Topological Recursion and Crepant Transformation Conjecture

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K-equivalence

Let $\mathcal{X}_\pm$ be a smooth complex algebraic variety
($\Rightarrow \mathcal{X}_\pm$ is a complex manifold, locally isomorphic to $\mathbb{C}^m$)
or a smooth (complex) Deligne-Mumford (DM) stack
($\Rightarrow \mathcal{X}_\pm$ is a complex orbifold, locally isomorphic to $\mathbb{C}^m/G$, where G is a finite group acting holomorphically on $\mathbb{C}^m$) .

The canonical line bundle $K_{\mathcal{X}_\pm} = \Lambda^m T^*_{\mathcal{X}_\pm}$ is an algebraic (holomorphic) (orbifold) line bundle over $\mathcal{X}_\pm$.

Following C.-L. Wang, we say $\mathcal{X}_+$ and $\mathcal{X}_-$ are **K-equivalent** if there exists a smooth variety/DM stack $\mathcal{Y}$ and birational maps $f_\pm : \mathcal{Y} \to \mathcal{X}_\pm$ such that $f_+^* K_{\mathcal{X}_+} = f_-^* K_{\mathcal{X}_-}$.

$$\begin{tikzcd}
 & \mathcal{Y} \arrow[dl, "f_+"'] \arrow[dr, "f_-"] & \\
\mathcal{X}_+ \arrow[rr, dashed, leftrightarrow, "\text{Crepant}", "\text{Transformation}"'] & & \mathcal{X}_-
\end{tikzcd}$$

Crepant Transformation Conjecture (CTC)

The **Crepant Transformation Conjecture** (CTC) was first proposed by Y. Ruan around 2001, and later refined/extended by Bryan-Graber, Coates-Iritani-Tseng, Iritani, Coates-Ruan, etc.

CTC relates Gromov-Witten (GW) invariants of $\mathcal{X}_+$ and $\mathcal{X}_-$.
GW invariants of $\mathcal{X}$ are virtual counts of parametrized holomorphic curves in $\mathcal{X}$.

In this talk, we will describe CTC for **symplectic toric Calabi-Yau 3-orbifold**. Here we say a complex manifold/orbifold $\mathcal{X}$ is **Calabi-Yau** if $K_{\mathcal{X}}$ is the trivial holomorphic line bundle $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$.
(Borisov-Chen-Smith: smooth toric DM stacks)

Symplectic toric Calabi-Yau 3-orbifolds

Let $G = (\mathbb{C}^*)^{\mathfrak{p}}$ act on $\mathbb{C}^{\mathfrak{p}+3}$ linearly and faithfully:

$$\lambda \cdot (z_1, \ldots, z_{\mathfrak{p}+3}) = (\rho_1(\lambda) z_1, \ldots, \rho_{\mathfrak{p}+3}(\lambda) z_{\mathfrak{p}+3})$$

where $\lambda = (\lambda_1, \ldots, \lambda_{\mathfrak{p}}) \in G$, $(z_1, \ldots, z_{\mathfrak{p}+3}) \in \mathbb{C}^{\mathfrak{p}+3}$, and $\rho_i : G \to \mathbb{C}^*$ are G-characters which satisfy the Calabi-Yau condition

$$\prod_{i=1}^{\mathfrak{p}+3} \rho_i(\lambda) = 1.$$

The action of $G = (\mathbb{C}^*)^{\mathfrak{p}}$ restricts to a Hamiltonian action by the maximal compact subgroup $G_{\mathbb{R}} = U(1)^{\mathfrak{p}}$ of G on

$$\left(\mathbb{C}^{\mathfrak{p}+3}, \frac{\sqrt{-1}}{2} \sum_{i=1}^{\mathfrak{p}+3} dz_i \wedge d\bar{z}_i\right)$$

Symplectic toric Calabi-Yau 3-orbifolds

If $\rho_i(\lambda) = \prod_{a=1}^{\mathfrak{p}} \lambda_a^{Q_i^a}$ (where $Q_i^a \in \mathbb{Z}$), then (up to the addition to a constant vector in $\mathbb{R}^p$) the moment map of the $G_{\mathbb{R}}$-action is given by

$$\mu : \mathbb{C}^{\mathfrak{p}+3} \longrightarrow \mathbb{R}^{\mathfrak{p}}, \quad z \mapsto (\mu^1(z), \ldots, \mu^{\mathfrak{p}}(z))$$

where $\mu(z) = \frac{1}{2} \sum_{i=1}^{\mathfrak{p}+3} Q_i^a |z_i|^2$. Given a regular value $\theta \in \mathbb{R}^{\mathfrak{p}}$,

$$\mathcal{X}_\theta = [\mu^{-1}(\theta)/G_{\mathbb{R}}]$$

is a toric Calabi-Yau 3-orbifold with a Kähler form ω_θ; if the $G_{\mathbb{R}}$-action on $\mu^{-1}(\theta)$ is free then $(\mathcal{X}, \omega_\theta)$ is a 3-dimensional Käher manifold. The coarse moduli space

$$X_\theta = \mu^{-1}(\theta)/G_{\mathbb{R}} = (\mathbb{C}^{\mathfrak{p}+3})^{\theta-ss}/G$$

is a simplicial toric Calabi-Yau 3-fold.

