Holomorphic Morse inequalities, old and new

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

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Introduction and goals

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Important problems in algebraic or analytic geometry

- Find upper and lower bounds for the dimensions of cohomology groups $h^q(X, L^{\otimes m} \otimes \mathcal{F})$ where $\mathcal{F}$ is a coherent sheaf, asymptotically as $m \to +\infty$, e.g. in terms of $\theta = \Theta_{L,h}$.

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- (Harder question ?) In case $q = 0$ and $\mathcal{F}$ is invertible (say), try to analyze the base locus of $H^0(X, L^{\otimes m} \otimes \mathcal{F})$, i.e. the set of common zeroes of all holomorphic sections.

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Holomorphic Morse inequalities (D-, 1985) provide workable answers in terms of the q-index sets of the curvature form.

Holomorphic Morse inequalities: main statement

The *q-index set* of a real $(1,1)$-form θ is defined to be

$$X(\theta,q) = \{x \in X \mid \theta(x) \text{ has signature } (n-q,q)\}$$

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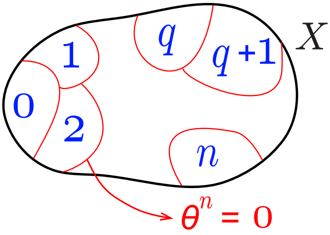
Set also $X(\theta, \le q) = \bigcup_{0 \le j \le q} X(\theta, j)$.

$X(\theta, q)$ and $X(\theta, \le q)$ are open sets.

$\mathrm{sign}(\theta^n) = (-1)^q$ on $X(\theta, q)$.



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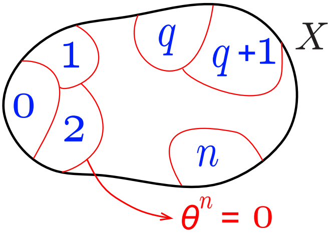

Theorem (D-, 1985)

Let $\theta = \Theta_{L,h}$ and $r = \operatorname{rank} \mathcal{F}$. Then, as $m \to +\infty$

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes \mathcal{F}) \leq r \frac{m^n}{n!} \int_{X(\theta, \leq q)} (-1)^q \theta^n + o(m^n).$$

Strategy of proof and consequences

The proof proceeds by considering the $\overline{\partial}$-complex and looking at the spectral theory of $\overline{\square} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ acting on sections of $L^{\otimes m} \otimes \mathcal{F}$.

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$$m\theta = im\sum_{j,k} \theta_{jk} dz_j \wedge d\overline{z}_k = i\sum_{j,k} \theta_{jk} d\zeta_j \wedge d\overline{\zeta}_k$$

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- $h^q(X, L^{\otimes m} \otimes \mathcal{F}) \geq r \dfrac{m^n}{n!} \displaystyle\int_{\bigcup_{q-1 \leq j \leq q+1} X(\theta,j)} (-1)^q \theta^n - o(m^n).$

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- For $q = 0$, $h^0(X, L^{\otimes m} \otimes \mathcal{F}) \ge r \dfrac{m^n}{n!} \displaystyle\int_{X(\theta,\le 1)} \theta^n - o(m^n).$

Singular version of holomorphic Morse inequalities

We assume here that L is equipped with a possibly singular metric $h = e^{-\varphi}$ were φ is quasi-psh with analytic singularities, i.e. locally

$$\varphi(z) = c \log \sum_j |g_j(z)|^2 + u(z), \quad g_j \text{ holomorphic}, \ u \in C^\infty, \ c > 0.$$

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Then L^2 estimates involve multiplier ideal sheaves $\mathcal{I}(m\varphi) \subset \mathcal{O}_X$

$$\mathcal{I}(m\varphi)_x = \Big\{ f \in \mathcal{O}_{X,x}\,;\ \exists U \ni x \text{ s.t. } \int_U |f|^2 e^{-m\varphi} dV < +\infty \Big\}.$$

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Theorem (L. Bonavero 1996 – proof based on blowing up)

The same estimates as above are still valid, when one considers instead the twisted cohomology groups

$$H^q(X, L^{\otimes m} \otimes \mathcal{I}(m\varphi) \otimes \mathcal{F})$$

and Morse integrals in the complement of $\Sigma = \varphi^{-1}(-\infty) =$ singular set of $\theta = \Theta_{L,h}$:

$$\int_{X(\theta,q) \smallsetminus \Sigma} (-1)^q \theta^n.$$

Algebraic versions of Morse inequalities

Assume here that X is projective algebraic / $\mathbb{C}$, and that $L = \mathcal{O}_X(A - B)$ where A and B are ample (or nef) $\mathbb{Q}$-divisors (such that $A - B$ is integral).

