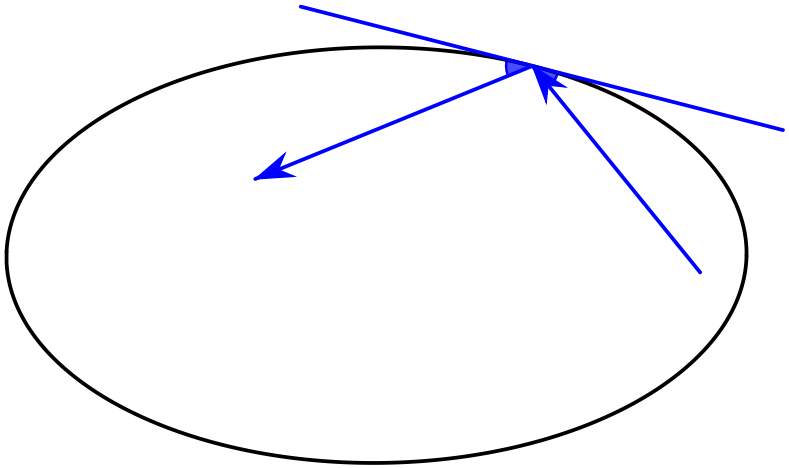
Billiards in conics revisited

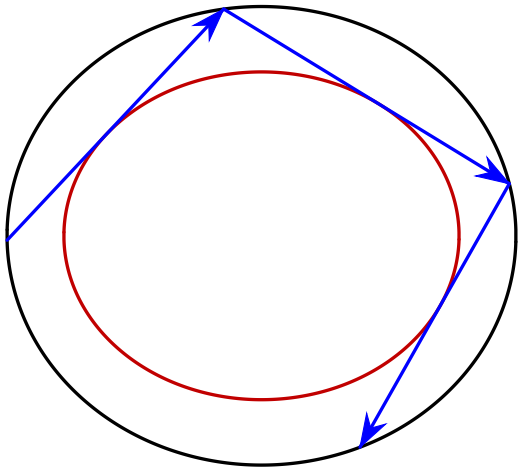
Stony Brook University

October 2023

Birkhoff billiards

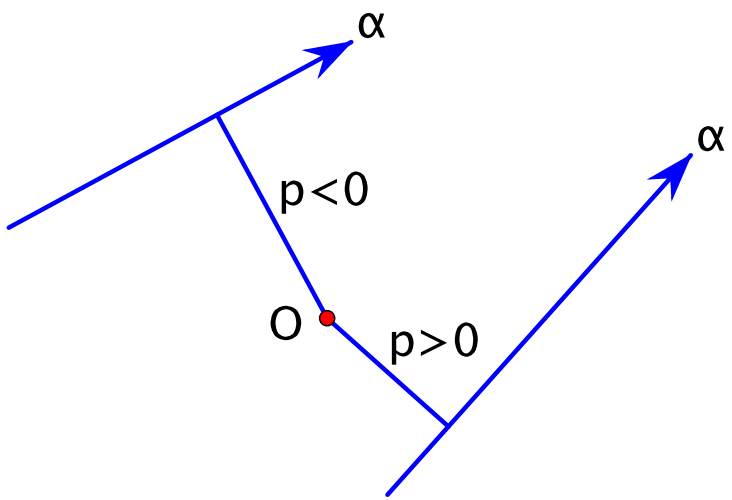
Motion of a free mass point with elastic reflection off the boundary (or reflection of rays of light from an ideal mirror).





Existence of caustics: if the billiard is strictly convex and sufficiently smooth (KAM theory, Lazutkin's theorem, 1973).

