On positivity of a class of conformal covariant operators

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- On (M^d, g), k an integer, $(P^d_{2k})_g$ is a class of differential operator of order $2k$ with leading symbol $(-\Delta_g)^k$; with the conformal covariant property that under conformal change of metric $g_w = e^{2w} g$, we have

$$(P^d_{2k})_{g_w}(\phi) = e^{-\frac{d+2}{2}w}(P^d_{2k})_g(e^{\frac{d-2}{2}w}\phi)$$

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  for all smooth functions ϕ defined on M.
- When $k = 1$, P^d_2 is the conformal Laplace or Yamabe operator:

$$(P^d_2)_g = -\Delta_g + \frac{d-2}{4(d-1)} R_g$$

  where R_g is the scalar curvature of the metric g.

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- When $d > 2k$, $(P^d_{2k})_g(1) := \frac{d-2}{2}(Q^d_{2k})_g$, e.g. when $k = 1$, $d > 2$, $Q^d_2 = \frac{1}{2(d-1)}R$ while when $k = 1$, $d = 2$, Q^2_2 is defined to be the Gaussian curvature.

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- When $d = 2k$, Branson's curvature $Q_{2k} = Q^{2k}_{2k}$ is also defined. When (M^d, g) is locally conformally flat, $C_d\chi(M) = \int (Q_d)_g dv_g$; in general, $\int (Q_d)_g dv_g$ is a conformal invariant.

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- when $\gamma = \frac{1}{2}$,

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is the Robin boundary operator, where H is the mean curvature and $Q_1 = \frac{1}{n}H$.
- On the flat case, $\mathbb{R}^{n+1}_+$, $g_+ = \frac{dy^2+dx^2}{y^2}$, where $y > 0, x \in \mathbb{R}^n$, for all $\gamma > 0$, we have

$$P_{2\gamma} = (-\Delta_x)^\gamma, Q_{2\gamma} = 0.$$

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- Note when $\gamma = 1$, R (scalar curvature) > 0 implies P_2 (the conformal Laplace operators) > 0.
- Theorem (Gursky ('99)) On (M^4, g) closed manifolds, $Y(M, g) > 0$ and $\int (Q_4)_g dv_g > 0$ implies that $(P_4)_g > 0$. In particular, when $R_g > 0$, $(Q_4)_g > 0$ then $(P_4)_g > 0$.

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- Formulas of P_4^d, Q_4^d by Paneitz ('83) and GJMS,

$$(P_4^d)_g = (-\Delta_g)^2 + div(a_d Ric_g + b_d R^g)D + \frac{(d-4)}{2} {Q_4^d}_g$$

$${Q_4^d}_g = c_1(-\Delta_g R + c_2 R_g^2 - c_3 |\text{traceless} Ric_g|^2)$$

where c_1, c_2, c_3 are positive dimensional constants.

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  On (M^d, g) closed manifolds, if $R_g > 0$, and $(Q_4^d)_g > 0$, then $(P_4^d)_g > 0$ when $d \geq 4$.
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  They have made an in-depth study of P_4 operators, proved strong maximal principle ($P_4 u > 0$ implies $u > 0$).
- Hang-Yang ('14) ($d = 3$) When $R_g > 0$ and $(Q_4^3)_g > 0$, Green's function of $(P_4^3)_g$ is negative.

Main Theorems

- Positivity of $P_{2\gamma}$ as boundary operators of conformal compact Einstein manifolds On (X^{n+1}, M^n, g_+) Poincare Einstein manifold, where $M^n = \partial X$, the conformal infinity boundary. We (J. Case - Chang) have two results:

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- Theorem 2: When $1 < \gamma < 2$, $n \geq 4$, $R_{(M, g_0)} > 0$ and $Q^n_{2\gamma} > 0$ implies $P^n_{2\gamma} > 0$. When $n = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

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- Key step in the proof: The "right" choice of the conformal compactified Einstein metric in X^{n+1}.