Example

$G = \mathbb{C}^*$, $G_{\mathbb{R}} = U(1)$, $\mathfrak{p} = 1$.

$$\lambda \cdot (z_1, z_2, z_3, z_4) = (\lambda z_1, \lambda z_2, \lambda z_3, \lambda^{-3} z_4).$$

$$\mu : \mathbb{C}^4 \longrightarrow \mathbb{R}, \quad (z_1, z_2, z_3, z_4) \mapsto \frac{1}{2}\left(|z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2\right).$$

$$X_\theta = \mu^{-1}(\theta)/U(1) = \begin{cases} \left((\mathbb{C}^3 - \{0\}) \times \mathbb{C}\right)/\mathbb{C}^* = \mathcal{O}_{\mathbb{P}^2}(-3) = K_{\mathbb{P}^2}, & \theta > 0; \\ \left(\mathbb{C}^3 \times (\mathbb{C} - \{0\})\right)/\mathbb{C}^* = \mathbb{C}^3/\mathbb{Z}_3, & \theta < 0. \end{cases}$$

$\mathcal{X}_+ = K_{\mathbb{P}^2} = X_+$ is a symplectic toric Calabi-Yau 3-manifold
$\mathcal{X}_- = [\mathbb{C}^3/\mathbb{Z}_3]$ is a symplectic toric Calabi-Yau 3-orbifold
$X_- = \mathbb{C}^3/\mathbb{Z}_3$ is a simpicial toric Calab-Yau 3-fold

$$\begin{tikzcd} K_{\mathbb{P}^2} \arrow[rr, dashed, leftrightarrow] \arrow[dr, "\text{toric crepant resolution}"'] & & {[\mathbb{C}^3/\mathbb{Z}_3]} \arrow[dl] \\ & \mathbb{C}^3/\mathbb{Z}_3 & \end{tikzcd}$$

Inertia Stack and Chen-Ruan Orbifold Cohomology

$$X = \mu^{-1}(\theta)/G_{\mathbb{R}} = U/G, \quad U = (\mathbb{C}^{\mathfrak{p}+3})^{\theta-ss}, \quad \mathcal{X} = [U/G].$$

The **inertia stack** of $\mathcal{X}$ is

$$I\mathcal{X} = \{(z,b) \in U \times G : b \cdot z = z\}/G = \bigcup_{b \in B} \mathcal{X}_b = \mathcal{X}_0 \cup \underbrace{\bigcup_{b \in B \setminus \{1\}} \mathcal{X}_b}_{\text{twisted sectors}}$$

where $B = \{b \in G : U^b \text{ is non-empty}\}$, $\mathcal{X}_b = [U^b/G]$, and $\mathcal{X}_0 = [U/G] \cong \mathcal{X}$. If $\mathcal{X} = X$ is smooth then $I\mathcal{X} = \mathcal{X}_0 = X$.

If $x \in \mathcal{X}_b$, b acts on $T_x\mathcal{X}$ with weights $e^{2\pi\sqrt{-1}\epsilon_j}$, where $\epsilon_j \in [0,1)$. $\mathrm{age}(b) := \epsilon_1 + \epsilon_2 + \epsilon_3 \in \{0,1,2\}$. As a graded vector space over $\mathbb{C}$, the **Chen-Ruan orbifold cohomology** of $\mathcal{X}$ is

$$H^*_{\mathrm{CR}}(\mathcal{X}) = \bigoplus_{b \in G} H^*(\mathcal{X}_b)[2\mathrm{age}(b)] = \mathbb{C}\mathbf{1}_0 \oplus H^2_{\mathrm{CR}}(\mathcal{X}) \oplus H^4_{\mathrm{CR}}(\mathcal{X}),$$

where $\deg \mathbf{1}_0 = 0$, $\dim_{\mathbb{C}} H^2_{\mathrm{CR}}(\mathcal{X}) = \mathfrak{p} \geq \dim_{\mathbb{C}} H^4_{\mathrm{CR}}(\mathcal{X}) =: \mathfrak{g}$.

Example

$$\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3], \quad I\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_\zeta \cup \mathcal{X}_{\zeta^2},\ \zeta = e^{2\pi\sqrt{-1}/3}.$$

$$\mathcal{X}_0 = \mathcal{X}, \quad \mathcal{X}_\zeta = \mathcal{X}_{\zeta^2} = [0/\mathbb{Z}_3] = B\mathbb{Z}_3.$$

$$\mathrm{age}(\zeta) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1, \quad \mathrm{age}(\zeta^2) = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2.$$

$$H^*_{\mathrm{CR}}(\mathcal{X}) = \underbrace{\mathbb{C}\mathbf{1}_0}_{H^0} \oplus \underbrace{\mathbb{C}\mathbf{1}_{\frac{1}{3}}}_{H^2} \oplus \underbrace{\mathbb{C}\mathbf{1}_{\frac{2}{3}}}_{H^4}.$$

$$H^*_{\mathrm{CR}}(K_{\mathbb{P}^2}) = H^*(K_{\mathbb{P}^2}) = \underbrace{\mathbb{C}\mathbf{1}}_{H^0} \oplus \underbrace{\mathbb{C}H}_{H^2} \oplus \underbrace{\mathbb{C}H^2}_{H^4}.$$

In general:

- $H^2_{\mathrm{CR}}(\mathcal{X}) = H^2(\mathcal{X}) \oplus \bigoplus_{b \in B_1} \mathbb{C}\mathbf{1}_b$,
  where $B_1 = \{b \in B : \mathrm{age}(b) = 1\}$.
- If $\mathcal{X}_+$ and $\mathcal{X}_-$ are related by a single toric wall-crossing (e.g. $\mathcal{X}_+ = K_{\mathbb{P}^2}$, $\mathcal{X}_- = [\mathbb{C}^3/\mu_3]$) then $H^*_{\mathrm{CR}}(\mathcal{X}_+) \cong H^*_{\mathrm{CR}}(\mathcal{X}_-)$ as graded vector spaces over $\mathbb{C}$.