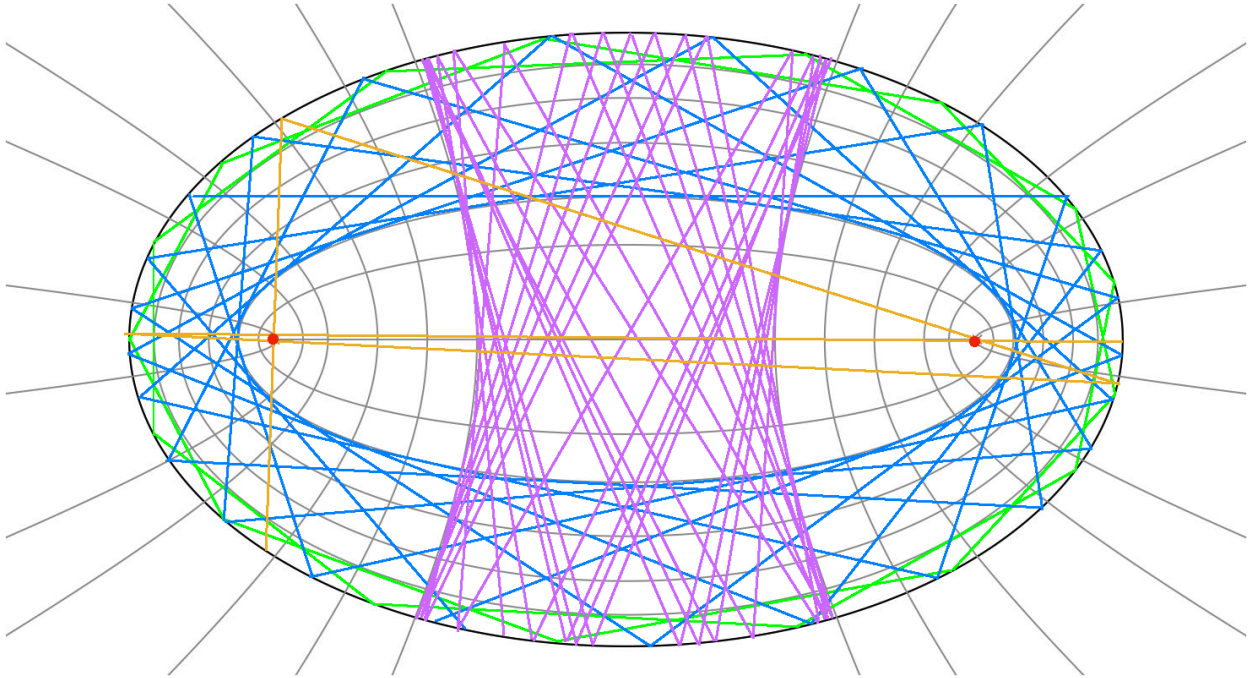
Symplectic structure on oriented lines (rays of light)



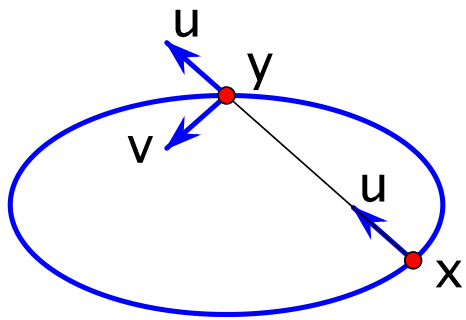
$$\omega = dp \wedge d\alpha.$$

Likewise in higher dimensions (and in Riemannian, or even Finsler, manifolds). *The billiard (optical) reflection is a symplectic map.*

Billiard in ellipse



Theorem: *Billiard in ellipse is completely integrable; the confocal ellipses (and hyperbolas) are its caustics.*



Proof: Let the ellipse be $(Ax, x) = 1$. Claim: $(Ax, u) = (Ay, v)$ (the Joachimsthal integral). Indeed,

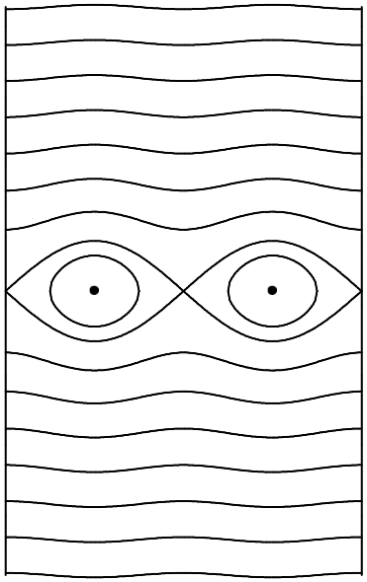
$$(Ax + Ay, y - x) = (Ax, y) - (Ax, x) + (Ay, y) - (Ay, x) = 0,$$

hence $(Ax, u) = -(Ay, u)$. Also, Ay is a normal at point y, therefore

$$(Ay, u + v) = 0, \quad \text{hence} \quad (Ay, u) = -(Ay, v).$$

The same argument works in any dimension.

