A personal tribute to Louis Nirenberg: February 28, 1925-January 26, 2020.

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Preface: California meets New York 1972.

I first met Louis Nirenberg in person in 1972 when I became a Courant Instructor after my Ph.d work at Stanford. He was already a celebrated mathematician and a suave sophisticated New Yorker, athough he was born in Hamilton, Canada and grew up in Montreal. I was born in Brooklyn, New York and a post-doc at Courant was a return home for me. I had been greatly influenced by my time at Stanford in the late sixties by the hippie culture and the political anti-war activism. Louis accepted and welcomed me without hesitation as he did with almost everyone he met.

Louis' early work

During the twenty year period period 1953-1973 he produced an incredible body of work in many fields of analysis, real and complex geometry starting with his thesis work solving the classical Weyl and Minkowski problems in the smooth category.

Since this is the Geometry festival, I will mostly talk about Louis' work inspired by geometric ideas and I will give some background history since I personally find it interesting as I get older.

Louis loved to collaborate and I apologize for omitting many other important results of his from the last twenty five years, many of which were in collaboration with his brilliant and devoted student Yanyan Li.

Fortunately there have been many celebrations of his work in print on the occasion of the many honors and awards he received, the last being the Abel prize (2015).

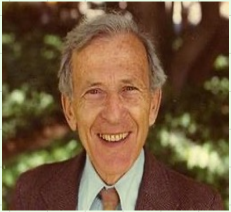
Louis' work on elliptic systems finds some novel applications.

However I do want to describe some novel applications of his early work with Agmon and Douglis. They introduced a very general notion of an elliptic system that is hard to understand and somewhat mysterious for the non-expert.

I will start with two examples of what we call " free boundary regularity" contained in a joint paper of Louis with David Kinderlehrer and myself.

An example of Lewy.

The first example is due to Hans Lewy. Louis was a great admirer of Hans Lewy and was well acquainted with his work.



Let $u, v \in C^2(\Omega \cup S)$ be solutions of the system

$$\Delta u = 0, \;\; \Delta v + \lambda(x) v = 0 \text{ in } \Omega, \tag{1}$$

$$u = v, \;\; u_{x_n} = v_{x_n} \;\; \text{on } S. \tag{2}$$

Analysis of the Lewy example.

where $\lambda(x) \neq 0$ is analytic and $\partial\Omega$ is partly contained in the hyperplane $S = \{x_n = 0\}$. Lewy proved for $n = 2$ that u and v extend analytically across S. This is very surprising since the boundary conditions, which say that u and v share the same boundary conditions are "not coercive". To show that coercivity is present, we introduce the "splitting", $w = u - v$ and rewrite the system (1), (2) in terms of u and w:

$$\Delta u = 0, \tag{3}$$

$$\Delta w + \lambda(x)(w + u) = 0, \tag{4}$$

$$w = w_{x_n} = 0 \text{ on } S. \tag{5}$$

Analysis of the Lewy example continued.

Using the ADN general theory, we assign the weights $s = 0$ to equation (3) and s= -2 to equation (4) and the weight $t_u = 2$ and $t_w = 4$ to the unknowns. These weight are "consistent" and with this choice, the principal part of the system is

$$\Delta u = 0, \quad (6)$$

$$\Delta w + \lambda(x) u = 0, \quad (7)$$

which is certainly elliptic.

Analysis of the Lewy example continued.

The crucial point to note is that now the boundary conditions (5) are coercive for the system. Indeed we can eliminate u entirely from (7) since $u = -(1/\lambda)\Delta w$ and obtain

$$\Delta^2 w + 2\lambda\nabla(1/\lambda)\cdot\nabla\Delta w + \Delta(1/\lambda)\Delta w = 0.$$

The boundary conditions $w = w_{x_n} = 0$ are now just Dirichlet conditions for this fourth order equation which are well known to be coercive. The end result is that u and v are analytic in $\Omega \cup S$ (or C^∞ if $\lambda \in C^\infty$). One point to emphasize is that it is not necessary to eliminate u and often not possible in other related problems.