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- Well-known result: f smooth on $\mathbb{R}^n$

$$\Delta_{x,y} U(x,y) \text{on} \mathbb{R}^{n+1}{}_{+} = (x, y | x \in \mathbb{R}^n, y > 0,$$
$$U\big|_{\mathbb{R}^n}(x) \;=\; f(x)$$

then

$$-U_y(x,0) = (-\Delta)^{\frac{1}{2}} f(x).$$

- Theorem (Caffarelli-Silvestre '06)

$0 < \gamma < 1$, $a = 1 - 2\gamma$,

$$(*) \begin{cases} \mathrm{div}(y^a \nabla U) & = 0 \text{ on } \mathbb{R}^{n+1}_{+} \\ U\big|_{\mathbb{R}^n} & = f. \end{cases}$$

Then

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- Applications to free-boundary problems, study of non-local minimal surface etc.

Work of Graham-Zworski

- On Conformal Compact Einstein Setting, a class of conformal covariant operators $P_{2\gamma}$ exists for

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- $P_{2\gamma} = (-\Delta)^{\gamma}$ in special setting of $(\mathbb{R}^{n+1}_{+}, \mathbb{R}^n, g_+)$, where

$$g_+ = \frac{dy^2 + dx^2}{y^2}$$

while

$$\bar{g} = y^2 g_+.$$

is the compactified metric.

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- Thus

$$g_+ = \frac{dy^2 + g_y}{y^2} \; in \; M \times (0, \epsilon).$$

Denote $\bar{g} = y^2 g_+$, then $\bar{g} = dy^2 + g_y$ on X , and $g_y\big|_{y=0} = g_0$ *on* M.

- On (X^{n+1}, M^n, g_+), given $f \in C^\infty(M)$
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- P_{2k} is the GJMS operators.

Examples

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- $(H^{n+1}/\Gamma, \Omega(\Gamma)/\Gamma, g_h)$, where Γ a Kleinian group.
- Schwarzschild space.

Theorem (C- Gonzalez '11)

On (X^{n+1}, M^n, g_+) *conformal compact Einstein setting, given a function* $f \in C^\infty(M)$*;*

$$(*)_s \qquad -\Delta_{g_+} u - s(n-s)u = 0 \text{ on } X$$

$$\updownarrow \qquad\qquad s = \frac{n}{2} + \gamma$$

$$(*)'_s \qquad -div_{\bar{g}}(\rho^a \nabla_{\bar{g}} U) + E(\rho, a)U = 0 \text{ on } X$$

$$U = \rho^{s-n} u \qquad U\Big| = f$$

where $\bar{g} = \rho^2 g_+$, $0 < \gamma \leq \frac{n}{2}$, ρ*: any totally geodesic defining function.*

We can express $P_{2\gamma} f$ *in terms of boundary behavior of* U*.*

- In general, the expression of $E(\rho, a)$ is complicated, but in the special case when $\gamma = \frac{1}{2}$, $a = 0$, $E(\rho, a) = C_n R_{\bar{g}}$, the equation becomes

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$$-\Delta_{g+} v - s(n-s)v = 0 \quad (*)_s$$

and v is Possion operator on data $f \equiv 1$. Note if $v > 0$ on X^{n+1}, one can define $\rho* = v^{\frac{1}{n-s}}$ then $E(\rho*, a) = 0$, where $s = \frac{n}{2} + \gamma$ and $a = 1 - 2\gamma$.

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- It turns out when $0 < \gamma < 1$, $v > 0$ if and only if $\lambda_1(-\Delta_+) > \frac{n^2}{4} - \gamma^2$.

Proof of Theorem 1

Lemma: When $0 < \gamma < 1$, $(a = 1 - 2\gamma)$ and

$$\rho* = y + d_\gamma Q_{2\gamma} y^{1+2\gamma} + O(y^3) > 0.$$

Given f, $u =$ solution of Poission equation with data f, $U := (\rho*)^{\gamma - \frac{n}{2}} u$. Then for some positive c_γ, we have

$$P_{2\gamma} f(x) = c_\gamma \lim_{y \to 0} (\rho^*)^a \frac{\partial U}{\partial n}(x) + \frac{n - 2\gamma}{2} Q_{2\gamma} f(x),$$

for $x \in \partial X$.