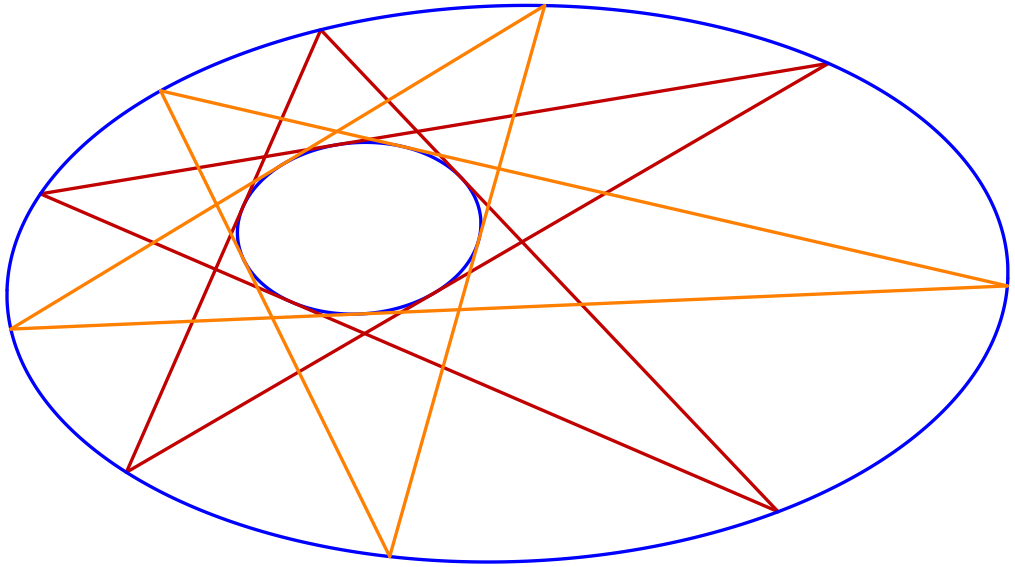
Arnold-Liouville theorem



Using invariance of the area form, one defines coordinates on the invariant curves, in which the billiard map is a parallel translation. Equivalently, the Hamiltonian vector field of the Joachimsthal integral is invariant under the billiard map.

First corollaries

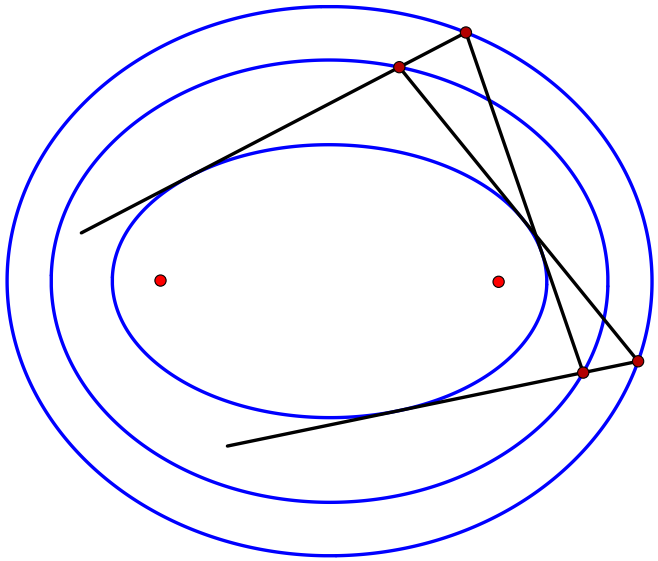
Poncelet porism:

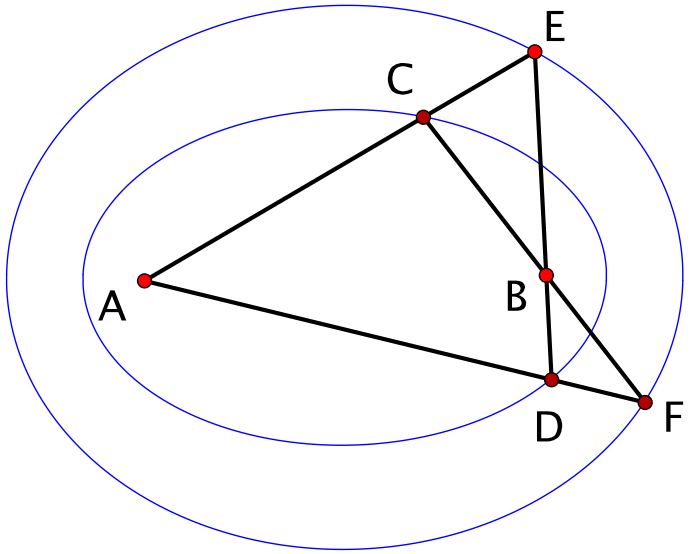




R. Schwartz, S. Tabachnikov. *Centers of mass of Poncelet polygons, 200 years after*. Math. Intelligencer, v. 38, No 2 (2016), 29-34. (*Hint*: in part, the paper is a hoax!)