The smoothness of the liquid edge.

Consider a configuration of three minimal surfaces in R^3 meeting along a $C^{1+\alpha}$ curve, the liquid edge γ, at equal angles of $2\pi/3$.

Such a configuration represents one of the stable singularities that soap films can form (Plateau, J.C.C Nitsche, Jean Taylor).

Formulation of the free boundary problem.

With the origin of a system of coordinates on this curve, let us represent the three surfaces as graphs over the tangent plane to one of them at the origin.

Denote by Γ the orthogonal projection of γ on this plane. Two of the functions u^1, u^2 will be defined on one side Ω^+ of Γ while the third u^3 will be defined on the opposite side Ω^- of Γ.

We now suppose $\Omega^\pm \subset R^n$ and Γ is a $C^{1+\alpha}$ hypersurface. Suppose that the graphs of u^1 and u^2 meet at angles μ_1, μ_2.

.

Then we have an overdetermined system for the u^i, $i = 1, 2, 3$:

$$
\begin{aligned}
Mu^1 = Mu^2 &= 0 \text{ in } \Omega^+, && (8)\\
Mu^3 &= 0 \text{ in } \Omega^-, && (9)\\
u^1 = u^2 &= u^3 \text{ on } \Gamma, && (10)\\
\frac{\nabla u^j \cdot \nabla u^3 + 1}{\sqrt{1+|\nabla u^j|^2}\sqrt{1+|\nabla u^3|^2}} &= \cos\mu_j \quad j = 1, 2, && (11)\\
\nabla u^3(0) &= 0, && (12)
\end{aligned}
$$

where

$$Mu = (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1+|\nabla u|^2}) u_{x_i x_j}$$

is the minimal surface operator in nondivergence form.

The regularity theorem.

Theorem.. If three distinct minimal hypersurfaces in R^{n+1} meet along a $(n-1)$ dimensional $C^{1+\alpha}$ hypersurface γ at constant angles, then γ is analytic.

One has to introduce $w(x) = (u^2 - u^1)(x)$ and the transformation $y = (x', w(x))$, $x \in \Omega^+$, the so called zeroth order Legendre transform.

The mapping $x \to y$ transforms a neighborhood of 0 in Ω^+ into a neighborhood $U \subset \{y : y_n > 0\}$ and a portion of Γ into a $S \subset \{y : y_n = 0\}$. It has an inverse

$$x = (y', \psi(y)), \ \ y \in U \cup S.$$

which satisfies the simple transformation laws

$$w_n = \frac{1}{\psi_n},\ w_\alpha = -\frac{\psi_\alpha}{\psi_n},\ \alpha < n\ ,$$

$$\frac{\partial}{\partial x_n} = \frac{1}{\psi_n}\frac{\partial}{\partial y_n},\ \ \frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha} - \frac{\psi_\alpha}{\psi_n}\frac{\partial}{\partial y_n}\ .$$

We associate to this inverse a reflection mapping

$$x = (y', \psi(y) - Cy_n),\ \ y \in U \cup S,$$

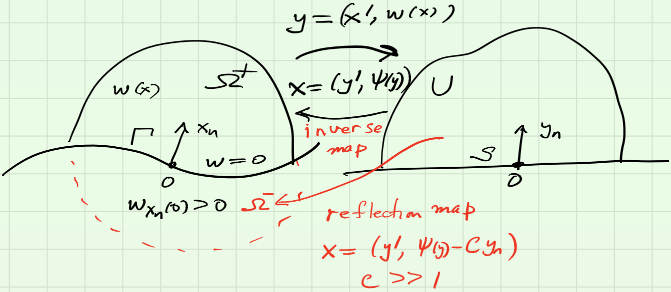
for $C > \sup_U |\nabla\psi(y)|$.