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Observation (D-, 1996)

In the above situation, the holomorphic Morse inequalities hold after replacing the q-index Morse integral by the intersection number $\binom{n}{q} A^{n-q} \cdot B^q$, and in particular (S. Trapani, 1995)

$$h^0(X, L^{\otimes m} \otimes \mathcal{F}) \geq r\,\frac{m^n}{n!}(A^n - nA^{n-1} \cdot B) - o(m^n).$$

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Proof. For $(1,1)$-forms $\alpha, \beta \geq 0$, elementary symmetric functions arguments yield

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Algebraic proof by F. Angelini (1996), via exact sequence arguments.

Algebraic Morse inequalities of Benoît Cadorel

Definition of adapted stratifications (projective case)

- An "adapted stratification" for L over X is a collection of non singular projective schemes $S = (S_j)$, $\dim S_j = j$, $S_n = X$, together with proper birational morphisms ψ_j of S_j onto the support $|D_j| = \psi_j(S_j)$ of a divisor D_j of S_{j+1}, such that, when putting $\Phi_j = \psi_{n-1} \circ \cdots \circ \psi_j : S_j \to X$, the pull-back $\Phi_j^* L$ satisfies $\Phi_j^* L \simeq \mathcal{O}_{S_j}(D_{j-1}) = \mathcal{O}_{S_j}(D_{j-1}^+ - D_{j-1}^-)$.

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Theorem (Cadorel, December 2019)

$$\sum_{0\le j\le q}(-1)^{q-j}h^j(X,L^{\otimes m}\otimes\mathcal{F}) \le \frac{(-1)^q r\, m^n}{n!}\deg c_1(L,S)^n_{[\le q]} + O(m^{n-1}).$$

Considerations and questions about base loci

Let (L, h) be a hermitian line bundle over X. If we assume that $\theta = \Theta_{L,h}$ satisfies $\int_{X(\theta,\leq 1)} \theta^n > 0$, then we know that L is big, i.e. that $h^0(X, L^{\otimes m}) \geq c\, m^n$, for $m \geq m_0$ and $c > 0$,

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Unsolved problem

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We would need for instance to be able to check the positivity of Morse integrals $\int_{Z(\theta_{|Z}, \leq 1)} \theta^{n-p}$ for Z irreducible, $\operatorname{codim} Z = p$.

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Conjecture

Let X be a compact complex manifold and $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ a Bott-Chern class, represented by closed real $(1,1)$-forms modulo $\partial\overline{\partial}$ exact forms. Assume α pseudoeffective, and set

$$\mathrm{Vol}(\alpha) = \sup_{T=\alpha+i\partial\overline{\partial}\varphi\geq 0} \int_X T_{ac}^n, \quad T \geq 0 \text{ current, } n = \dim X.$$

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Then

$$\mathrm{Vol}(\alpha) \geq \sup_{\theta\in\{\alpha\},\ \theta\in C^\infty} \int_{X(\theta,\leq 1)} \theta^n$$

where

$X(\theta,q) = q$-index set of $\theta = \big\{x\in X\,;\,\theta(x) \text{ has signature } (n-q,q)\big\}$.

Conjecture on volumes of (1,1)-classes

Conjectural corollary (transcendental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be nef classes. Then $\mathrm{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta$.

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Observation (BDPP, 2004)

The volume estimate holds if X has deformation approximations by projective manifolds X_ν of maximal Picard number $\rho(X_\nu) = h^{1,1}$.

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The volume estimate holds if X has deformation approximations by projective manifolds X_ν of maximal Picard number $\rho(X_\nu) = h^{1,1}$.

Theorem 1 (Xiao 2015, Popovici 2016)

If $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$, then $\alpha - \beta$ is a big class, i.e. $\mathrm{Vol}(\alpha - \beta) > 0$.

Conjecture on volumes of (1,1)-classes

Conjectural corollary (transcendental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be nef classes. Then $\mathrm{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta$.

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Theorem 2 (Witt-Nyström & Boucksom 2019)

The transcendental volume estimate holds if X is projective.