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- Proof:

  With the choice of $\rho*$, $g* = (\rho*)^2 g+$,we have $U := (\rho*)^{\gamma - \frac{n}{2}} u$ and $U|_M = f$, U satisfies the PDE

$$-div_{g*}((\rho*)^a \nabla_{g*} U) = 0.$$

  Hence

$$\int_X (-div_{g*}((\rho*)^a \nabla_{g*} U) \; U = 0$$

  and

$$\int_M (\rho*)^a \frac{\partial U}{\partial n} U = \int_X (\rho*)^a |\nabla U|^2 \geq 0.$$

  We then apply the Lemma to finish the proof.

Outline of the Proof of Theorem 2

- Theorem 2: When $1 < \gamma < 2$, $d \geq 4$, $R_{(M^d, g_0)} > 0$ and $Q^d_{2\gamma} > 0$ implies $P^d_{2\gamma} > 0$. When $d = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

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- Step 4: Apply the extension theorem and some proof similar to that of Gursky-Malchiodi to establish the theorem.

Extension Theorem in flat case when $\gamma > 1$

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- On $\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$, Denote

$$\Delta_a U = y^{-a} div(y^a \nabla U) = \Delta U + \frac{a}{y}\frac{\partial U}{\partial y}$$

Then

$$\Delta_a U = 0, \ \ with \ U|_{\mathbb{R}^n} = f$$

where $a = 1 - 2\gamma$, iff

$$(\Delta_b)^2 U = 0, \ \ with \ U|_{\mathbb{R}^n} = f, \ \ and \ lim_{y \to 0} y^b \frac{\partial U}{\partial y} = 0$$

with $b = 3 - 2\gamma$.

Extension Theorem in flat case when $\gamma > 1$

In this case

- $$\int_{\mathbb{R}^n} (-\Delta_x f)^\gamma f dx = c_{n,\gamma} \int_{\mathbb{R}^{n+1}_+} (\Delta_b U)^2 y^b dxdy.$$

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- $$(-\Delta_x f)^\gamma (x) = c_{n,\gamma} \lim_{y \to 0} y^b \frac{\partial}{\partial y} \Delta_b U(x,y).$$

  We have the "renormalized energy", e.g. when $\gamma = \frac{3}{2}$, $a = 1 - 2\gamma = -2$, $b = 3 - 2\gamma = 0$, then

$$\lim_{\epsilon \to 0} \left(- \int_{\mathbb{R}^n} \int_{y \geq \epsilon} |\nabla U|^2 y^{-2} dxdy + \frac{1}{\epsilon} \int_{y=\epsilon} |\nabla_x f|^2 dx \right) \quad = \frac{1}{2} \int_{\mathbb{R}^n} \int_{y \geq 0}$$

Extension Theorem on Poincare Einstein setting

- Notation: Given a number $m \in \mathbb{R}$, ϕ a function defined on (X, g), (F, h) a metric space of dimension m; on the metric measure space $(X, g, e^{-\phi} dv_g)$, denote $P_{2k,\phi}^{m}$ the GJMS operators on the warped product space

$$(X \times_{e^{-\phi}} F^m, g \oplus e^{-\frac{2\phi}{m}} h)$$

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- In this notion, when $m = \infty$, $Ric^m_\phi = Ric + \nabla^2 \phi$ the Bakry-Emery Ricci tensor, Δ operator is replaced by $\Delta_\phi := \Delta - \nabla\phi\, \nabla$.

