Deformation quantization, and obstructions to the existence of closed star products

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1. Star product

A **star product** on a Poisson manifold M of dimension $n = 2m$ is an associative product $*$ on the space $C^\infty(M)[[\nu]]$ of formal power series in ν with coefficients in $C^\infty(M)$ (**formal functions**) such that if we write

$$f * g := \sum_{r=0}^{\infty} \nu^r C_r(f, g) \quad \text{for } f, g \in C^\infty(M)$$

then

1. the C_r's are bidifferential ν-linear operators,
2. $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \{f, g\}$,
3. the constant function 1 is a unit for $*$ (i.e. $f * 1 = f = 1 * f$).

Giving a star product is referred to as a **deformation quantization**.
(Bayen-Flato-Fronsdal-Lichnérowicz-Sternheimer, 1978)

Recall that a symplectic form ω is a closed nondegenerate 2-form.
It induces the Poisson bracket

$$\{f, g\} := -\omega(X_f, X_g)$$

for f, $g \in C^\infty(M)$
and vector field X_f uniquely determined by $\imath(X_f)\omega = df$.

There are known **constructions of star products** for symplectic manifolds by De Wilde-Lecompte (1983), Fedesov (1994), Omori-Maeda-Yoshioka (1991), and for general Poisson manifolds by Kontsevich (2003 (1997 peprint)).

This talk takes up **Fedosov's star product for compact symplectic manifolds.**

Example : Moyal star product (local models on Darboux charts)

Consider the vector space $\mathbb{R}^{2n}$ endowed with standard symplectic structure

$$\omega_{\text{std}} := \frac{1}{2}(\omega_{\text{std}})_{ij} dx^i \wedge dx^j.$$

The Moyal star product of f and $g \in C^\infty(\mathbb{R}^{2n})$ is defined by:

$$\begin{aligned}(f *_{\text{Moyal}} g)(x) &:= \left(\exp\left(\frac{\nu}{2}\Lambda^{ij}\partial_{y^i}\partial_{z^j}\right) f(y)g(z)\right)\Big|_{y=z=x} \\ &= \sum_{r=0}^{+\infty}\left(\frac{\nu}{2}\right)^r \frac{1}{r!}\Lambda^{i_1 j_1}\dots\Lambda^{i_r j_r}\frac{\partial^r f}{\partial x^{i_1}\dots\partial x^{i_r}}(x)\frac{\partial^r g}{\partial x^{j_1}\dots\partial x^{j_r}}(x),\end{aligned}$$

where Λ^{ij} denotes the coefficients of the inverse matrix of $(\omega_{\text{std}})_{ij}$.

The **trace of a star product** is an algebra character

$$\mathrm{Tr} : C^\infty(M)[[\nu]] \to \mathbb{R}[\nu^{-1}, \nu]]$$

satisfying

$$\mathrm{Tr}([f, g]_*) = 0.$$

Fact [Fedosov, Nest–Tsygan, Gutt–Rawnsley]
Any star product $*$ on a symplectic manifold (M, ω) admits a trace.
Any trace is given by an L^2-pairing with a formal function $\rho \in C^\infty(M)[\nu^{-1}, \nu]]$:

$$\mathrm{Tr}(F) = \frac{1}{\nu^m} \int_M F\rho \; dv.$$

The formal function ρ **is called the trace density**. It is unique up to multiplication by a formal constant, i.e. an element of $\mathbb{R}[\nu^{-1}, \nu]]$.

A star product is said to be (strongly) **closed**
if the integration functional is a trace,
equivalently, if the **trace density is a formal constant**,
i.e. $\rho \in \mathbb{R}[\nu^{-1}, \nu]]$.

Closed star products were considered by Connes-Flato-Sternheimer to study the relation between the cyclic cohomology and Hochschild cohomology (1992).

Omori-Maeda-Yoshioka proved the existence of a closed star product on any symplectic manifold (1992) (based on their construction using the notion of "Weyl manifold").

