Existence of minimal hypersurfaces

André Neves

(Joint with Fernando Marques)

Imperial College
London

Motivation

Poincaré's second best question (1905)

Does every 2-sphere has a closed geodesic?

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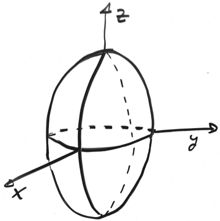

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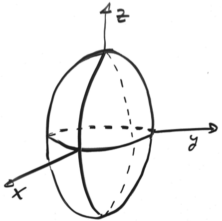

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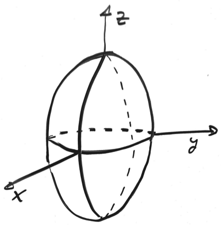

- Also work by Kilingenberg, Ballman, Jost, Taimanov, Grayson.
- Franks, Bangert, (1992) Every (S^2, g) has an infinite number of closed geodesics.

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- Jost, (1989) Every (S^3, g) admits 4 minimal embedded spheres.
- White, (1991) 3-spheres with positive Ricci curvature have one minimal embedded torus.
- White Conjecture: Any 3-sphere has five minimal embedded tori.

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- Khan–Markovic, (2012) Every compact hyperbolic 3-manifold has incompressible surfaces of arbitrarily high genus and thus an infinite number of immersed minimal surfaces.
- Kapouleas outlined an approach to theorem above when $n = 2$ based on desingularization or doubling methods.

Some questions

- With the round metric, $\omega_i(S^3) = 4\pi$ for $i = 1, ..., 4$.

  Charles Nurser can show that $\omega_i(S^3) = 2\pi^2$ for $i = 5, 6, 7, 8$ and $2\pi^2 < \omega_9(S^3) \leq 8\pi$.

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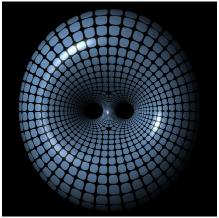

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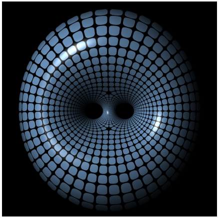

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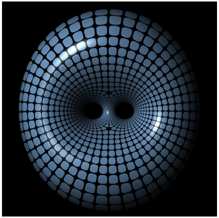

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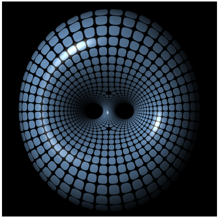

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- Weyl Law for the k-width (Gromov Conjecture):

$$\lim_{k\to\infty} \frac{\omega_k(M)}{k^{\frac{1}{n+1}}} = \alpha(n)(vol\, M)^{\frac{n}{n+1}}.$$

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- Consider the first $k+1$-eigenfunctions $\phi_0, \phi_1, \ldots, \phi_k$ on M and

$$\omega_k(M) \leq \bar{\omega}_k(M) = \sup_{(a_0,\ldots,a_k)\in\mathbb{R}^{k+1}} vol\{a_0\phi_0 + \ldots + a_k\phi_k = 0\}.$$

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- Conjecture: The minimal hypersurface realizing $\omega_k(M)$ should be "well" approximated by $\{\phi_k = 0\}$.

  Is Σ_k becoming equidistributed?

  Is the first betti number of $\{\phi_k = 0\}$ proportional to k?