GH}\left((M, g), (N, h)\right) < \epsilon, \tag{4}$$

with $m = 2n$. Then the normalized Ricci flow initiated from (M, g) exists immortally and converges to a Calabi-Yau metric (N^n, h', J').
Remark: This relies on the work of Dai-Wang-Wei 2007.

Theorem[W. 2020]: Suppose (M_i^n, g_i, J_i) is a sequence of Kähler manifolds of complex dimension n satisfying

$$\inf_{x \in M_i} I(B(x, g_i)) \geq (1-\epsilon) I_{2n}, \quad \mathrm{diam}(M, g) \leq D, \quad \sigma_a \leq R \leq \sigma_b.$$

By taking subsequence if necessary, we have

$$(M_i, g_i, J_i) \xrightarrow{C^{1,\alpha}-\text{Cheeger}-\text{Gromov}} (M_\infty, g_\infty, J_\infty),$$

where $(M_\infty, g_\infty, J_\infty)$ is a $C^{1,\alpha}$-Kähler manifold, $0 < \alpha < 1$.

Let (M, g) be a complete Riemannian manifold of dimension m, and Ω be a connected, open subset of M with smooth boundary. Then we can regard $(\Omega, \partial\Omega, g)$ as a smooth manifold with boundary. Let a be a smooth function on $\bar{\Omega}$, and τ be a positive constant. Then we define

$$\mathscr{S}(\Omega) := \left\{ \varphi \middle| \varphi \in W_0^{1,2}(\Omega), \quad \varphi \geq 0, \quad \int_\Omega \varphi^2 dv = 1 \right\}, \tag{5}$$

$$\begin{aligned} \mathcal{W}^{(a)}(\Omega, g, \varphi, \tau) &:= -m - \frac{m}{2}\log(4\pi\tau) \\ &+ \int_\Omega \left\{ \tau \left(a\varphi^2 + 4|\nabla\varphi|^2 \right) - 2\varphi^2 \log\varphi \right\} dv, \end{aligned} \tag{6}$$

Then we define

$$\boldsymbol{\mu}^{(a)}(\Omega, g, \tau) := \inf_{\varphi \in \mathscr{S}(\Omega)} \mathcal{W}^{(a)}(\Omega, g, \varphi, \tau), \tag{7}$$

$$\boldsymbol{\nu}^{(a)}(\Omega, g, \tau) := \inf_{s \in (0,\tau]} \boldsymbol{\mu}^{(a)}(\Omega, g, s), \tag{8}$$

$$\boldsymbol{\nu}^{(a)}(\Omega, g) := \inf_{\tau \in (0,\infty)} \boldsymbol{\mu}^{(a)}(\Omega, g, \tau). \tag{9}$$

By the result of O. Rothaus, we know that for each smooth function "a" and each positive number $\tau > 0$, $\boldsymbol{\mu}^{(a)}(\Omega, g, \tau)$ is achieved by a function $\varphi \in W_0^{1,2}(\Omega)$ whenever Ω is bounded. Let $\{(M, g(t)), 0 \leq t < T\}$ be a Ricci flow solution. Let $a = R$. Perelman showed that

$$\frac{d}{dt}\boldsymbol{\mu}^{(R)}(M, g(t), \Lambda - t) \geq 0$$

for each positive Λ. This monotonicity is the basis of his no-local-collapsing theorem.

Theorem(W-, 2017): Let $A \geq 1000m$ be a large constant. Let $\{(M^m, g(t)), 0 \leq t \leq T\}$ be a Ricci flow solution satisfying

$$t \cdot Rc(x,t) \leq (m-1)A, \quad \forall\, x \in B_{g(t)}\left(x_0, \sqrt{t}\right),\ t \in (0,T]. \tag{10}$$

Then we have

$$\boldsymbol{\mu}(\Omega'_T, g(T), \tau_T) - \boldsymbol{\mu}(\Omega_0, g(0), \tau_T + T) \geq -A^{-2} \tag{11}$$

for every $\tau_T \in (0, A^2 T)$. Here $\Omega'_T = B_{g(T)}\left(x_0, 8A\sqrt{T}\right)$ and $\Omega_0 = B_{g(0)}\left(x_0, 20A\sqrt{T}\right)$.

We define

$$\Omega_t := B_{g(t)}\left(x_0, 20A - 2A\sqrt{t}\right), \quad \Omega'_t := B_{g(t)}\left(x_0, 10A - 2A\sqrt{t}\right). \tag{12}$$

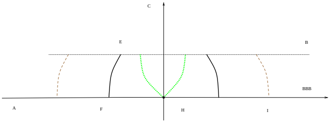

图: Different domains for almost monotonicity

Theorem[Pseudo-locality]:

For any $\alpha \in (0,1)$, there are positive constants $\varepsilon_P = \varepsilon_P(m,\alpha)$ and $\delta_P = \delta_P(m,\alpha)$ such that if (M,g) is a Ricci flow solution define for $t \in [0,T]$ with each time slice $(M,g(t))$ being a complete Riemannian manifold, and if one of the conditions holds for $p \in M$:

1. (Perelman, 2002) $R_{g(0)} \geq -1$ on $B_{g(0)}(p,1)$ and $I_{B_{g(0)}(p,1)} \geq (1-\delta_P)I_m$, or
2. (Tian-Wang, 2012) $Rc_{g(0)} \geq -\delta_P g(0)$ on $B_{g(0)}(p,1)$ and $\left|B_{g(0)}(p,1)\right|_{g(0)} \geq (1-\delta_P)\omega_m$,

where I_m and ω_m stands for the isoperimetric constant and volume of the m-Euclidean unit ball, respectively, and I_Ω denotes the isoperimetric constant for the domain $\Omega \subset M$, then

$$\forall t \in (0, \varepsilon_P^2], \qquad \sup_{B_{g(t)}(p, \varepsilon_P)} \left|Rm_{g(t)}\right|_{g(t)} \leq \alpha t^{-1} + \varepsilon_P^{-2}. \tag{13}$$

Theorem[Improved version of pseudo-locality, W., 2020]:
For each $\alpha \in (0, \frac{1}{100m})$, there exists $\delta = \delta(\alpha, m)$ with the following properties.
Suppose $\{(M^m, g(t)), 0 \le t \le T\}$ is a Ricci flow solution satisfying

$$\inf_{0<t\le T} \boldsymbol{\mu}\left(B_{g(0)}\left(x_0, \delta^{-1}\sqrt{t}\right), g(0), t\right) \ge -\delta^2. \tag{14}$$

Then for each $t \in (0, T]$ and $x \in B_{g(t)}\left(x_0, \alpha^{-1}\sqrt{t}\right)$, we have

$$t|Rm|(x,t) \le \alpha, \tag{15}$$

$$\inf_{\rho\in(0,\alpha^{-1}\sqrt{t})} \rho^{-m} \left|B_{g(t)}(x,\rho)\right|_{dv_{g(t)}} \ge (1-\alpha)\omega_m, \tag{16}$$

$$t^{-\frac{1}{2}} \cdot inj(x,t) \ge \alpha^{-1}. \tag{17}$$

In particular, (15),(16) and (17) hold whenever the following condition is satisfied:

$$\nu\left(B_{g(0)}\left(x_0,\delta^{-1}\sqrt{T}\right), g(0), T\right) \ge -\delta^2. \tag{18}$$

Under the condition that Rc uniformly bounded below, if (M_i, g_i) converges to a smooth manifold (M_∞, g_∞) in the Gromov-Hausdorff topology, then $(M_i, g_i(1))$ converges to (M_∞, g_∞) in the smooth topology(up to diffeomorphism). Then we can apply various stability theorems. This is very closely related to Jim Isenberg's talk. (See for example, the work of Isenberg-Bahuaud-Guenther.)

The compactness of Kähler manifolds depends on an improved pseudo-locality theorem in the Kähler setting.

Theorem[KRF on Fano manifold, Chen-W. 2020, arXiv:1405.6797]:
Let X be a projective manifold with $-K_X$ ample. Then the Kähler Ricci flow

$$\frac{\partial \omega}{\partial t} = -\mathrm{Rc}(\omega) + \omega \tag{19}$$

has uniformly bounded diameter and converges to a unique singular Kähler-Ricci-Soliton metric ω_{KRS} in the sense of Gromov-Hausdorff as $t \to \infty$.
Remark: This solves a conjecture of Hamilton-Tian 1997. The key observation is to transform the question of time-slice compactness to space-time compactness, and to use "good"-function level sets to estimate distance distortion.

The limit in the previous Theorem is a conifold shrinking Ricci soliton.

Theorem[Compactness of conifold shrinking-Ricci-soliton moduli, Li-Li-W. Huang-Li-W. 2018]:

Let (M_i, p_i, g_i, f_i) be a sequence of non-collapsed, non-compact Ricci shrinkers. Then by taking subsequence if necessary, we have

$$(M_i, p_i, g_i, f_i) \xrightarrow{\hat{C}^\infty-\text{Cheeger}-\text{Gromov}} (X, p, d, f),$$

where (X, p, d, f) is a Riemannian conifold shrinking Ricci soliton.

Theorem[KRF on general type manifold, W. 2018, arXiv:1706.06485]:
Let X be a projective manifold with K_X big and nef. Then the Kähler Ricci flow

$$\frac{\partial \omega}{\partial t} = -Rc(\omega) - \omega \tag{20}$$

has uniformly bounded diameter and converges to the unique singular Kähler-Einstein metric ω_{KE} on X_{can} in the sense of Gromov-Hausdorff as $t \to \infty$.
Remark: This solves a conjecture of Song-Tian 2009.

Theorem[Rigidity of the first Betti number, Huang-W. 2020]: Given positive integer m and a closed manifold N, there is a small constant $\delta_B(m, N) \in (0, 1)$ with the following properties. If (M^m, g) satisfies $Rc \geq -(m-1)g$ and $d_{GH}(M, N) < \delta_B$, then

1. $b_1(M) - b_1(N) \leq m - k$; and
2. if the equality holds, then M is diffeomorphic to an $(m-k)$-torus bundle over N.

Remark: This generalize the Colding-Gromov rigidity theorem of b_1, which is the case $N = \{pt\}$. The key is the Ricci flow smoothing technique based on the pseudo-locality on the covering space.

Question:

Suppose (M^4, g) is a Riemannian manifold satisfying

$$\begin{cases} Rc \geq 3, \\ |M|_{dv_g} > \frac{3}{4}|S^4|_{g_{round}}. \end{cases}$$

Then the normalized Ricci flow initiated from (M^4, g) exists immortally and converges to a round metric g_∞ exponentially fast.

Thank you for your attention!