Example : Berezin-Toeplitz star product (related to Kähler geom)

For an ample line bundle $L \to M$ with a Kähler metric in $c_1(L)$, there is a star product known as **Berezin-Toeplitz star product**.

To a function $F \in C^\infty(M)$, one can associate a **Toeplitz operator** $T_F^k \in End(H^0(M, L^k))$ defined by

$$T_F^k : H^0(M, L^k) \to \Gamma(M, L^k) \to H^0(M, L^k) : s \mapsto Fs \mapsto \Pi^k(Fs),$$

for $\Pi^k : \Gamma(M, L^k) \to H^0(M, L^k)$ being the L^2-projection.

By Bordemann-Meinrenken-Schlichenmaier, there are bi-differential operators C_j such that

$$\left\| T_F^k \circ T_G^k - \sum_{j=0}^{j=N-1} \left(\frac{1}{k}\right)^j T^k_{C_j(F,G)} \right\|_{Op} \leq K_N(F,G) \left(\frac{1}{k}\right)^N$$

for $F, G \in C^\infty(M)$.

Definition The **Berezin-Toeplitz (BT for short) star product** $*_{BT}$ is defined by

$$F *_{BT} G := \sum_{j=0}^{\infty} \nu^j C_j(F,G) \text{ for } F,G \in C^\infty(M).$$

By Bordemann-Meinrenken-Schlichenmaier, the trace of the BT star product is given by

$$\mathrm{tr}^{*_{BT}}(F) := \sum_{j=0}^{\infty} \nu^{j-m} \int_M \tau_j(F) \frac{\omega^m}{m!}$$

where

$$\left| Tr\left(T_F^k\right) - \sum_{j=0}^{j=N-1} \left(\frac{1}{k}\right)^{j-m} \int_M \tau_j(F) \frac{\omega^m}{m!} \right| \leq \tilde{K}_N(F) \left(\frac{1}{k}\right)^{N-m}$$

for linear differential operators τ_j on $C^\infty(M)$, with $\tau_0 = Id$.

But for the **Bergman kernel** ρ_k

$$Tr^k\left(T_F^k\right) = \int_M F(x)\rho_k(x)\frac{\omega^m}{m!}$$

and ρ_k has the well-known asymptotic expansion

$$\left\|\rho_k - \sum_{i=0}^{s} a_i k^{m-i}\right\|_{C^r} \leq C_{s,r}k^{m-s-1},$$

with a_1 the **scalar curvature**.
Thus

$$\tau_j(F) = a_j F,$$

i.e. **the trace density for $*_{BT}$ coincides with the asymptotic expansion of the Bergman kernel as a formal function in $\nu = 1/k$.**

In this talk, we consider **Fedosov star product** constructed on symplectic manifolds.

The Fedosov star product is defined
given a **symplectic connection** ∇ and
a **closed formal** 2-**form** $\Omega \in \nu\Omega^2(M)[[\nu]]$,
and thus we denote it by $*_{\nabla,\Omega}$.

Here, a symplectic connection means
a torsion free connection making ω parallel.

Fedosov's construction roughly goes as follows:

Step 1 : Using the symplectic connection we can construct a flat connection of the "Weyl algebra bundle" W. (The curvature lies in the center of the Weyl algebra.)

Step 2 : Flat sections (parallel sections) in $\Gamma(W)$ form an algebra.

Step 3 : There is a one-to-one correspondence between the set of those flat sections and $C^\infty(M)[[\nu]]$, which induces a star product.

It is known that any star product on a symplectic manifold
is equivalent to a Fedosov star product.
The equivalence is given by

$$1 + \sum_{k \geq 1} \nu^k T_k$$

with T_k differential operators.

In this talk, we study closedness of Fedosov star products naturally attached to symplectic or Kähler manifolds.