Gromov-Witten invariants

$$H^2_{\mathrm{CR}}(\mathcal{X}) = \underbrace{H^2(\mathcal{X})}_{\mathfrak{p}'} \oplus \bigoplus_{i=1}^{\mathfrak{p}-\mathfrak{p}'} \mathbb{C}\mathbf{1}_{b_i}, \quad \text{where } B_1 = \{b_1, \ldots b_{\mathfrak{p}-\mathfrak{p}'}\}.$$

Given $i_1, \ldots, i_\ell \in \{1, \ldots, \mathfrak{p} - \mathfrak{p}'\}$,

$$\langle \mathbf{1}_{b_{i_1}} \cdots \mathbf{1}_{b_{i_\ell}} \rangle^{\mathcal{X}}_{g,d'} \in \mathbb{Q}$$

is the virtual number of holomorphic maps $f : (C, x_1, \ldots, x_\ell) \to \mathcal{X}$, where C is a (nodal) orbicurve of genus g, $f_*[C] = d' \in H_2(X; \mathbb{Z})/\mathrm{torsion} = \mathbb{Z}^{\mathfrak{p}'}$, $f(x_j, \zeta) \in \mathcal{X}_{b_{i_j}}$.

$$F^{\mathcal{X}}_g(\tau) = \sum_d e^{\sum_{a=1}^{\mathfrak{p}'} d_a \tau_a} \prod_{a=\mathfrak{p}'+1}^{\mathfrak{p}} \frac{\tau_a^{d_a}}{d_a!} \langle \mathbf{1}_{b_1}^{d_{\mathfrak{p}'+1}} \cdots \mathbf{1}_{b_{\mathfrak{p}-\mathfrak{p}'}}^{d_{\mathfrak{p}}} \rangle^{\mathcal{X}}_{g,d'}$$

where $\tau = (\tau_1, \ldots, \tau_{\mathfrak{p}})$, $d = (d_1, \ldots, d_{\mathfrak{p}})$ extended degree.
In particular, if $\mathcal{X}$ is smooth then $\mathfrak{p}' = \mathfrak{p}$ and

$$F^{\mathcal{X}}_g(\tau) = \sum_d \prod_{a=1}^{\mathfrak{p}} Q_a^{d_a} N^{\mathcal{X}}_{g,d}, \text{ where } Q_a = e^{\tau_a} \text{ and } N^{\mathcal{X}}_{g,d} = \langle\ \rangle^{\mathcal{X}}_{g,d}.$$

Open Gromov-Witten Invariants

virtual counts of parametrized holomorphic curves in X with boundaries in $L \subset X$

Aganagic-Vafa Lagrangian

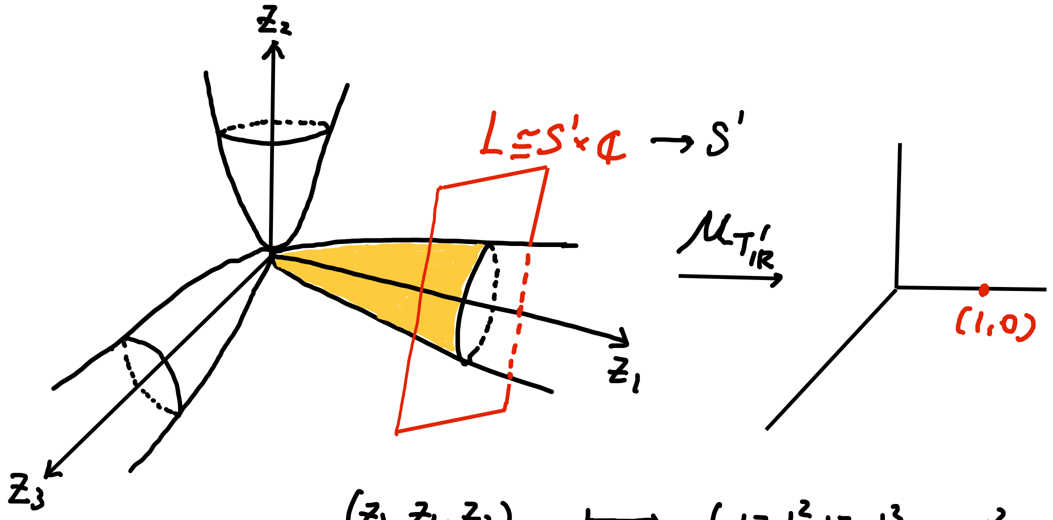
generalization of Harvey-Lawson SLag in $\mathbb{C}^3$