Now define

$$\phi^+(y) = u^1(x),\ x \in \Omega^+,\ \ \phi^-(y) = u^3(x),\ \ x \in \Omega^-.$$

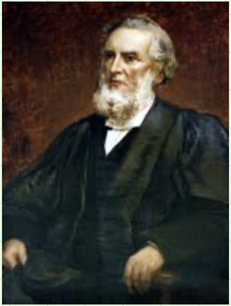
Note that $u^2(x) = (w + \phi^+)(y) = y_n + \phi^+(y)$.

Then $\phi^+(y)$, $\phi^-(y)$, $\psi(y)$ satisfy (the technical part of the argument) a coercive elliptic system in the ADN sense which implies the analyticity.



The influence of Alexandrov and Serrin on Nirenberg's work.

A central question in differential geometry has been to classify all possible "soap bubbles", or more formally put: classify all closed constant mean mean curvature hypersurfaces in R^{n+1} (or more generally in special Riemannian manifolds M^{n+1}). Amazingly, the first such result was proven by J. H. Jellett in 1853.



A starshaped closed surface M in R^3 with constant positive mean curvature is the standard round sphere.

A modern presentation of his proof, which extends to R^{n+1} goes as follows. Let $X : M^n \to R^{n+1}$ be the position vector and N the outward normal to M. We assume $X(M)$ has constant mean curvature $H > 0$ and is starshaped about the origin, i.e. $X \cdot N \geq 0$. If Δ_M is the Laplace-Beltrami operator and A the second fundamental form of M, then

$$\begin{aligned}
\Delta_M X &= -nHN, &(13)\\
\Delta_M N &= -|A|^2 N, &(14)\\
\Delta_M |X|^2/2 &= n - nHN, &(15)\\
\Delta_M X \cdot N &= nH - |A|^2 X \cdot N. &(16)
\end{aligned}$$

Then (15) (16) imply

$$\Delta_M(H|X|^2/2 - X \cdot N) = (|A|^2 - nH^2) X \cdot N \geq 0.$$

Integrating over M we conclude $|A|^2 = nH^2$ or M is totally umbilic, so a sphere.

Heinz Hopf visits the USA.

In his visit to Courant and Stanford in 1955-1956, Hopf lectures on various topics in differential geometry in the large and presents his ingenious proof that if M is a closed immersed constant mean surface in R^3 with genus 0, then M is a round sphere.

He also sketched the proof of a new result (unpublished at the time) of Alexandrov:

A closed embedded hypersurface M^n in R^{n+1} of constant mean curvature is a round sphere.

Hopf goes on to speculate, " it is my opinion that this proof by Alexandrov, especially the geometric part, opens up important new aspects in differential geometry in the large".

An important paper of Jim Serrin.

Jim Serrin (1926-2012) was a brilliant innovator in elliptic theory, the calculus of variations, fluids and mechanics.

In 1971, Serrin published a paper in which he showed that if u is a solution of the overdetermined (free boundary) problem

$$\Delta u = -1 \text{ in } \Omega,\ u = 0,\ u_\nu = c < 0 \text{ on } \partial\Omega,$$

(with Ω a C^2 domain) then Ω is a ball.

The important part of Jim's paper was that he used Alexandrov's method of moving planes *in the pde context* and developed a significant improvement of Hopf's boundary point lemma as well as a continuous (sweeping) use of the maximum principle.

These techniques were simplified and improved by Gidas, Ni and Nirenberg in their first paper. Most importantly, Louis recognized that Serrin's idea could be used when the domain has symmmetry to show that solutions inherit symmetry (for appropriate elliptic operators).

Symmetry for solutions of elliptic pde; motivating questions.

Many problems in both Yang-Mills theory, astrophysics and reaction-diffusion equations are modeled by equations of the form $\Delta u + f(u) = 0$. For example the power nonlinearity $f(u) = u^\alpha$ occurs frequently. The range $1 \leq \alpha < \frac{n+2}{n-2}$ is called the subcritical range because of the Sobolev embedding theorem while the range $\alpha > \frac{n+2}{n-2}$ is called supercritical. The case $\alpha = \frac{n+2}{n-2}$ is particularly important because the equation becomes conformally invariant. This is the conformally flat case of the famous Yamabe problem.

