`Hodge Theory’ from Floer homology:

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Definition: FOOO = Fukaya-Oh-Ohta-Ono

Hodge Theory in complex geometry

X Kahler maniflds

$\Omega^{p,q}(X):$ (p,q)-forms on X

$$\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$$

$$\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$$

$$\bar{\partial} + u\partial : \bigoplus_{p,q} \Omega^{p,q}(X)[[u]] \rightarrow \bigoplus_{p,q} \Omega^{p,q}(X)[[u]]$$

$$H(\bar{\partial} + u\partial) = \frac{\mathrm{Ker}\left(\bar{\partial} + u\partial\right)}{\mathrm{Im}\left(\bar{\partial} + u\partial\right)} \cong H(X) \otimes \mathbb{C}[[u]]$$

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

$$\int \; : \bigoplus_{p,q} \Omega^{p,q}(X) \to \mathbb{C}$$

$$\int \; : H(\bar{\partial} + u\partial) \to \mathbb{C}[[u]]$$

$$\int \partial u \wedge v \pm \int u \wedge \partial v = 0$$

$$\int \bar{\partial} u \wedge v \pm \int u \wedge \bar{\partial} v = 0$$

$$\langle \;\; \rangle : H(\bar{\partial} + u\partial)^{\otimes 2} \to \mathbb{C}[[u]]$$

$$\langle u, v \rangle = \int u \wedge v$$

There is a family version of this story : Variation of Hodge structure

Mirror symmetry

$$X$$

$$X^\vee$$

$$H(\bar{\partial} + u\partial)$$

Symplectic manifold

$$\int : H(\bar{\partial} + u\partial) \to \mathbb{C}[[u]]$$

Quantum cohomology

$$\wedge$$

Yukawa coupling

(actually need to identify (k,0)-form
with n-k poly vector field. using holomorphic
n form.)

Mirror symmetry

X $X^\vee$

Hodge theory of X

Something similar to Hodge theory ?

?

Derived category of coherent sheaves

Cyclic filtered A infinity category

FOOO

Homological mirror symmetry
(Kontsevitch)

Lagrangian Floer theory

Something similar to Hodge theory ?

$\uparrow$?

Cyclic filtered A infinity category

There are proposal by

Kontsevitch-Soibelman-Kazarkov-Pantev-Auroux ....

K. Saito – A. Takahashi

etc.

A differential graded algebra

(special case of A infinity algebra,
the story works in the same way for filtered A infinity algebra)

$$HH(A) = H(CH(A), \delta^H)$$ Hochshild complex

$$\delta^H + uB : CH(A) \otimes \mathbb{C}[[u]] \to CH(A) \otimes \mathbb{C}[[u]]$$

$\Updownarrow$

$$\bar{\partial} + u\partial : \bigoplus_{p,q} \Omega^{p,q}(X)[[u]] \to \bigoplus_{p,q} \Omega^{p,q}(X)[[u]]$$

B

Connes' opeartor

$$H(\delta^H + uB) = HC(A)$$

Cyclic homology.

Example

$$A = \Omega M$$ de-Rham complex of a manifold M

$$HH(A) = H(LM)$$ homology of free loop space of M

$$HC(A) = H_{S^1}(LM)$$ S^1 equivariant homology of free loop space of M

$$B : H(LM) \to H(LM)$$

$$P \xrightarrow{\sigma} LM \qquad S^1 \times P \to LM$$

$$t, x \to t\sigma(x)$$

Remark

$$A = \Omega M$$

$$HH(A) = H(LM)$$

$$HC(A) = H_{S^1}(LM)$$

In this example

$$HC(A) \neq HH(A) \otimes \mathbb{C}[[u]]$$

No reasonable non-degenerate pairing on $HH(A) = H(LM)$

Reason

This corresponds to the case when symplectic manifold is

T^*M non-compact.

$$\delta^{H} + uB : CH(A) \otimes \Lambda[[u]] \to CH(A) \otimes \Lambda[[u]]$$

$\Updownarrow$

$$\bar{\partial} + u\partial : \bigoplus_{p,q} \Omega^{p,q}(X)[[u]] \to \bigoplus_{p,q} \Omega^{p,q}(X)[[u]]$$

B

Connes' opeartor

This talk:

There are cases that this proposal actually works and gives Mirror.