Entire curves in projective varieties and hyperbolicity

- The goal is to study (nonconstant) entire curves $f : \mathbb{C} \to X$ drawn in a projective variety/$\mathbb{C}$. The variety X is said to be Brody ($\Leftrightarrow$ Kobayashi) hyperbolic if there are no such curves.

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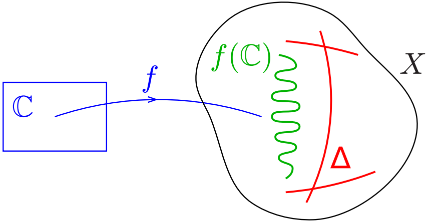

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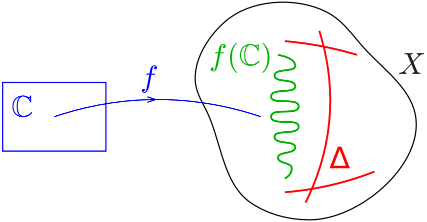

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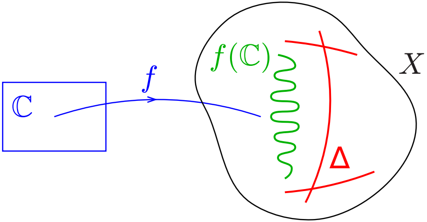

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- The strategy is to show that under suitable conditions, such entire curves must satisfy algebraic differential equations.

k-jets of curves and k-jet bundles

Let X be a nonsingular n-dimensional projective variety over $\mathbb{C}$.

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Definition of k-jets

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$$f(t) = x + t\xi_1 + t^2\xi_2 + \cdots + t^k\xi_k + O(t^{k+1}), \quad t \in D(0, \varepsilon) \subset \mathbb{C},$$

and $x = f(0) \in U$, $\xi_s \in \mathbb{C}^n$, $1 \le s \le k$.

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Notation

Let J^kX be the bundle of k-jets of curves, and $\pi_k : J^kX \to X$ the natural projection, where the fiber $(J^kX)_x = \pi_k^{-1}(x)$ consists of k-jets of curves $f_{[k]}$ such that $f(0) = x$.

Algebraic differential operators

Let $t \mapsto z = f(t)$ be a germ of curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet at any point $t = 0$. Look at the $\mathbb{C}^*$-action induced by *dilations* $\lambda \cdot f(t) := f(\lambda t)$, $\lambda \in \mathbb{C}^*$, for $f_{[k]} \in J^k X$.

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Taking a (local) connection ∇ on T_X and putting $\xi_s = f^{(s)}(0) = \nabla^s f(0)$, we get a trivialization $J^k X \simeq (T_X)^{\oplus k}$ and the $\mathbb{C}^*$ action is given by

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Here, we assume the coefficients $a_{\alpha_1 \alpha_2 \ldots \alpha_k}(x)$ to be holomorphic in x, and view P as a differential operator $P(f) = P(f\,;\,f', f'', \ldots, f^{(k)})$,

$$P(f)(t) = \sum a_{\alpha_1 \alpha_2 \ldots \alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}.$$

Graded algebra of algebraic differential operators

In this way, we get a graded algebra $\bigoplus_m E_{k,m}(X)$ of differential operators. As sheaf of rings, in each coordinate chart $U \subset X$, it is a pure polynomial algebra isomorphic to

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By filtering by the partial degree of $P(x; \xi_1, \ldots, \xi_k)$ successively in $\xi_k, \xi_{k-1}, \ldots, \xi_1$, one gets a multi-filtration on $E_{k,m}(X)$ such that the graded pieces are

$$G^\bullet E_{k,m}(X) = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1} T_X^* \otimes \cdots \otimes S^{\ell_k} T_X^*.$$

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where $T_X^*\langle\Delta\rangle$ is the logarithmic tangent bundle, i.e., the locally free sheaf generated by $\frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p}, dz_{p+1}, \ldots, dz_n$.

Projectivized jets and direct image formula

Green Griffiths bundles

Consider $X_k := J^k X/\mathbb{C}^* = \mathrm{Proj} \bigoplus_m E_{k,m}(X)$. This defines a bundle $\pi_k : X_k \to X$ of weighted projective spaces whose fibers are the quotients of $(\mathbb{C}^n)^k \smallsetminus \{0\}$ by the $\mathbb{C}^*$ action

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and let $\mathcal{O}_{X_k\langle\Delta\rangle}(1)$ be the corresponding tautological sheaf, so that

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Generalized Green-Griffiths-Lang conjecture

Generalized GGL conjecture

If (X, Δ) is a log pair of general type, in the sense that $K_X + \Delta$ is big, then there is a proper algebraic subvariety $Y \subsetneq X \smallsetminus \Delta$ containing all entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$.