Question 1. Suppose we consider the Dirichlet problem

$$\Delta u + f(u) = 0,\ u \geq 0 \text{ in } B = B_1(0),\ u = 0 \text{ on } \partial B.$$

Is u radially symmetric, i.e $u = u(|x|)$?

There is a simple counterexample if we allow u to change sign. Take $f(u) = \lambda_k u$ where λ_k is the kth Dirichlet eigenvalue of B. Then the eigenfunctions are not radially symmetric.

Question 2.

Let $\Delta u + f(u) = 0$, $u \geq 0$ in R^n. Is u radially symmetric? The famous equation $\Delta u + u^{\frac{n+2}{n-2}}$, $u > 0$ in R^n has the explicit solutions

$$u(x) = (\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x - x_0|^2})^{\frac{n-2}{2}} \ \lambda > 0.$$

On the other hand the subcritical equation $\Delta u + u^{\alpha} = 0$, $u \geq 0$ in R^n has only the trivial solution (Gidas-Spruck).

Question 3 (singular solutions).

There is also a complicated family of singular solutions of the form $u(x) = r^{-\frac{n-2}{2}}\psi(t)$ where $r = |x|$, $t = -\log r$ and $\psi(t)$ is a periodic translation invariant solution of the ODE:

$$\psi'' - (\frac{n-2}{2})^2\psi + \psi^{\frac{n+2}{n-2}} = 0.$$

The simplest singular solution is

$$u(x) = \frac{k}{|x|^{\frac{n-2}{2}}},\ k = (\frac{n-2}{2})^{\frac{n-2}{2}}.$$

Thus the classification of singular solutions is quite challenging.

A general symmetry result of Nirenberg and collaborators.

Nirenberg refined his original method with Gidas and Ni (for bounded domains) in a paper with Berestycki by incorporating the Alexandrov maximum principle into the argument.

Theorem Let Ω be a bounded domain which is convex in the e_1 direction and symmetric with respect to the plane $x_1 = 0$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies

$$\Delta u + f(u) = 0,\ u \geq 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where $f(t)$ is Lipschitz. Then $u(x_1, x') = u(-x_1, x')$ and $u_{x_1}(x) < 0$ for any x with $x_1 > 0$.

Alexandrov's generalized gradient map and Monge-Ampere measure

Since the ideas of Alexandrov may not be so familiar to geometers, we give a brief sketch as they are now a fundamental part of elliptic theory.

Alexandrov introduced the normal mapping (subdifferential) of $u \in C^0(\Omega)$, $\Omega \subset R^n$:

$$\partial u(x_0) = \{p \in R^n : u(x) \geq u(x_0) + p \cdot (x - x_0)\},$$

$$\text{If } E \subset \Omega,\ \partial u(E) = \cup_{x \in E} \partial u(x).$$

Of course, $\partial u(x_0)$ may be empty.
The (lower) contact set $\Gamma := \{x \in \Omega : \partial u(x) \neq \emptyset\}$.
If $u \in C^1$ and $x \in \Gamma$, then $\partial u(x) = \nabla u(x)$.
If $u \in C^2$ and $x \in \Gamma$, then $\nabla^2 u \geq 0$.

Example. $\Omega = B_r(0)$, $u(x) = \frac{h}{R}(|x| - R)$. Then

$$\partial u(x) = \begin{cases} \frac{h}{R}\frac{x}{|x|} & x \neq 0 \\ \overline{B_{\frac{h}{R}(0)}} & x = 0 \end{cases}$$

Theorem. If $u \in C^0(\Omega)$, then
$\mathcal{S} = \{E \subset \Omega : \partial u(E)$is Lebesgue measurable is a Borel σ algebra.
The set function $\mathcal{M}u(E) = |\partial u(E)||$ is called the Monge-Ampere measure associated to u and if $u \in C^2(\Omega)$ is convex,

$$\mathcal{M}u(E) = \int_E \det(D^2 u).$$

Application to the Isoperimetric inequality (Cabré).