We can define paring $(CH(A) \otimes \Lambda[[u]])^{\otimes 2} \to \Lambda[[u]]$

which has all the required properties.

(X,ω) Symplectic manifold

$L \subset X$ Lagrangian submanifold

$(\Omega L, d, \wedge)$ differential graded algebra (de Rham complex)

$\Downarrow$ $\leftarrow$ deformation by pseudo-holomorphic discs [FOOO]

$(\Omega L, \{m_k\})$ A infinity algebra

$\Downarrow$ Consider many Lagrangian submanifolds [FOOO]

A infinity category $F(X)$

Suppose X compact

Conjecture (Kontsevitch, Seidel ....)

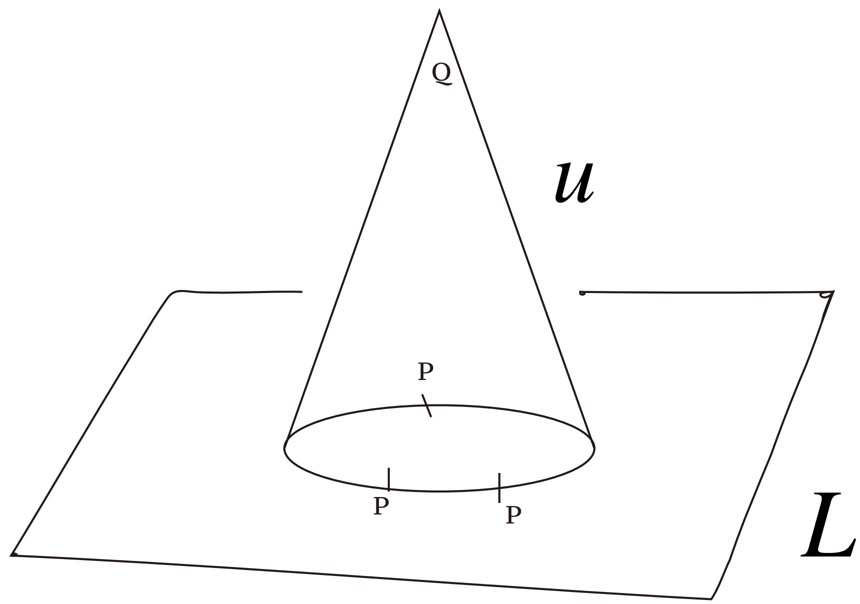
$$HH(F(X)) \cong H(X)$$ in certain cases.

More precisely $$p_* : CH(F(X)) \to S_*(X)$$

chain map, always. Called open closed map. ([FOOO] general case.)

$$p_* : CH(F(X)) \to S_*(X)$$

is expected to induce an isomorphism in cohomology in a good case.



$$\left\langle p_*(P,\ldots,P),Q\right\rangle_{PD_X}$$

Proposition $p_* \circ B = 0$

Corollary 1 p_* induces

$$pc_* : HC(F(X)) \to H_{S^1}(X) = H(X) \otimes \Lambda[[u]]$$

Corollary 2 If $p_* : HH(F(X)) \to H(X)$

is isomorphism then

$$pc_* : HC(F(X)) \to H_{S^1}(X)$$ is an isomorphism.

and $$HC(F(X)) \cong HH(F(X)) \otimes \Lambda[[u]]$$

Remark

$$HC(A) \cong HH(A) \otimes \Lambda[[u]]$$

is proved for A infinity category A
which is `smooth and compact'.

by Kaledlin (in purely algebraic method.)