Harvey-Lawson SLag

$$L = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - |z_2|^2 = 1 = |z_1|^2 - |z_3|^2 \ , \ \mathrm{Im}(z_1 z_2 z_3) = 1 \}$$

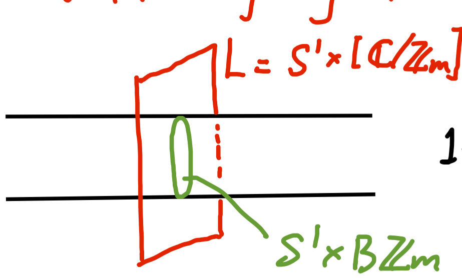
Compact CY torus

$$\begin{array}{ccccc} T'_{\mathbb{R}} & \subset & T_{\mathbb{R}} & \subset & T \\ \| & & \| & & \| \\ U(1)^2 & & U(1)^3 & & (\mathbb{C}^*)^3 \end{array}$$



$$(z_1, z_2, z_3) \longmapsto (|z_1|^2 - |z_3|^2, |z_2|^2 - |z_3|^2)$$

Aganagic-Vafa Lagrangian



$\frac{2\pi}{m}$

1-dim'l orbit $\subset \mathbb{C}^* \times B\mathbb{Z}_m$

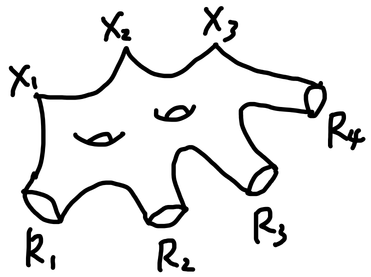
Given $i_1,\dots,i_\ell \in \{1,\dots,p-p'\}$

$\mu_1,\dots,\mu_n \in \mathbb{Z}$, $\quad k_1,\dots,k_n \in \mathbb{Z}_m$

$\langle 1_{b_{i_1}} \cdots 1_{b_{i_\ell}} \rangle^{\mathcal{X},(L,f)}_{g,d',(\mu_1,k_1)\cdots(\mu_n,k_n)} \in \mathbb{Q}$ are virtual counts

of holomorphic maps $u: (\Sigma, x_1,\dots,x_\ell, \partial\Sigma) \longrightarrow (\mathcal{X}, L)$

Σ: genus g $\qquad \partial\Sigma = \coprod_{i=1}^{n} R_i$



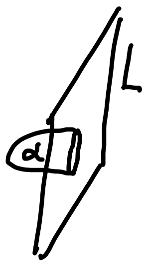
$u(x_j,\zeta) \in \mathcal{X}_{b_{i_j}}$

$u_*[R_i] = (\mu_i, k_i) \in H_1(L;\mathbb{Z}) = \mathbb{Z}\times\mathbb{Z}_m$

L homotopic to $S^1 \times B\mathbb{Z}_m$

$u_*[\Sigma] = d' + \left(\sum_{i=1}^{n}\mu_i\right)\alpha$

$d' \in H_2(\mathcal{X};\mathbb{Z})$, $\quad \alpha \in H_2(\mathcal{X},L)$



Counts depend on the framing $f \in \mathbb{Z}$

$$F^{\mathcal{X},(L,f)}_{g,n\,(k_1,\dots,k_n)}(\tau, \hat{X}_1,\dots,\hat{X}_n) \qquad \tau = (\tau_1,\dots,\tau_p)$$

$$= \sum_{d}\sum_{\mu} \prod_{a=1}^{p'} e^{d_a\tau_a} \prod_{a=p'+1}^{p} \frac{\tau_a^{d_a}}{d_a!} \prod_{i=1}^{n} \hat{X}_i^{\mu_i} \langle 1_{b_1}^{d_{p'+1}} \cdots 1_{b_{p-p'}}^{d_p} \rangle^{\mathcal{X},(L,f)}_{g,d',(\mu_1,k_1)\cdots(\mu_n,k_n)}$$

$d = (d_1,\dots,d_p)$ extended degree $\to$ A-model closed string coordinates $\tau = (\tau_1,\dots,\tau_p)$

$\mu = (\mu_1,\dots,\mu_n)$ winding numbers $\to$ A-model open string coordinates $(\hat{X}_1,\dots,\hat{X}_n)$

$H_L = \bigoplus_{k=0}^{m-1} \mathbb{C}e_k \cong \mathbb{C}^m \qquad F^{\mathcal{X},(L,f)}_{g,n} = \sum_{k_1\cdots k_n=0}^{m-1} F^{\mathcal{X},(L,f)}_{g,n\,(k_1,\dots,k_n)} e_{k_1}\otimes\cdots\otimes e_{k_n} \in H_L^{\otimes n}.$

CTC for toric CY 3-orbifolds

- Coates-Iritani-Jiang (2014): genus-zero equivariant CTC for K-equivalent toric DM stack $\mathcal{X}_+, \mathcal{X}_-$ related by a single toric wall-crossing

  any dim

  not necessarily Calabi-Yau

$\Rightarrow$ relating $F_0^{\mathcal{X}_+}$ and $F_0^{\mathcal{X}_-}$

$\mathcal{X}_+, \mathcal{X}_-$ K-equivalent toric CY 3-orbifolds related by a single toric wall-crossing