Given Ω, solve the Neumann problem

$$\Delta u = 1 \text{ in } \Omega$$
$$u_\nu = c = \frac{|\Omega|}{|\partial\Omega|} \text{ on } \partial\Omega$$

Claim: The normal map $\partial u(\Gamma)$ contains a ball of radius c.
For if a plane $p \cdot x + k$ is moved up from $-\infty$ has first contact at on $\partial\Omega$, then $|p| \geq |\nabla u| \geq c$. Hence,

$$\omega_n c^n \leq \int_{\Gamma_u} \det D^2 u \leq \int_{\Gamma_u} (\frac{\Delta u}{n})^n \leq \frac{1}{n^n}|\Omega| \text{ , or}$$

$$|\partial\Omega| \geq n\omega_n^{\frac{1}{n}} |\Omega|^{\frac{n-1}{n}} \ .$$

A simple version of the ABP maximum principle.

Lemma.

Assume $\Delta u + c(x)u \geq f$ in Ω, $c(x) \leq 0$, $u \leq 0$ on $\partial\Omega$.
Then $M := \sup_\Omega u \leq Cd\|f^-\|_{L^n(\Omega)}$, $d = \mathrm{diam}(\Omega)$.

Proof. Assume $M > 0$ is achieved at an interior point $x_1 \in \Omega$. Let $v = -u^+$. Then $v < 0$ on Γ_v, the lower contact set of v and $-M = v(x_1)$, $v = 0$ on $\partial\Omega$. Hence

$$\Delta v(x) = -\Delta u(x) \leq -c(x)v - f \leq f^-, \ \ x \in \Gamma_v.$$

Claim: $B(0, M/d) \subset \partial v(\Gamma_v)$.

Let $|p| < M/d$ and translate the plane $p \cdot x + k$ up from $-\infty$ until there is a first contact with the graph of v at a point $x_0 \in \overline{\Omega}$. Then $x_0 \notin \partial\Omega$ for otherwise $v(x_0) = 0$, and so

$$-M = v(x_1) \geq p \cdot (x_1 - x_0) \geq -|p|d > -M, \text{ a contradiction.}$$

Therefore $p = \nabla v(x_0)$ proving the Claim. Hence

$$\omega_n (M/d)^n \leq \int_{\Gamma_v} \det(D^2 v) \leq \int_{\Gamma_v} (\Delta v/n)^n \leq \|f^-\|^n_{L^n(\Omega)},$$

and the Lemma follows.

The maximum principle for domains of small volume.

We now prove the ABP maximum principle for $\Delta u + c(x)u \geq 0$ and drop the assumption $c(x) \leq 0$. To accomplish this we write $c = c^+ - c^-$, $\Delta u - c^- u \geq -c^+ u^+$. Applying the Lemma gives

$$M \leq Cd\|c^+ u^+\|_{L^n(\Omega)} \leq Cd\|c^+\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{n}} M \leq M/2$$

for $|\Omega| \leq (2Cd\|c^+\|_{L^\infty(\Omega)})^{-n}$. We have proved with no assumption on the sign of $c(x)$:

Theorem Assume $\Delta u + c(x)u \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega$. Then there is a positive constant δ depending on n, $\text{diam}(\Omega)$, $\|c^+\|_{L^\infty(\Omega)}$ so that if $|\Omega| \leq \delta$, then $u \leq 0$ in Ω.

Other types and methods for symmetry results

Caffarelli, Gidas and Spruck: asymptotic symmetry (measure theoretic, no growth assumptions)

Berestycki and Nirenberg (and papers with Caffarelli): sliding method, symmetry in a half-space

Congming Li: extensions to fully nonlinear

Yanyan Li and M. Zhu, Yanyan Li: method of moving spheres, conformally invariant

and many many more...

The Monge-Ampère boundary value problem.

The Monge-Ampère equation $\det u_{ij} = \psi(x, u, \nabla u) > 0$ is the prototypical fully nonlinear elliiptic pde. To see when it is elliptic, we linearize at a point x_0 , i.e.

$$L\phi = \frac{d}{dt}\det[(u_{ij})(x_0) + t\phi_{ij}]|_{t=0} = A^{ij}\phi_{ij},$$

where A^{ij} is the cofactor matrix of $u_{ij}(x_0)$. For ellipticity we need $(A^{ij}) > 0$, that is, u convex.