Since confocal ellipses share their caustics, *the reflections from confocal ellipses commute*.



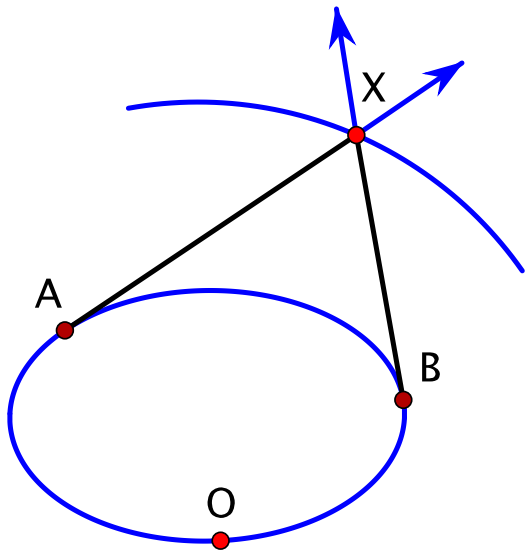


The most elementary problem of elementary geometry:

$$AC + CB = AD + DB \iff AE + EB = AF + FB.$$

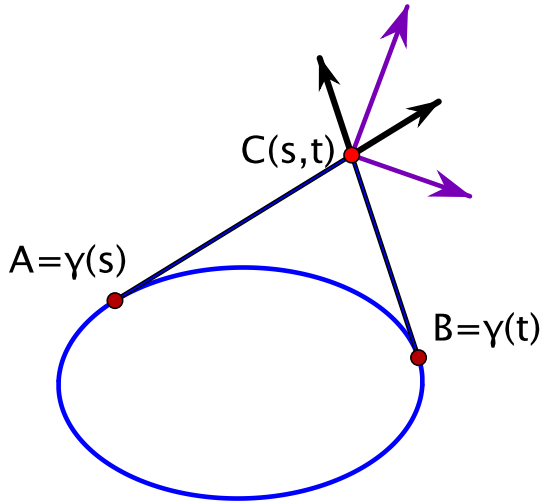
String construction

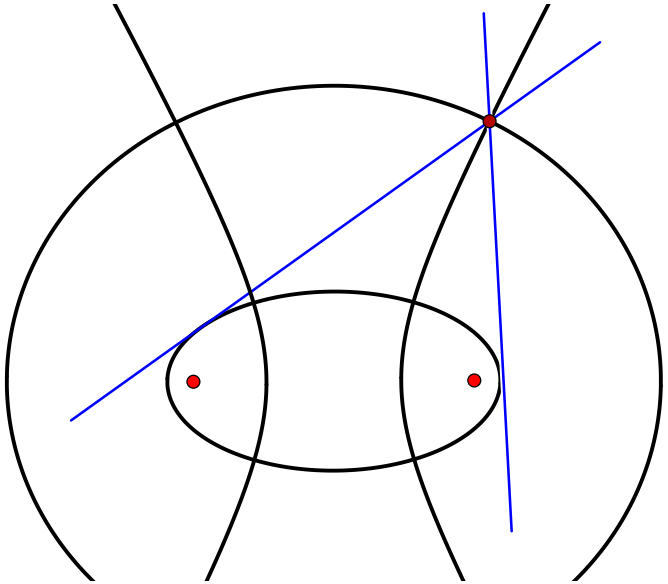
Recovering the billiard table from its caustic: $OA + AX + X\overset{\frown}{B} + \overset{\frown}{BO} = const$. Indeed, let $f(X) = |XA| + \overset{\frown}{AO}; g(X) = |XB| + \overset{\frown}{BO}$:



Theorem [Graves]: *Wrapping a closed non-elastic string around an ellipse produces a confocal ellipse.*