- J. Zhou (2008)

$\mathbb{Z}_m$ acts on $\mathbb{C}^2$ $\quad \zeta\cdot(z_1,z_2) = (\zeta z_1, \zeta^{-1} z_2)$

$\mathbb{C}^2/\mathbb{Z}_m$ $\quad A_{m-1}$-surface singularity

$\widetilde{\mathbb{C}^2/\mathbb{Z}_m}$ $\quad$ toric crepant resolution of $\mathbb{C}^2/\mathbb{Z}_m$

$$\begin{array}{ccc} \mathcal{X}_+ = \mathbb{C}\times\widetilde{\mathbb{C}^2/\mathbb{Z}_m} & & \mathcal{X}_- = \mathbb{C}\times[\mathbb{C}^2/\mathbb{Z}_m] \\ & \searrow \quad \swarrow & \\ & \mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_m & \end{array}$$

$$F_g^{\mathcal{X}_+}(\tau_1^+,\dots,\tau_{m-1}^+) \longleftrightarrow F_g^{\mathcal{X}_-}(\tau_1^-,\dots,\tau_{m-1}^-) \qquad \text{all genus } g$$

analytic continuation + change of variables

- D. Ross (2013) generalized this result to toric CY 3-orbifolds with transverse A-singularities ($\Leftarrow$ Brini-Cavalieri-Ross, MOOP, Ross-Zong)

- Coates-Iritani, Lho-Pandharipande (2018)

  $\mathcal{X}_+ = K_{\mathbb{P}^2}$, $\quad \mathcal{X}_- = [\mathbb{C}^3/\mathbb{Z}_3]$ $\qquad \zeta \cdot (z_1, z_2, z_3) = (\zeta z_1, \zeta z_2, \zeta z_3)$

(c) $\quad F_g^{\mathcal{X}_+}(\tau^+) \longleftrightarrow \{ F_{g'}^{\mathcal{X}_-}(\tau^-),\ g' \leq g \}$

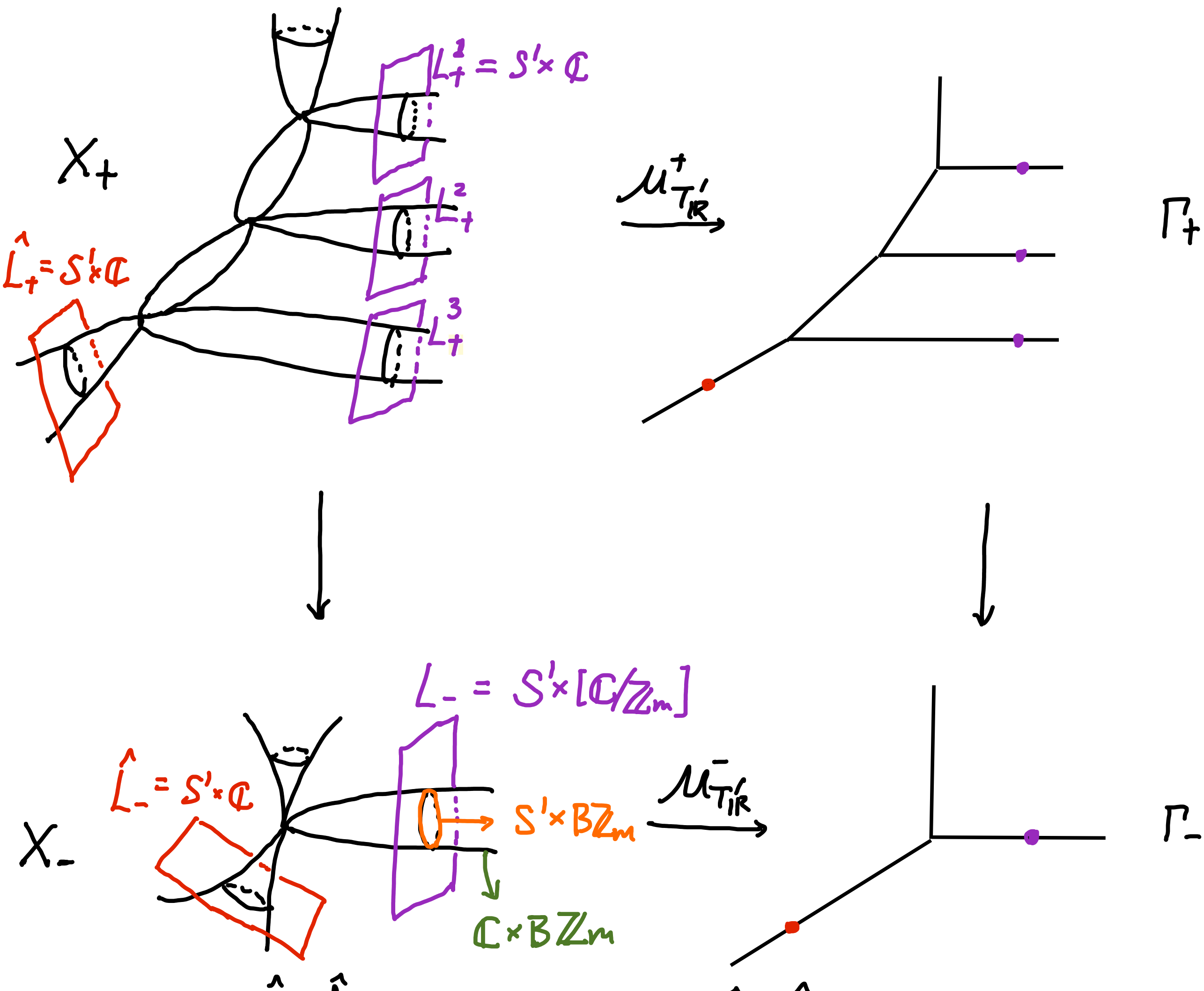
analytic continuation
\+ change of variables
\+ symplectic transformation