On a compact symplectic manifold, we fix the de Rham class $[\omega_0]$ of the symplectic form and a formal second cohomology class $[\Omega_0] \in \nu H^2(M, \mathbb{R})[[\nu]]$.

We study the following problem:

Problem : Can one find a triple (ω, ∇, Ω) consisting of a symplectic form $\omega \in [\omega_0]$, a symplectic connection ∇ with respect to ω and $\Omega \in [\Omega_0]$ such that $*_{\nabla,\Omega}$ is closed?

Let G be a compact connected Lie group
acting effectively on a compact symplectic manifold M
preserving the symplectic form ω, a closed formal 2-form $\Omega \in \nu\Omega^2(M)[[\nu]]$
and a symplectic connection ∇ so that the Fedosov star product $*_{\nabla,\Omega}$ is G-invariant.

We identify a Lie algebra element $X \in \mathfrak{g}$ with a vector field on M by the action of G.

To define the **quantum moment map**, regard $\omega - \Omega$ as the "**quantum symplectic form**".

If a vector field X satisfies

$$i(X)(\omega - \Omega) = df_X$$

for some formal function $f_X \in C^\infty(M)[[\nu]]$, we call X a **quantum Hamiltonian vector field**,
and also call f_X the **quantum Hamiltonian function** of X.

Given a symplectic form ω and a closed formal 2-form Ω, a G-equivariant map $\mu : M \to \mathfrak{g}^*[[\nu]]$ is a **quantum moment map** if $\mu_X := \langle \mu, X \rangle \in C^\infty(M)[[\nu]]$ is a quantum Hamiltonian function of $X \in \mathfrak{g}$, i.e.

$$i(X)(\omega - \Omega) = d\mu_X.$$

If there is a quantum moment map, we say that G-action on (M, ω, Ω) is **quantum-Hamiltonian**.

Quantum moment maps are not unique, and any two of them differ by a formal constant.

Thus, we can assume the quantum moment map is **normalized** so that

$$\int_M \mu_X (\omega - \Omega)^m = 0.$$

Given a quantum-Hamiltonian G-space (M, ω_0, Ω_0), we denote by $\mathcal{C}^G([\omega_0], [\Omega_0])$ the space consisting of all triples (ω, Ω, ∇) such that

(a) (M, ω, Ω) is a quantum-Hamiltonian G-space,

(b) ω is cohomologous to ω_0 and there is a smooth path $\{\omega_s\}_{0 \leq s \leq 1}$ consisting of G-invariant symplectic forms joining ω_0 and ω in the cohomology class $[\omega_0]$ (so that **Moser's theorem** can be applied),

(c) Ω is cohomologous to Ω_0, and

(d) ∇ is a G-invariant symplectic connection with respect to ω.

The trace of a star product $*$ on a symplectic manifold (M, ω) can be **normalized** as follows.

On a contractible Darboux chart U we have an **equivalence**

$$B : (C^\infty(U)[[\nu]], *) \to (C^\infty(U)[[\nu]], *_{\mathsf{Moyal}})$$

of $*|_{C^\infty(U)[[\nu]]}$ with the Moyal star product $*_{\mathsf{Moyal}}$ satisfying

$$Bf *_{\mathsf{Moyal}} Bg = B(f * g).$$

The **normalization condition** is

$$\mathsf{Tr}(f) = \frac{1}{(2\pi\nu)^m} \int_M Bf \, \frac{\omega^m}{m!}.$$

B has been expressed explicitly and studied by Fedosov and Gutt-Rawnsley.

Theorem : Let (M, ω_0, Ω_0) be a quantum-Hamiltonian G-space and consider a triple (ω, Ω, ∇) in $\mathcal{C}^G([\omega_0], [\Omega_0])$. For $X \in \mathfrak{g}$, let μ_X be the quantum Hamiltonian function of X with respect to $\omega - \Omega$ with normalization

$$\int_M \mu_X (\omega - \Omega)^m = 0.$$

Then the **normalized trace** $\mathrm{Tr}^{*_{\nabla,\Omega}}(\mu_X)$ of the Fedosov star product $*_{\nabla,\Omega}$ is **independent of the choice of** (ω, Ω, ∇) in $\mathcal{C}^G([\omega_0], [\Omega_0])$.