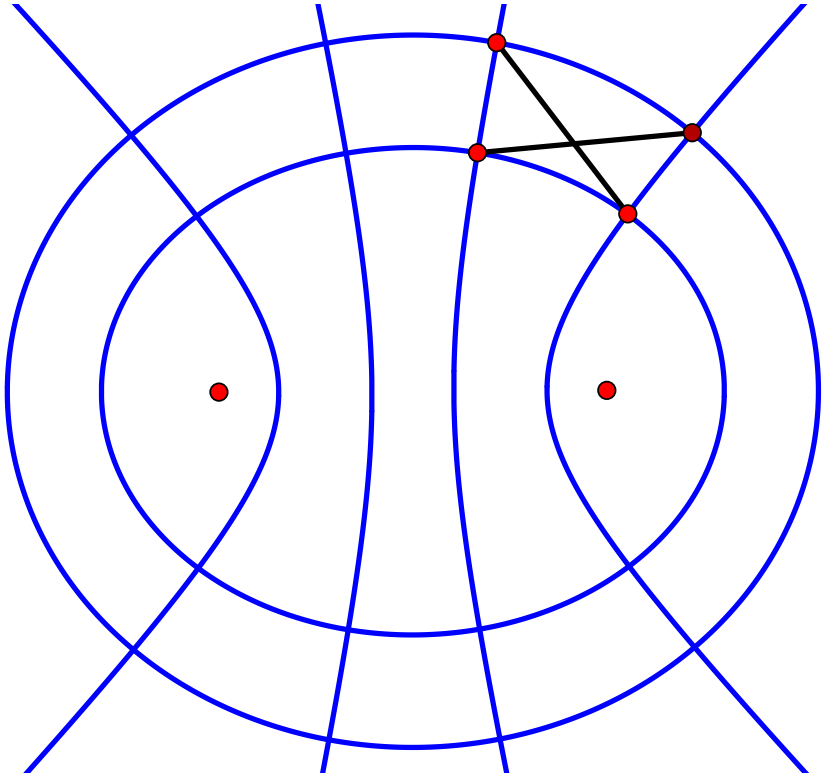
Coordinates outside of an ellipse:





The confocal ellipses are $\{t-s = \text{const}\}$, the confocal hyperbolas are $\{t+s = \text{const}\}$. The billiard reflection in a confocal ellipse is $t \mapsto t + c$, and in a confocal hyperbola $t \mapsto c - t$.

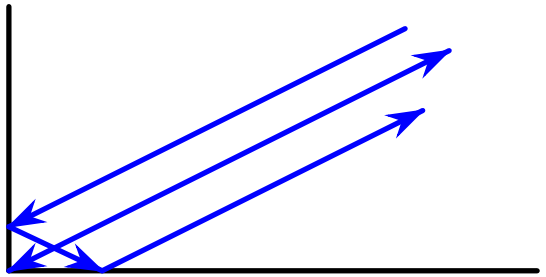
Ivory's lemma



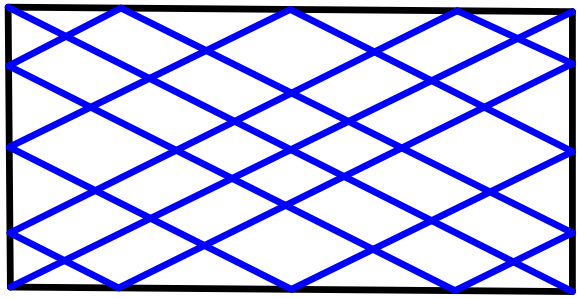
On the attraction of homogeneous ellipsoids. Phil. Trans. Royal Soc. London **99** (1809), 345–372.