- Fang-L-Yu-Zong (in progress) generalize (c) to K-equivalent toric CY 3-orbifolds related by a single toric wall-crossing

Open CTC for toric CY 3-orbifolds relative to Aganagic-Vafa Lagrangians

- Brini-Cavalieri-Ross (2013)

$$\mathcal{X}_+ = \mathbb{C}\times\widetilde{\mathbb{C}^2/\mathbb{Z}_m} \qquad \mathcal{X}_- = \mathbb{C}\times[\mathbb{C}^2/\mathbb{Z}_m]$$



$$F_{g,n}^{\mathcal{X}_+,(\hat{L}_+,\hat{f}_+)} \longleftrightarrow F_{g,n}^{\mathcal{X}_-,(\hat{L}_-,\hat{f}_-)}$$

analytic continuation + change of variables

$$F_{g,n}^{\mathcal{X}_+,\coprod_{j=1}^{m}(L_+^j,f_+^j)} \longleftrightarrow F_{g,n}^{\mathcal{X}_-,(L_-,f_-)}$$

analytic continuation + change of variables + $U^{\otimes n}$ ($U\in GL(m,\mathbb{C})$)

- Ke-Zhou (2015)

$\mathcal{X}_+, \mathcal{X}_-$ toric CY 3-orbifolds + conditions on $\mathcal{X}_+ \dashleftarrow\dashrightarrow \mathcal{X}_-$

$L_+ = L_- = S^1 \times \mathbb{C}$

$$F_{0,1}^{\mathcal{X}_+,(L_+,f_+)} \longleftrightarrow F_{0,1}^{\mathcal{X}_-,(L_-,f_-)}$$

analytic continuation + change of variables

- S. Yu (2020)

$\mathcal{X}_+, \mathcal{X}_-$ toric CY 3-orbifolds related by a single wall-crossing

(1) If $L_+ = L_- = S^1 \times [\mathbb{C}/\mathbb{Z}_m]$ then

$$F_{0,1}^{\mathcal{X}_+,(L_+,f_+)} \longleftrightarrow F_{0,1}^{\mathcal{X}_-,(L_-,f_-)}$$

analytic continuation + change of variables

(2) If $L_- = S^1 \times [\mathbb{C}/\mathbb{Z}_m]$

$L_+^1 = S^1 \times [\mathbb{C}/\mathbb{Z}_{m_1}]$ $\quad L_+^2 = S^1 \times [\mathbb{C}/\mathbb{Z}_{m_2}]$ $\quad m_1 + m_2 = m$

$$\begin{bmatrix} F_{0,1}^{\mathcal{X}_+(L_+^1,f_+^1)} \\ F_{0,1}^{\mathcal{X}_+(L_+^2,f_+^2)} \end{bmatrix} = U_{m_1,m_2} F_{0,1}^{\mathcal{X}_-,(L_-,f_-)}$$

analytic continuation + change of variables

$\rotatebox{90}{$\in$}$ $GL(m,\mathbb{C})$ depending only on m_1, m_2

- Fang-L-Yu-Zong (in progress)

(O1) If $L_+ = L_- = S^1 \times [\mathbb{C}/\mathbb{Z}_m]$ then

$$F_{g,n}^{\mathcal{X}_+,(L_+,f_+)} \longleftrightarrow \left\{ F_{g',n'}^{\mathcal{X}_-,(L_-,f_-)} \right\}_{(g',n') \in I_{g,n}}$$

finite

analytic continuation

+ change of variables

+ symplectic transformation

(O2) If $L_- = S^1 \times [\mathbb{C}/\mathbb{Z}_m]$

$L_+^1 = S^1 \times [\mathbb{C}/\mathbb{Z}_{m_1}]$, $L_+^2 = S^1 \times [\mathbb{C}/\mathbb{Z}_{m_2}]$

$$F_{g,n}^{\mathcal{X}_+,(L_+^1,f_+^1)(L_-^2,f_-^2)} \longleftrightarrow \left\{ F_{g',n'}^{\mathcal{X}_-,(L_-,f_-)} \right\}_{(g',n') \in I_{g,n}}$$

analytic continuation + change of variables

+ symplectic transformation + $U^{\otimes n}$ $\quad U \in GL(m,\mathbb{C})$

Our proof relies on the Remodeling Conjecture

Conjectured by Bouchard-Klemm-Mariño-Pasquetti (BKMP)

proved in full generality by Fang-L-Zong

The BKMP Remodeling Conjecture can be viewed as a version of all-genus open-closed mirror symmetry