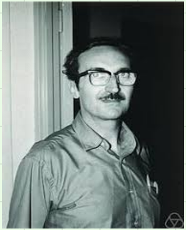
The boundary value problem for the Monge-Ampère operator is classically formulated as follows. Let $\Omega \subset R^n$ be a smooth strictly convex domain and let ϕ, $\psi > 0$ be smooth. Find a strictly convex solution $u \in C^\infty(\overline{\Omega})$ of the boundary value problem

$$\begin{aligned} \det(u_{ij}) &= \psi(x, u, \nabla u) \text{ in } \Omega, \\ u &= \phi \text{ on } \partial\Omega. \end{aligned} \tag{17}$$

Some history.

In 1971 Pogorelov showed that for the special case $\phi \equiv 0$, $\psi = \psi(x) > 0$, there is a unique weak solution u in the sense of Alexandrov and moreover $u \in C^\infty(\Omega)$.

His proof uses his famous interior second derivative estimate and Calabi's ingenious interior estimates for third derivatives (in the metric $ds^2 = u_{ij}dx_i dx_j$).



In 1974 Louis announced at the International Congress in Vancouver his joint work with Calabi providing a solution of (20) in complete generality. Unfortunately the argument contained a gap and fell through.

The classical existence theorem

Louis, Luis Caffarelli and myself, and independently Nikolai Krylov proved the now classical existence existence theorem:

Theorem. Suppose Ω is a strictly convex domain with Ω, ϕ, ψ smooth. Assume in addition that for the boundary data ϕ, there is a strictly convex subsolution $\underline{u}$, i.e.

$$\begin{aligned} \det\left(\underline{u}_{ij}\right) &\geq \psi(x, \underline{u}, \nabla \underline{u}) \text{ in } \Omega, \\ \underline{u} &= \phi \text{ on } \partial\Omega. \end{aligned} \tag{18}$$

Then there exists a strictly convex solution $u \in C^\infty(\overline{\Omega})$ to (17). If $\psi_u \geq 0$, the solution is unique.

Remarks

There are two main difficulties in proving the existence of smooth strictly convex solutions via the continuity method in which one tries to prove apriori estimates for any admissible solution. The first one is to show that the second normal derivative is apriori bounded on the boundary (assuming all other are apriori bounded), that is , $u_{nn} \leq C$ on $\partial\Omega$. The standard way to do this is from the equation $\det D^2 u = \psi$. In a suitable frame one can write

$$A^{nn} u_{nn} = O(1)$$

where A^{ij} is the cofactor matrix of u_{ij} and then solve for u_{nn}. Thus one needs to prove strict convexity of the solution at the boundary.

In the simplest case $\phi = 0$, assuming strict convexity of the domain, this boils down to showing $u_n \geq c_0 > 0$ for e_n the outer unit normal.

The second major problem is that C^2 estimates do not suffice because the equation is fully nonlinear and not uniformly elliptic. One need to obtain global $C^{2+\alpha}$ estimates.

This can be done strictly in the interior of Ω using the Evans-Krylov theorem or Calabi's third derivative estimates. However the global estimates required a new idea.

A geometric point of view suggests a stronger theorem

The following simple example points the way to a stronger result.

Example. Let Γ_1, Γ_0 be strictly convex smooth closed codimension 2 hypersurfaces in parallel planes, say $x_{n+1} = 1, 0$ respectively. Is there a hypersurface S of constant Gauss curvature $K_0 > 0$ for K_0 sufficiently small? Intuitively the answer is clearly yes.