Extension Theorem on Poincare Einstein setting

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- (1) On $(X^{n+1}, \partial X, g_+)$, C.C.E. with $Ric_{g+} = -n$, When $s = \frac{n}{2} + \gamma$, $g = \rho^2 g+$, the $(*)_s$ equation

$$-\Delta_{g+} u - s(n-s)u = 0, \ \ on \ X$$

can be re-written as $(*)''$

$$P^m_{2,\phi} U = 0 \ on \ \ X$$

where (F^m, h) is chosen to be the (sphere) with $Ric_h = (m-1)h$, $U = \rho^{s-n} u$ and $g = \rho^2 g_+$, $m = 1 - 2\gamma$ and $e^{-\phi} = \rho^m$.

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- With this notion, we have the extension theorem on Poincare Einstein manifolds.

Step 3, the right choice of compactified metric

- Recall if $v = v_s$ satisfies the Possion equation with Dirichlet data $f \equiv 1$, under the condition $R_{\partial X, g_0} > 0$, we have $v > 0$ on X. Denote $\rho* = v^{\frac{1}{n-s}}$, and $g = g* = (\rho*)^2 g_+$, $s = \frac{n}{2} + \gamma$.

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- (b) When $1 < \gamma < 2$, $(Q_4)_{\phi_2}^{m_2} = 0$, where $m_2 = 1 - 2\gamma$, $e^{-\phi_2} = (\rho*)^{m_2}$

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- (c) When $\gamma > 1$, $R_{g*}|_{\partial X} = c_\gamma R(\partial X, g_0) > 0$, where $c_\gamma > 0$.

Step 3, the right choice of compactified metric

- Another crucial property:

  Lemma: Under the assumption $R_{\partial X, g_0} > 0$, for all $s \geq \frac{n}{2} + 1$, $R_{g*} > 0$ on X.

  Proof: Due to property (b) above, we have the PDE for $R = R(g*)$,

$$\Delta_{\phi_2} R = c_1 R^2 - c_2 |E|^2,$$

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- It turns out when $s = n + 1$, $c_1 = 0$, the metric has been studied before by J. Lee '95, where the equation by maximal principle together with property (c) gives $R_{g*} > 0$.
- We can now run a "continuity" argument on the parameter s starting at $s = (n + 1)$, together property (c), apply strong maximal principle to conclude $R_{g*} > 0$ on X for all $s \geq \frac{n}{2} + 1$.

Proof of Theorem 2

When $1 < \gamma < 2$, we will show that when $R_{(\partial X, g_0)} > 0$ and $Q_{2\gamma} > 0$, implies $P_{2\gamma} > 0$.

Proof:

Given f defined on ∂X, by Extension theorem

$$\int_{\partial X} (P_{2\gamma} f)\ f dv_{g_0} = \frac{n-2\gamma}{2} \int_{\partial X} (Q_{2\gamma} f)\ f dv_{g_0} \\ + c_\gamma \text{ Energy term of } (P_4)^{m_2}_{\phi_2}.$$

We apply the fact $R_{g*} > 0$, together with an argument similar to that of Grusky-Malchiodi to prove the 4-th order energy term is non-negative, and which together with $Q_{2\gamma} > 0$ establishes the result.

Some discussion

- Theorem

  (J. Qing - Guillarmou '10) On (X^{n+1}, M^n, g^+) *C.C.E. manifolds with* $n+1 > 3$, $Y(M^n, g_0) > 0$ *iff the first real scattering pole* $\leq \frac{n}{2} - 1$.

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  - Schoen-Yau: If M is of positive scalar curvature, then $\delta(\Gamma) \doteqdot$ Hausdroff dim of $S^n \setminus \Omega(\Gamma)$, then $\delta(\Gamma) \leq \frac{n}{2} - 1$.
  - Work of *Sullivan – Patterson*, *P.Perry* etc.

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- Work of Gonzalez-Qing '12 studied the $Q_{2\gamma}$ equation and related positive mass problem when $0 < \gamma < 1$. When $\gamma = \frac{1}{2}$, $Q_1 = cH$, the mean curvature. In general, is there a geometric description of the fractional Q curvature?