$(\mathcal{X}, \omega_0)$ toric CY 3-orbifold + (L, f) framed AV Lagrangian

$\theta = (\theta_1 \cdots \theta_p)$

$\longrightarrow$ spectral curve $\mathcal{S} = (C, \log X, \log Y, B)$

- $C = \{ H(X, Y, q) = 0 \} \subset (\mathbb{C}^*)^2$ mirror curve

$$H(X,Y,q) = \sum_{(m,n) \in P} a_{m,n}(q) X^m Y^n \qquad q = (q_1, \dots, q_p)$$

- $\bar{C} \subset \mathbb{P}_\Delta \leftarrow$ projective toric surface

$\downarrow$

compactified mirror curve

$$\text{genus}(\bar{C}) = \mathfrak{g} = \dim H^4_{CR}(\mathcal{X})$$

- $B = B(p_1, p_2)$ meromorphic symmetric 2-form on $\bar{C} \times \bar{C}$

double pole along $\Delta \subset \bar{C} \times \bar{C}$, holomorphic on $\bar{C} \times \bar{C} - \Delta$

In local holomorphic coordinate near $(p, p) \in \Delta$

$$B = \Big(\frac{1}{(z_1 - z_2)^2} + \underset{\substack{\downarrow \\ \text{holomorphic and symmetric}}}{h(z_1, z_2)} \Big) dz_1 dz_2$$

$$\int_{p_1 \in \alpha_i} B(p_1, p_2) = 0 \qquad \{\alpha_i, \beta_j\} \text{ symplectic basis of } H_1(\bar{C})$$

Chekhov-Eynard-Orantin Topological Recursion

Initial data : $\omega_{0,1} = \log Y \frac{dX}{X}$, $\omega_{0,2} = B$

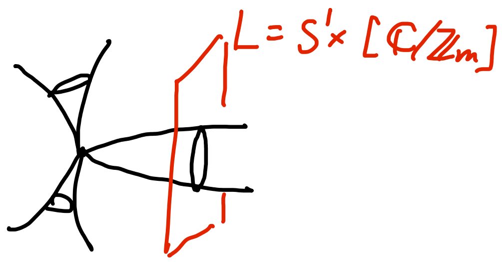
$2g-2+n>0$, $n>0$ $\quad \omega_{g,n}(p_1 \cdots p_n)$ defined recursively by taking residue at the $2g-2+n$ ramification points of the simply-branched cover $X: C \to \mathbb{C}^*$

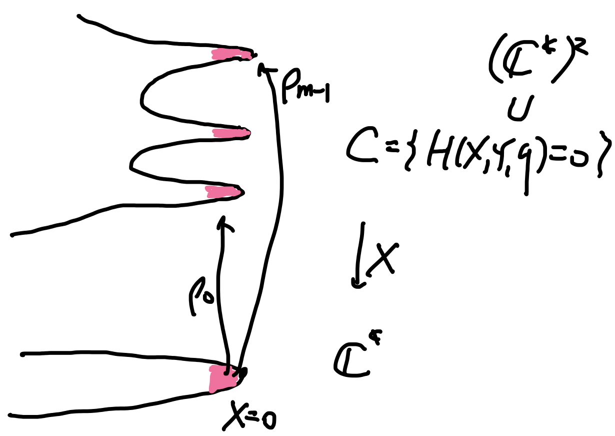
genus g, n punctures

$\omega_{g,n}$ meromorphic symmetric n-form on $(\bar{C})^n$

holomorphic on $(\bar{C} - \mathrm{Crit}(X))^n$

$\omega_{g,0}(q)$ obtained by $\omega_{g,1}$





On D^n $\quad (\rho_{j_1} \times \cdots \times \rho_{j_n})^* \omega_{g,n}$

$$= \underbrace{W_{g,n,(j_1,\ldots,j_n)}(q, X_1, \ldots, X_n)}_{\text{holomorphic and symmetric in } X_1 \cdots X_n} dX_1 \cdots dX_n$$

Theorem (Fang-L-Zong)

$g \geq 2$ $\quad F_g^{\mathcal{X}}(\tau) = \omega_{g,0}(q)$ under the closed mirror map

$$\tau_a = T^a(q) \qquad a = 1, \dots, p$$

$$= \begin{cases} \log q_a + A_a(q) & a = 1, \dots, p' \\ q_a + A_a(q) & a = p'+1, \dots, p \end{cases}$$

$A_a(q)$ holomorphic $\quad A_a(0) = 0$

$2g-2+n > 0$

$$d_{\tilde{X}_1} \cdots d_{\tilde{X}_n} F_{g,n,(k_1,\dots,k_n)}^{\mathcal{X},(\mathcal{L},f)}(\tau, \tilde{X}_1, \dots \tilde{X}_n)$$

$$= \sum_{j_1 \cdots j_n = 0}^{m-1} \prod_{i=1}^{n} C_{k_i}^{j_i} \, W_{g,n,(j_1 \cdots j_n)}(q, X_1 \cdots X_n)\, dX_1 \cdots dX_n$$

under closed mirror map $\quad \tau_a = T^a(q) \quad a = 1, \dots, p$

and open mirror map $\quad \log \hat{X}_i = \log X_i + A_0(q)$

$A_0(q)$ holomorphic $\quad A_0(0) = 0$

It remains to relate $\omega_{g,n}^{+}$ and $\omega_{g,n}^{-}$.