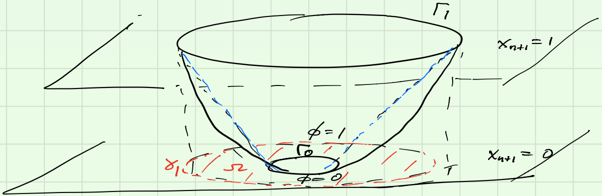
Let's specialize further and suppose the parallel projection of Γ_1, call it γ_1 contains Γ_0. Then it is not difficult too see that if a solutions exists it is of the form $S = \text{graph}(u)$ over the annulus Ω with outer boundary γ_1 and inner boundary Γ_0.

Strictly convex curves in parallel planes.

Thus u satisfies

$$\det(u_{ij}) = K_0(1 + |\nabla u|^2)^{\frac{n+2}{2}} \text{ in } \Omega,$$

$$u = 1 \text{ on } \gamma_1, \; u = 0 \text{ on } \Gamma_0.$$



The subsolution existence theorem

It is not difficult to show that there is a unique smooth solution as expected even though the classical existence theorem does not apply since here Ω is an annulus. Bo Guan and I proved the subsolution existence theorem which holds in great generality. We state only the simplest version to compare with the classical theorem.

Theorem. Suppose Ω, ϕ, ψ smooth and assume ithat for the boundary data ϕ, there is a locally strictly convex subsolution $\underline{u} \in C^2(\overline{\Omega})$, i.e.

$$\begin{aligned} \det(\underline{u}_{ij}) &\geq \psi(x, \underline{u}, \nabla \underline{u}) \text{ in } \Omega, \\ \underline{u} &= \phi \text{ on } \partial\Omega. \end{aligned} \tag{19}$$

Then there exists a locally strictly convex solution $u \in C^\infty(\overline{\Omega})$ to (20). If $\psi_u \geq 0$, the solution is unique. Moreover any admissible solution satisfies the apriori estimate $\|u\|_{C^{2+\alpha}(\Omega)} \leq C$ for a controlled constant C.

Implicitly defined fully nonlinear elliptic pde.

The work of CNS on Monge-Ampere equations has a natural and important extension to implicitly defined fully nonlinear pde. Let $A = (a_{ij})$ be a symmetric nxn matrix (or more generally a natural tensor on a Riemannian manifold) and define

$$F(A) = f(\lambda_1, \ldots, \lambda_n),$$

where the λ_i are the eigenvalues of A and $f(\lambda)$ is a symmetric function (say smooth for simplicity). Then $F(A)$ will also be smooth. When $A = (u_{ij}(x))$, $x \in \Omega$, and $f(\lambda) = \Pi \lambda_i$, $F(A) = \det u_{ij}(x)$, and we recover the Monge-Ampere operator which is elliptic when λ lies in the positive cone $\Gamma_n = \{\lambda \in R^n : \lambda_i > 0\}$.

What happens in general?

Ellipticity and concavity.

Let f is defined in an open symmetric convex cone Γ with vertex at the origin with $\Gamma_n^+ \subset \Gamma$.

Proposition. Assume that $f_i := f_{\lambda_i} > 0\,\forall i$ and that f is concave. Then $F^{ij} := \frac{\partial F}{\partial a_{ij}} = f_i \delta_{ij}$ when A is diagonal so the linearized operator $L = F^{ij}\nabla_i\nabla_j$ is elliptic. Moreover F is a concave function of A.

The concavity condition is very important for the regularity theory of fully nonlinear elliptic pde.

Garding's theory of hyperbolic polynomials.

Louis was an incredibly thorough mathematician (i.e he didn't miss much) with a vast knowledge of pde literature and knew Garding's work on hyperbolic polynomials. As the name suggests, Garding's beautiful theory was related to hyperbolic pde but ultimately had many important algebraic consequences and is important in convex analysis (see the 2013 CPAM paper of Harvey-Lawson).

Definition. A homogeneous polynomial $p(\lambda)$ of degree m in R^n is called hyperbolic with respect to a direction $a \in R^n$ (notation hyp a) if for all $\lambda \in R^n$, the polynomial $p(ta + \lambda)$ has exactly k real roots. Then $p(a) \neq 0$ and we may assume $p(a) > 0$. It is easily checked (essentially Rolle's theorem) that

$$q(\lambda) := \sum a_j \frac{\partial}{\partial \lambda_j} p(\lambda)$$

is also hyp a.