Pairing (FOOO and Abouzaid-FOOO)

$$\langle\ \rangle_{\mathrm{res}} : CH(F(X))^{\otimes 2} \to \Lambda$$

$$CH(H(L)) = \bigoplus_k H(L)^{\otimes k}$$

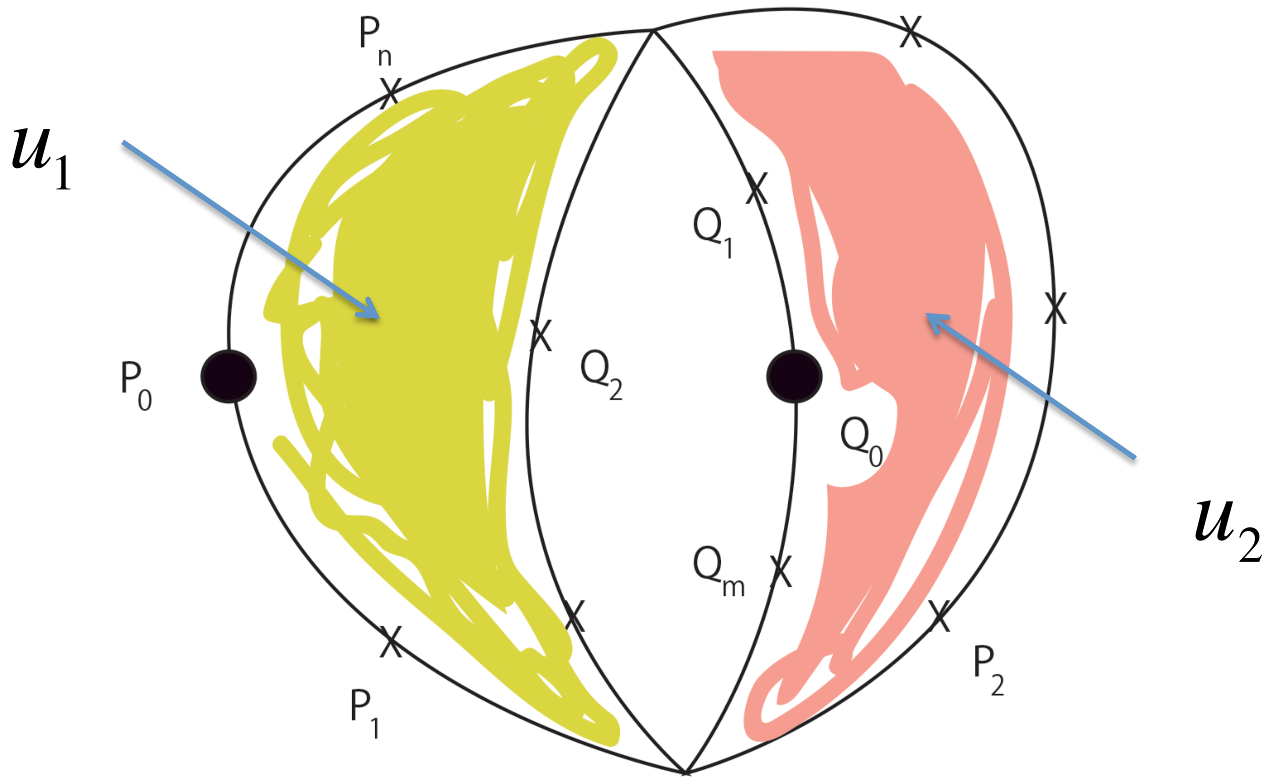
$$a, b \in H(L) \subset CH(H(L))$$

$$\langle a,b\rangle_{\mathrm{res}} = \sum_{i(1),i(2),j(1),j(2)} g^{i(1)j(2)} \left\langle a, e_{i(1)} \cup e_{i(2)} \right\rangle_{PD_L} \left\langle b, e_{j(1)} \cup e_{j(2)} \right\rangle_{PD_L}$$

e_i basis of H(L)

$$(g^{ij}) = (g_{ij})^{-1} \qquad g_{ij} = \left\langle e_i, e_j \right\rangle_{PD_L}$$

$\cup$ Product structure in Floer homology (cup product plus deformation by holomorphic disc.)



$$\left\langle P_0 \otimes \cdots \otimes P_n, Q_0 \otimes \cdots \otimes Q_m \right\rangle_{\text{res}}$$

Proposition $$\langle \delta_H x, y \rangle_{\mathrm{res}} = \pm \langle x, \delta_H y \rangle_{\mathrm{res}}$$

$$\langle Bx, y \rangle_{\mathrm{res}} = \pm \langle x, By \rangle_{\mathrm{res}}$$