The proof relies on the works on Fedosov and Gutt-Rawnsley on B.

Hence, one can define a symplectic invariant :

Definition We define a character $\mathsf{Tr}^{[\omega_0],[\Omega_0]} : \mathfrak{g} \to \mathbb{R}[\nu^{-1}, \nu]]$ by

$$\mathsf{Tr}^{[\omega_0],[\Omega_0]}(X) := \mathsf{Tr}^{*\nabla,\Omega}(\mu_X)$$

where the right hand side is given by the above Theorem with normalization

$$\int_M \mu_X (\omega - \Omega)^m = 0.$$

In the particular case $\Omega = 0$, we obtain an
obstruction to $\exists$ of closed Fedosov star products :

Theorem Let (M, ω_0) be a compact symplectic manifold.
If there exists a closed Fedosov star product $*_{\nabla,0}$ for $(\omega, 0, \nabla)$ in $\mathcal{C}^G([\omega_0], 0)$ then $\mathrm{Tr}^{[\omega_0],0}$ vanishes.

Expanding $\mathrm{Tr}^{[\omega_0],0}(X)$ in terms of power series in ν we obtain a **series of integral invariants** obstructing the existence of closed Fedosov star products

(i.e. L^2-inner product of the Hamiltonian function and the trace density).

Fact (La Fuente-Gravy): The trace density of $*_{\nabla,0}$ is given by

$$\rho^{\nabla,0} := 1 + \frac{\nu^2}{24}\mu(\nabla) + O(\nu^3)$$

where $\mu(\nabla)$ is the **Cahen-Gutt momentum** (explained later) of the symplectic connection ∇ given by

$$\mu(\nabla) := (\nabla^2_{(p,q)} \mathrm{Ric}^\nabla)^{pq} - \frac{1}{2}\mathrm{Ric}^\nabla_{pq}\mathrm{Ric}^{\nabla\, pq} + \frac{1}{4}\mathrm{R}^\nabla_{pqrs}\mathrm{R}^{\nabla\, pqrs},$$

where R^∇ is the curvature of ∇ and $\mathrm{Ric}^\nabla(\cdot,\cdot) := \mathrm{tr}[V \mapsto R^\nabla(V,\cdot)\cdot]$ is the Ricci tensor.

Thus, the closedness of $*_{\nabla,0}$ implies the constancy of the Cahen-Gutt momentum $\mu(\nabla)$.

This work was originally motivated by the study of Cahen-Gutt moment map of the space of symplectic connections with the action $\mathrm{Ham}(\omega)$.

Comparing it with BT star product, we expect $\mu(\nabla)$ should play a similar role as the **scalar curvature** in Kähler geometry.

List of similarities with cscK

The integral of $\mu(\nabla)$ is the Pontrjagin number $p_1 \cdot [\omega]^{m-2}$.
c.f. $\int_M Scal(\omega)\omega^m = -c_1(K_M) \cdot [\omega]^{m-1}$.

In an earlier work, La Fuente-Gravy obtained a "Futaki invariant" obstructing $\exists$ of a Kähler metric with constant $\mu(\nabla)$.

Indeed, for the Kähler case, the ν^{2-m}-term of $\mathrm{Tr}^{[\omega_0],0}$ is exactly La Fuente-Gravy's Futaki invariant.

Futaki-Ono obtained a different derivation of La Fuente-Gravy's Futaki invariant using Cahen-Gutt moment map similarly to the Donaldson-Fujiki picture.

La Fuente-Gravy's Futaki invariant in Kähler setting coincides with one of the obstructions to asymptotic Chow semi-stability.