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Fundamental vanishing theorem

[Green-Griffiths 1979], [D- 1995], [Siu-Yeung 1996], ...
Let A be an ample divisor on X. Then, for all global jet differential operators on (X, Δ) with coefficients vanishing on A, i.e. $P \in H^0(X, E_{k,m}(X, \Delta) \otimes \mathcal{O}(-A))$, and for all entire curves $f : \mathbb{C} \to X \setminus \Delta$, one has $P(f_{[k]}) \equiv 0$.

Proof of the fundamental vanishing theorem

Simple case. First consider the compact case ($\Delta = 0$), and assume that f is a Brody curve, i.e. $\|f'\|_\omega$ bounded for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

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Logarithmic case. In the logarithmic case, one can use instead a "Poincaré type metric" ω. Removing the hypothesis f' bounded is more tricky. One possible way is to use the Ahlfors lemma and some representation theory.

Probabilistic cohomology estimate

Theorem (D-, PAMQ 2011 + recent work for logarithmic case)

Fix A ample line bundle on X, and hermitian structures $(T_X\langle\Delta\rangle, h)$, (A, h_A) with $\omega_A = \Theta_{A,h_A} > 0$.

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$$L_{k,\varepsilon} = \mathcal{O}_{X_k\langle\Delta\rangle}(1) \otimes \pi_k^*\mathcal{O}_X\Big(-\frac{1}{kn}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\varepsilon A\Big), \quad \varepsilon \in \mathbb{Q}_+.$$

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Then for m sufficiently divisible, we have a lower bound

$$h^0(X_k, L_{k,\varepsilon}^{\otimes m}) = h^0\Big(X, E_{k,m}(X,\Delta)\otimes\mathcal{O}_X\Big(-\frac{m\varepsilon}{kn}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)A\Big)\Big)$$
$$\geq \frac{m^{n+kn-1}}{(n+kr-1)!}\frac{(1+\frac{1}{2}+\cdots+\frac{1}{k})^n}{n!\,(k!)^n}\Big(\int_{X(\eta,\leq 1)}\eta_\varepsilon^n - \frac{C}{\log k}\Big).$$

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Corollary

If $K_X+\Delta$ is big and $\varepsilon>0$ is small, then η_ε can be taken >0, so $h^0(X_k, L_{k,\varepsilon}^{\otimes m}) \geq C_{n,k,\eta,\varepsilon}\, m^{n+kn-1}$ with $C_{n,k,\eta,\varepsilon}>0$, for $m\gg k\gg 1$.

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Theorem (D-, PAMQ 2011 + recent work for logarithmic case)

Fix A ample line bundle on X, and hermitian structures $(T_X\langle\Delta\rangle, h)$, (A, h_A) with $\omega_A = \Theta_{A,h_A} > 0$. Let $\eta_\varepsilon = \Theta_{K_X+\Delta,\det h^*} - \varepsilon\omega_A$ and

$$L_{k,\varepsilon} = \mathcal{O}_{X_k\langle\Delta\rangle}(1) \otimes \pi_k^* \mathcal{O}_X\Big(-\frac{1}{kn}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\varepsilon A\Big), \quad \varepsilon \in \mathbb{Q}_+.$$

Then for m sufficiently divisible, we have a lower bound

$$h^0(X_k, L_{k,\varepsilon}^{\otimes m}) = h^0\Big(X, E_{k,m}(X,\Delta) \otimes \mathcal{O}_X\Big(-\frac{m\varepsilon}{kn}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)A\Big)\Big)$$
$$\geq \frac{m^{n+kn-1}}{(n+kr-1)!} \frac{(1+\frac{1}{2}+\cdots+\frac{1}{k})^n}{n!\,(k!)^n}\Big(\int_{X(\eta,\leq 1)} \eta_\varepsilon^n - \frac{C}{\log k}\Big).$$

Corollary

If $K_X + \Delta$ is big and $\varepsilon > 0$ is small, then η_ε can be taken > 0, so $h^0(X_k, L_{k,\varepsilon}^{\otimes m}) \geq C_{n,k,\eta,\varepsilon}\, m^{n+kn-1}$ with $C_{n,k,\eta,\varepsilon} > 0$, for $m \gg k \gg 1$.