Example. Since $\sigma_n(\lambda) = \Pi \lambda_i$ is hyp a for $a = (1, \ldots, 1)$, so are the elementary symmetric functions $\sigma_k(\lambda)$.

The Garding cone.

Let $\Gamma = \Gamma(P, a)$ denote the component in R^n containing a of the set where $p > 0$. Garding proved that Γ is a convex cone with vertex at the origin and that p is hyp b for all $b \in \Gamma$ Moreover $\Gamma(p, a) \subset \Gamma(q, a)$.

In particular the Garding cones for the elementary symmetric functions $\sigma_k(\lambda)$ are nested starting from the positive cone for $\sigma_n(\lambda)$ and ending with the half-space $\sum \lambda_i > 0$ for $\sigma_1(\lambda)$. Note also that for $k > 1$, $\sigma_k = 0$ on the positive λ_i axes.

The main result proved by Garding is an inequality which is equivalent to the statement that $p^{\frac{1}{m}}(\lambda)$ is concave in Γ. It follows easily that $p_{\lambda_i}(\lambda) > 0$ in Γ for all $i = 1, \ldots, n$.

Remarks.

There are interesting and important examples of "elliptic and concave" $f(\lambda)$ that do not arise from hyperbolic polynomials but are related to them.

Example. The homogeneous degree 1 quotients

$$f(\lambda) := \left(\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right)^{\frac{1}{k-l}}, \; k > l$$

is elliptic and concave in the Garding cone Γ_k for $\sigma_k(\lambda)$.

The Dirichlet problem.

$$F(D^2u(x)) \equiv f(\lambda(D^2u(x))) = \psi(x) \text{ in } \Omega,\ u = \varphi \text{ on } \partial\Omega,$$

where $f(\lambda)$ is elliptic and concave in a symmetric open convex cone Γ (with vertex at origin), containing the positive cone Γ_n and $\limsup_{\lambda\to\partial\Gamma} f(\lambda) \le \inf_{\overline{\Omega}} \psi$.

Our paper CNS3 assume also the additional structural condition: for all $C > 0$ and K compact, there exist $R = R(C, K)$ such that $f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \ge C,\ f(R\lambda) \ge C$ for all $\lambda \in K$

Theorem If there exists R large so that $(\kappa_1, \ldots, \kappa_{n-1}, R) \in \Gamma$ at each $x \in \partial\Omega$, there exists a unique admissible solution $u \in C^\infty(\overline{\Omega})$.

(Here $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$.)

Our theorem was hard to prove but is far from optimal. It also excludes the nice example

$$f(\lambda) := \left(\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right)^{\frac{1}{k-l}}, \ k > l$$

Over the years there has been significant improvements. For example in 2015 Bo Guan proved the essentially optimal result:

Theorem (Bo Guan) Assume only ellipticity and concavity (i.e no additional structure conditions). If there exists a admissible subsolution $\underline{u} \in C^2(\overline{\Omega})$, $u = \varphi$ on $\partial\Omega$, then there exists a unique admissible solution $u \in C^\infty(\overline{\Omega})$. If in addition $f(\lambda)$ satisfies

$$\sum \lambda_i f_{\lambda_i} \geq 0 \ \text{ on } \Gamma \cap \{\inf \psi \leq f \leq \sup \psi\},$$

we may take Ω to be a Riemannian manifold with smooth boundary.

Final thoughts.

The setting of implicitly defined fully nonlinear elliptic initiated in the paper CNS3 unleashed a tidal wave of research in fully nonlinear pde and geometric analysis (curvature flows, conformal geometry, complex geometry,...).

Louis Nirenberg's scholarship, insight and experience played a large role in this development and it is certainly one of his most enduring legacies.

Louis was also a great Ph.d advisor who produced 46 students and mentored numerous young mathematicians who are also a part of his legacy. We can all learn from him about the benefits of generosity, openness and collaboration.