F-Ono obtained Lichnerowicz-Matsushima type result for constant $\mu(\nabla)$ on compact Kähler manifolds.

Application to another Kähler setting.
On a compact Kähler G-manifold (M, ω_0, J) for a compact Lie group G preserving ω_0 and J, set $\mathcal{M}^G_{[\omega_0]}$ to be the space of G-invariant Kähler forms in the cohomology class of ω_0.

Consider the closed 2-form

$$\Omega(\omega) := \nu \operatorname{Ric}(\omega),$$

with $\operatorname{Ric}(\omega) := \operatorname{Ric}^\nabla(J\cdot, \cdot)$ being the Ricci form of the Kähler manifold (M, ω, J).

Problem(Kähler version) Can one find $\omega \in \mathcal{M}^G_{[\omega_0]}$ with Levi-Civita connection ∇ and Ricci form $\operatorname{Ric}(\omega)$ such that $*_{\nabla, \nu \operatorname{Ric}(\omega)}$ is closed?

A trace density for $*_{\nabla,\Omega(\omega)}$ has an expansion

$$\rho^{\nabla,\Omega(\omega)} = 1 - \frac{\nu}{2}\mathsf{Scal}_\omega + O(\nu^2),$$

(La Fuente-Gravy).

So a necessary condition for $*_{\nabla,\Omega(\omega)}$ to be closed is the existence of a constant scalar curvature Kähler metric.

The triple $(\omega, \Omega(\omega), \nabla)$ is in $\mathcal{C}^G([\omega_0], [\Omega(\omega_0)])$, and has the same quantum moment map, which we denote by $\tilde{\mu}_X$, normalized by

$$\int_M \tilde{\mu}_X \omega^m = 0.$$

Theorem Let (M, ω_0, J) be a compact Kähler manifold. Then

$$\mathrm{Tr}^{\mathcal{M}^G_{[\omega_0]}}(X) := \mathrm{Tr}^{*_{\nabla,\Omega(\omega)}}(\tilde{\mu}_X)$$

is independent of the choice of $\omega \in \mathcal{M}^G_{[\omega_0]}$. Moreover, if there exists a closed Fedosov star product $*_{\nabla,\Omega(\omega)}$ for $\omega \in \mathcal{M}^G_{[\omega_0]}$, then $\mathrm{Tr}^{\mathcal{M}^G_{[\omega_0]}}(X)$ vanishes.

Cahen-Gutt moment map:
On a symplectic manifold (M,ω), the space of symplectic connections $\mathcal{E}(M,\omega)$ is an affine space modeled on the set of all smooth sections $\Gamma(S^3(T^*M))$ of symmetric covariant 3-tensors:

$$\mathcal{E}(M,\omega) \cong \nabla + \Gamma(S^3(T^*M)), \qquad \omega_{i\ell}(\tilde{\Gamma}^{\ell}_{jk} - \Gamma^{\ell}_{jk}) \text{ symmetric.}$$

We assume M is a closed manifold.

On $\mathcal{E}(M,\omega)$ there is a natural symplectic structure $\Omega^{\mathcal{E}}$ defined at ∇ given by

$$\Omega^{\mathcal{E}}_{\nabla}(\underline{A},\underline{B}) = \int_M \omega^{i_1 j_1}\omega^{i_2 j_2}\omega^{i_3 j_3}\underline{A}_{i_1 i_2 i_3}\,\underline{B}_{j_1 j_2 j_3}\,\omega_m$$

for $\underline{A}$, $\underline{B} \in T_{\nabla}\mathcal{E}(M,\omega) \cong \Gamma(S^3(T^*M))$ where $\omega_m := \frac{\omega^m}{m!}$.

Theorem(Cahen–Gutt) The function μ on $\mathcal{E}(M, \omega)$ gives a moment map for the action of $\mathrm{Ham}(M, \omega)$.