Therefore, all $f : \mathbb{C} \to X \smallsetminus \Delta$ satisfy algebraic diff. equations.

A good Finsler metric on the k-jet bundle

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$$\Psi_{h_k}(f_{[k]}) := \Big(\sum_{1\le s\le k} \|\varepsilon_s \nabla^s f(0)\|_{h(x)}^{2b/s}\Big)^{1/b}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

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Letting $\xi_s = \varepsilon_s \nabla^s f(0)$, this can be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k}(1)$, and the curvature form of L_k is obtained by computing $\frac{i}{2\pi}\partial\overline{\partial} \log \Psi_{h_k}(f_{[k]})$ as a function of $(x, \xi_1, \ldots, \xi_k)$.

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Modulo negligible error terms of the form $O(\varepsilon_{s+1}/\varepsilon_s)$, this gives

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi}\sum_{1\le s\le k}\frac{1}{s}\frac{|\xi_s|^{2b/s}}{\sum_t |\xi_t|^{2b/t}}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\, dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor $-\Theta_{T_X,h}$ and $\omega_{\mathrm{FS},k}$ is the weighted Fubini-Study metric on the fibers of $X_k \to X$.

Evaluation of Morse integrals

The above expression can be simplified by using polar coordinates

$$x_s = |\xi_s|_h^{2b/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

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where $\omega_{\mathrm{FS},k}(\xi) = \frac{i}{2\pi b}\partial\overline{\partial}\log \sum_{1\le s\le k} |\xi_s|^{2b/s} > 0$ on fibers of $X_k \to X$.

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By holomorphic Morse inequalities, we need to evaluate an integral

$$\int_{X_k(\Theta_{L_h,h_k},\le 1)} \Theta_{L_k,h_k}^{N_k}, \quad N_k = \dim X_k = n + (kn-1),$$

and we have to integrate over the parameters $z \in X$, $x_s \in \mathbb{R}_+$ and u_s in the unit sphere bundle $\mathbb{S}(T_X, 1) \subset T_X$.

Probabilistic interpretation of the curvature

The signature of Θ_{L_k,h_k} depends only on the vertical terms, i.e.

$$\sum_{1\le s\le k} \frac{1}{s} x_s Q(u_s), \quad Q(u_s) := \frac{i}{2\pi} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j.$$

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After averaging over $(x_s) \in \Delta^{k-1}$ and computing the rational number $\int \omega_{\mathrm{FS},k}(\xi)^{nk-1} = \frac{1}{(k!)^n}$, what is left is to evaluate Morse integrals with respect to $(u_s) \in (\mathbb{S}(T_X, 1))^k$ of "horizontal" $(1,1)$-forms given by sums $\sum \frac{1}{s} Q(u_s)$, where (u_s) is a sequence of "random points" on the unit sphere.

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As $k \to +\infty$, this sum is asymptotically equivalent to a "Monte-Carlo" integral $\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \int_{u\in\mathbb{S}(T_X,1)} Q(u)\, du.$

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Now, $Q(u)$ quadratic form $\Rightarrow \int_{u\in\mathbb{S}(T_X,1)} Q(u)\, du = \frac{1}{n} \mathrm{Tr}(Q)$,

and we have $\mathrm{Tr}(Q) = \mathrm{Tr}(-\Theta_{T_X,h}) = \Theta_{\det T_X^*,\det h^*} = \Theta_{K_X,\det h^*}$.

The asserted Morse estimates follow.

A result on the base loci of jet differentials

Thorem (D-, 2021)

Let (X, Δ) be a pair of general type, i.e. such that $K_X + \Delta$ is big. Then there exists $k_0 \in \mathbb{N}$ and $\delta > 0$ with the following properties.

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with $k \ge k_0$, $\mathrm{ord}(P_j) = s_j$, $1 \le s_1 < \cdots < s_\ell \le k$, $\displaystyle\sum_{1 \le j \le \ell} \frac{1}{s_j} \le \delta \log k$.

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Then the Morse integrals $\int_{Z(L_{k,\varepsilon},\le 1)}\Theta^{\dim Z}_{L_{k,\varepsilon}}$ of

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are positive for $\varepsilon>0$ small, hence $H^0(Z,L_{k,\varepsilon}^{\otimes m})\ge c\,m^{\dim Z}$ for $m\gg 1$.

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Unfortunately, seems insufficient to show that $\dim\mathrm{IBs}(L_{k,\varepsilon})<n$.

Thank you for your attention!