Note $$\int \bar{\partial} u \wedge v = \pm \int u \wedge \bar{\partial} v$$

$$\int \partial u \wedge v = \pm \int u \wedge \partial v$$

Corollary 1 $\langle \ \rangle_{\mathrm{res}}$ extends to `higher residue pairing'

$$\langle \ \rangle_{\mathrm{res}} : HC(F(X))^{\otimes 2} \to \Lambda[[u]]$$

$$p_* : HH(F(X)) \to H(X)$$

Proposition (FOOO-AFOOO)

$$\langle x, y \rangle_{\mathrm{res}} = \langle p_* x, p_* y \rangle_{PD_X}$$

extends to `higher residue pairing

$$pc_* : HC(F(X)) \to H_{S^1}(X)$$

Corollary $$\langle x, y \rangle_{\mathrm{res}} = \langle pc_* x, pc_* y \rangle_{PD_X}$$

right hand side is $H_{S^1}(X) \otimes H_{S^1}(X) \to H_{S^1}(\mathrm{pt}) = \Lambda[[u]]$

How they work in the case of Toric – Landau-Ginzburg Mirror

A variant of Hodge Theory in complex geometry (K. Saitoh's theory of isolated singularity)

$$F : \mathbb{C}^n \to \mathbb{C}$$

holomorphic function with isolated critical point at 0

$$\Omega(\mathbb{C}^n)$$

differential forms on $\mathbb{C}^n$

$$F : \mathbb{C}^n \to \mathbb{C}$$

holomorphic function with isolated critical point at 0

$$\Omega(\mathbb{C}^n)$$

differential forms on $\mathbb{C}^n$

$\bar{\partial} + dW \wedge$	$\bar{\partial}$
∂	∂
Hodge structure in singularity theory	Hodge structure in Kahler manifold

$$\bar{\partial} + dW \wedge \qquad \Longleftrightarrow \qquad \bar{\partial}$$

$$\partial \qquad\qquad\qquad \partial$$

Hodge structure in singularity theory

Hodge structure in Kahler manifold

(Higher) residue pairing

$$\alpha, \beta \mapsto \int \alpha \wedge \beta$$

$$H(u\partial + \bar{\partial} + dW \wedge)$$

$$H(u\partial + \vec{\partial})$$

$$Jac(W) \otimes \Lambda[[u]]$$

$$H(X) \otimes \Lambda[[u]]$$

$$Jac(W) = \frac{O(\mathbb{C}^n)}{\left(\frac{\partial W}{\partial x_1}, \cdots, \frac{\partial W}{\partial x_n}\right)}$$

Jacobian ring

(Higher) residue pairing is defined on

$$H(u\partial + \overline{\partial} + dW \wedge)$$

$$\parallel$$

$$Jac(W) \otimes \Lambda[[u]]$$

$$\bar\partial + dW \wedge$$

$$\partial$$

Hodge structure in singularity theory

(Higher) residue pairing

$$H(u\partial + \bar\partial + dW \wedge)$$

$$Jac(W) \otimes \Lambda[[u]]$$

Such a structure appears in Lagrangian Floer theory of Toric manifold

Toric symplectic manifold

(X,ω) has symplectic T^n action

moment maps

$\pi:(X,\omega)\to P\subseteq T^n$ moment map

$\pi^{-1}(c)=T_c^n\simeq T^n$ Lagrangian torus

$$\pi^{-1}(c) = T_c^n \simeq T^n$$ Lagrangian torus

$$(\Omega T_c^n, \{m_k\})$$ Lagrangian Floer theory of T_c^n

$$(\Omega T_c^n, d, \wedge)$$ plus correction by pseudo-holomorphic disk

$$HH(\Omega T_c^n, d, \wedge) \cong H(LT^n) \cong \bigoplus_{\gamma \in \mathbb{Z}^n = \pi_1(T^n)} H(T^n)$$

$$HH(\Omega T_c^n, d, \wedge) \cong H(LT^n) \cong \bigoplus_{\gamma \in \mathbb{Z}^n = \pi_1(T^n)} H(T^n)$$