For $\mathcal{X} = \mathcal{X}_\pm$, we choose a basis $\{e_1, \dots, e_p\}$ of $H^2_{CR}(\mathcal{X})$ such that $\{e_1, \dots, e_g\}$ is a basis of $H^2_{CR,c}(\mathcal{X}) \subset H^2_{CR}(\mathcal{X})$

Let $\{e^1, \dots, e^g\}$ be the basis of $H^4_{CR}(\mathcal{X})$ dual to $\{e_1, \dots, e_g\}$

I-function (Coates-Corti-Iritani-Tseng; Cheong, Ciocan-Fontanine, Kim)

$$I_{\mathcal{X}}(q, z) = z 1_0 + \sum_{a=1}^{p} \underbrace{T^a(q)}_{\text{closed mirror maps}} e_a + \sum_{b=1}^{g} W_b(q) \frac{e^b}{z}$$

$p = g + n - 3$

C genus g, n punctures

There is $\mathbb{C}$-linear isomorphism

$$\pi: H_2((\mathbb{C}^*)^2, C; \mathbb{C}) \longrightarrow \mathcal{S}_{\mathcal{X}} = \mathrm{Span}\{1, T^1, \dots T^p, W_1, \dots, W_g\} \cong \mathbb{C}^{2g-2+n}$$

$$\pi(\Lambda) = \frac{1}{(2\pi\sqrt{-1})^2} \int_\Lambda \frac{dX}{X} \wedge \frac{dY}{Y}$$

Choose $A^0 = [T^2]$, $A^1, \dots A^p$ such that

$$\pi(A^0) = 1, \qquad \pi(A^a) = T^a, \quad 1 \le a \le p$$

$$\pi(B_b) = W_b, \quad 1 \le b \le g$$

$$H_2((\mathbb{C}^*)^2; C; \mathbb{C}) \to H_1(C; \mathbb{C}) \to H_1(\bar{C}; \mathbb{C}) \ni \alpha^a := \lambda(A^a)$$

$\beta_b := \lambda(B_b)$

Then $\{\alpha^1, \dots, \alpha^g, \beta_1, \dots, \beta_g\}$ is a symplectic basis of $H_1(\bar{C}; \mathbb{C})$

- analytic continuation + change of variables + symplectic transformation + $U \in GL(m, \mathbb{C})$ are determined by Yu's disk CTC and

$$\begin{tikzcd}
H_2((\mathbb{C}^*)^2, C_+; \mathbb{C}) \arrow[r, dash, "GM"] \arrow[d, "\pi_+", "\wr"'] & H_2((\mathbb{C}^*)^2, C_-; \mathbb{C}) \arrow[d, "\pi_-", "\wr"'] \\
S_{X_+} \arrow[r, "MB"] & S_{X_-}
\end{tikzcd}$$

GM: Gauss-Manin parallel transport

MB: Mellin-Barne analytic continuation

↳ computed by Borisov-Horja, generalized by Coates-Iritani-Jiang

Under the basis $\{1, T_\pm^{g+1}, \dots, T_\pm^{p}, T_\pm^{1}, \dots, T_\pm^{g}, W_1^\pm, \dots, W_g^\pm\}$

$\{A_\pm^0, A_\pm^{g+1}, \dots, A_\pm^{p}, A_\pm^{1}, \dots, A_\pm^{g}, B_1^\pm, \dots, B_g^\pm\}$

MB

GM

given by the $(2g+n-2)\times(2g+n-2)$ matrix

$$U_I^t = \left[\begin{array}{c|c|c} 1 & * & * \\ \hline 0 & * & * \\ \hline 0 & * & U_c^t \end{array}\right]$$

$Sp(2g, \mathbb{C})$

Under the basis $\{\alpha_\pm^1, \dots, \alpha_\pm^g, \beta_1^\pm, \dots \beta_g^\pm\}$

$GM: H_1(\bar{C}_+; \mathbb{C}) \to H_1(\bar{C}_-; \mathbb{C})$ is given by U_c^t

$U_c^t \in Sp(2g, \mathbb{C})$ determines the symplectic transformation

between $\omega_{g,n}^-$ $\leftarrow$ determined by $\{\alpha_i^- : 1 \le i \le g\}$

and $P(\omega_{g,n}^+)$ $\leftarrow$ determined by $\{GM(\alpha_i^-) : 1 \le i \le g\}$

analytic continuation

$$P(\omega_{g,n}^+) = \text{graph sum in terms of } \{\omega_{g',n'}^-\}_{(g',n')\in I_{g,n}}$$

If $\mathrm{Span}\{\alpha_i^- : 1 \le i \le g\} = \mathrm{Span}\{GM(\alpha_i^-) : 1 \le i \le g\}$

then $P(\omega_{g,n}^+) = \omega_{g,n}^-$

Special case: $g=0$ (e.g. $\mathcal{X}_- = \mathbb{C} \times [\mathbb{C}^2/\mathbb{Z}_m]$)

$$S_{\mathcal{X}_\pm} = \{1, T_\pm^1(q), \dots, T_\pm^p(q)\} \subseteq \mathbb{C}^{p+1} = \mathbb{C}^{n-2}$$

change of variables determined by U_I^t.