This follows from the formula

$$\frac{d}{dt}\Big|_{t=0} \int_M \mu(\nabla + tA)\, f\, \omega_m = \Omega^{\mathcal{E}}(\underline{L_{X_f}\nabla}, \underline{A}),$$

where

$$\underline{L_X \nabla} = (X^s R(\nabla, \omega)_{squt} + \nabla_q \nabla_u X^s\, \omega_{st})\, dx^q \otimes dx^u \otimes dx^t.$$

(Kähler case given later.)

Now we assume that M is a compact Kähler manifold and that ω is a fixed symplectic form. We set as in **Donaldson-Fujiki picture**

$$N = \{J \text{ integrable complex structure} \mid (M, \omega, J) \text{ is a Kähler manifold.}\}$$

La Fuente-Gravy considered the *Levi-Civita map* $lv : N \to \mathcal{E}(M, \omega)$ sending J to the Levi-Civita connection ∇^J of the Kähler manifold (M, ω, J). **Then** $lv^*\Omega^{\mathcal{E}}$ **gives a new symplectic structure on** N if (ω, J) has non-negative Ricci curvature (for nondegeneracy).

Lemma : If we choose local holomorphic coordinates $z^1, \cdots, z^m$ then for any real smooth function f we have

$$\begin{aligned} \underline{L_{X_f}\nabla^J} &= f_{ijk}dz^i \otimes dz^j \otimes dz^k + f_{\overline{ijk}}dz^{\bar{i}} \otimes dz^{\bar{j}} \otimes dz^{\overline{k}} \\ &+ f_{ij\overline{k}}dz^i \otimes dz^j \otimes dz^{\overline{k}} + f_{\overline{ij}k}dz^{\bar{i}} \otimes dz^{\bar{j}} \otimes dz^k \\ &+ f_{ik\bar{j}}dz^i \otimes dz^{\bar{j}} \otimes dz^k + f_{\overline{ik}j}dz^{\bar{i}} \otimes dz^j \otimes dz^{\overline{k}} \\ &+ f_{jk\bar{i}}dz^{\bar{i}} \otimes dz^j \otimes dz^k + f_{\overline{jk}i}dz^i \otimes dz^{\bar{j}} \otimes dz^{\overline{k}} \end{aligned}$$

where the lower indices of f stand for the covariant derivatives, e.g. $f_{ij\overline{k}} = \nabla_{\overline{k}}\nabla_j\nabla_i f$.

$$\underline{L_{X_f}\nabla^J} = 0 \Longrightarrow f_{\overline{ij}k} = 0 \iff f_{\overline{ij}} = 0 = f_{ij} \Longrightarrow \underline{L_{X_f}\nabla^J} = 0$$

Proposition 1. *For a real smooth function* f, $L_{X_f}\nabla^J = 0$ *if and only if* X_f *is a holomorphic Killing vector field.*

Corollary 2 (La Fuente-Gravy 2016)**.** *Let* (M, ω) *be a compact Kähler manifold, and* $\mathfrak{g}_{\mathbf{R}}$ *be the real reduced Lie algebra of holomorphic vector fields. We normalize the Hamiltonian functions* f *so that* $\int_M f\ \omega_m = 0$*. Then*

$$\mathrm{Fut}(\mathrm{grad}' f) := \int_M \mu(\nabla^J)\, f\ \omega_m$$

is independent of the choice of $J \in \mathcal{J}(M, \omega)$*.*

Theorem 3 (F-Ono 2018)**.**
If there exists a Kähler metric of non-negative Ricci curvature such that $\mu(\nabla)$ *is constant for the Cahen–Gutt moment map* μ *and the Levi-Civita connection* ∇ *then the* **reduced Lie algebra** $\mathfrak{g}$ *of holomorphic vector fields is* **reductive***.*

To show this we define Cahen–Gutt version of extremal Kähler metrics and prove the same structure theorem as the Calabi extremal Kähler metrics.