$$HH(\Omega T_c^n, d, \wedge) \cong \Gamma(\mathbb{C}^n; \Lambda^{*,0})$$

$H(T^n) \cong$ exterior algebra with n generatos

$$\sum_{\gamma=(\gamma_1,\cdots,\gamma_n)\in\mathbb{Z}^n} a_\gamma \gamma \quad \Longleftrightarrow \quad \sum_{\gamma=(\gamma_1,\cdots,\gamma_n)\in\mathbb{Z}^n} a_\gamma x_1^{\gamma_1} \cdots x_n^{\gamma_n}$$

$$HH(\Omega T_c^n, d, \wedge) \cong \Gamma(\mathbb{C}^n; \Lambda^{*,0})$$

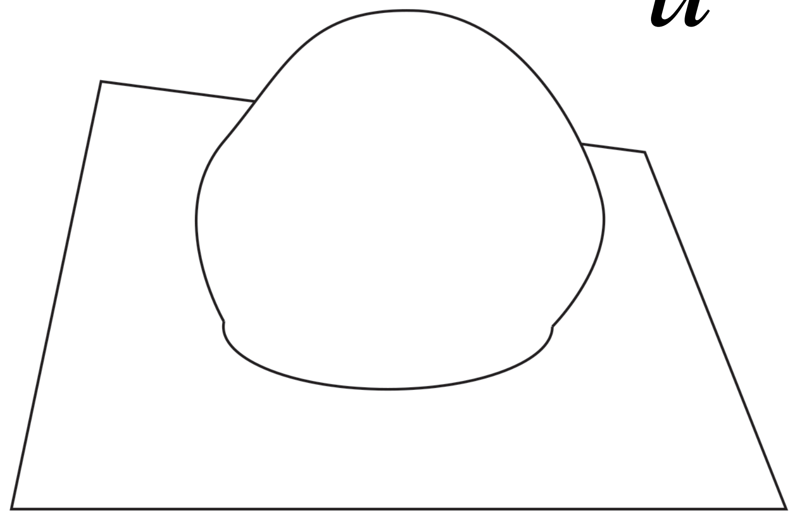
$$\Gamma(\mathbb{C}^n; \Lambda^{*,0}) \to \Gamma(\mathbb{C}^n; \Lambda^{*,0})$$

$$dW \wedge + \text{higher}$$

$$HH(\Omega T_c^n, \{m_k\}) \cong Jac(W)$$

W generating function of the counting holomorphic discs

u holomorphic disc



$$W(b) = \sum_u \pm T^{\int_{\partial D^2} u^* b}$$

$$b \in H^1(T^n)$$

We can calculate

$$B : HH(\Omega T_c^n, d, \wedge) \cong \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \to \Gamma(\mathbb{C}^n; \Lambda^{*,0})$$

and obtain:

$$B = \partial : \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \to \Gamma(\mathbb{C}^n; \Lambda^{*,0})$$

$$\partial$$

$$B = \partial : \Gamma(\mathbb{C}^n;\Lambda^{*,0}) \to \Gamma(\mathbb{C}^n;\Lambda^{*,0})$$

$$\overline{\partial} + dW \wedge$$

$$\Gamma(\mathbb{C}^n;\Lambda^{*,0}) \to \Gamma(\mathbb{C}^n;\Lambda^{*,0})$$

$$dW \wedge + \text{higher}$$

$$HH(\Omega T_c^n, \{m_k\}) \cong Jac_c(W)$$

Hodge structure in singularity theory

Hodge structure in Lagrangian Floer theory of toric manifold

Moreover in the case:

(X,ω) is a compact toric manifold:

$$p_* : \coprod_c HH(\Omega T_c^n, \{m_k\}) \cong \coprod_c Jac_c(W) \to H(X)$$

is an isomorphism.

so the story works in this case.

Summary:

- Using Floer homology and its A infinity structure
- we can find the same kinds of structure as Hodge theory
  on Hochshild and cyclic homology.
- Especially when open closed map is an isomorphism:

  It has a non-degenerate pairing (higher residue pairing) and
  Hodge to de Rham degeneration can be proved geometrically.

- In the case of toric manifold, the whole structure match with
  K. Saito's theory of singularity.