Ivory's lemma by way of billiards

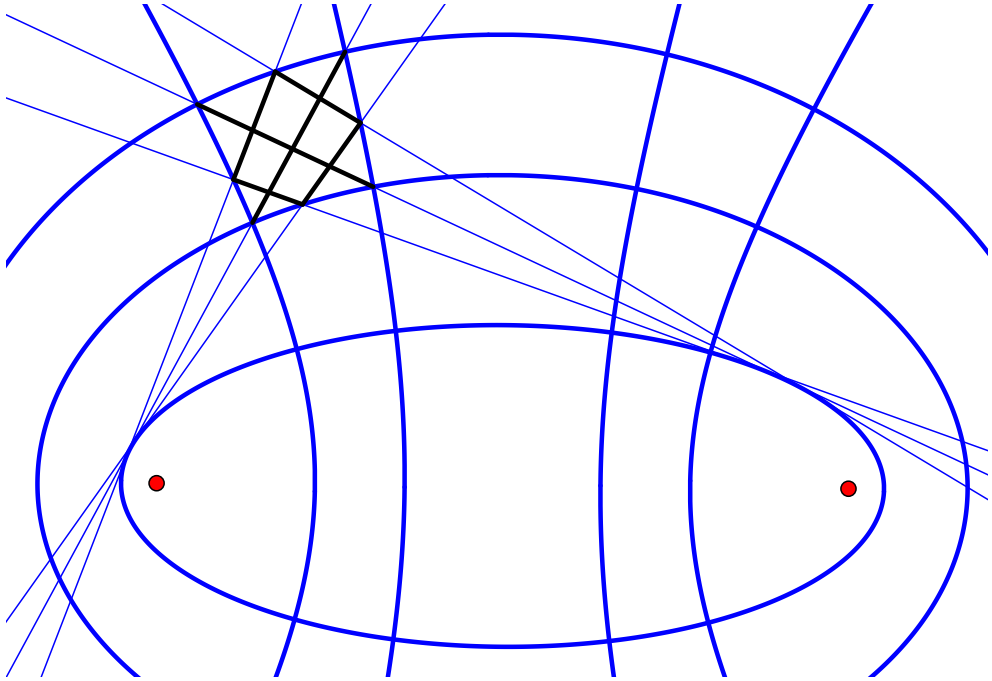
Corner reflector:



Diagonal of a rectangle are equal:

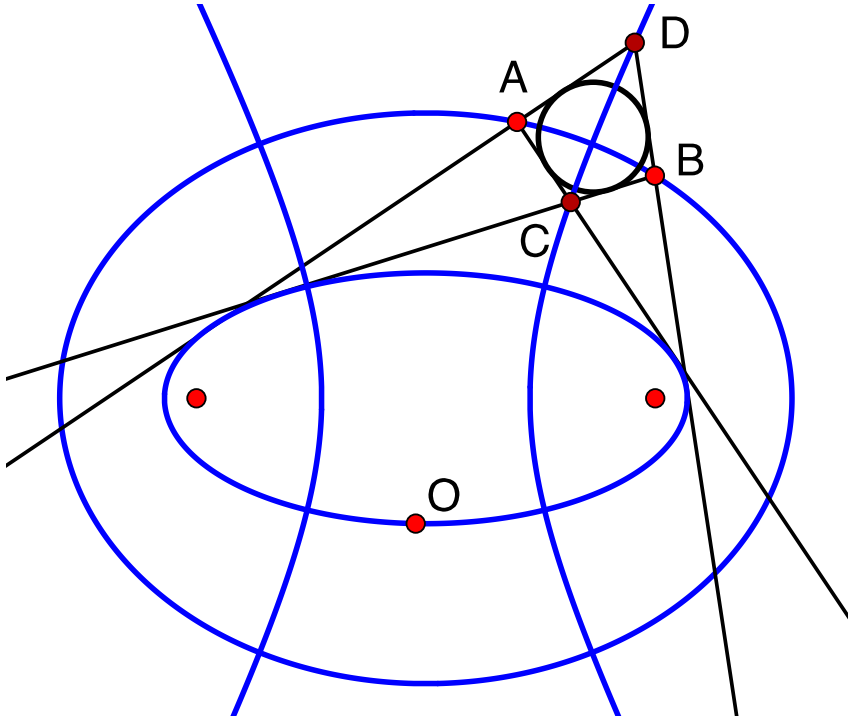


Likewise for confocal conics:



The composition of four reflections is a shift in the coordinate on the caustic, and it has a fixed point (a diagonal), hence it is the identity.

Chasles theorem



Coordinates s_1, s_2, t_1, t_2. Then $t_1 - s_1 = t_2 - s_2$, hence $t_1 + s_2 = t_2 + s_1$, and therefore points C and D lie on a confocal hyperbola.

Let f and g again be the distances from points to point O going around the ellipse. Then

$$f(A) + g(A) = f(B) + g(B), \quad f(C) - g(C) = f(D) - g(D),$$

hence

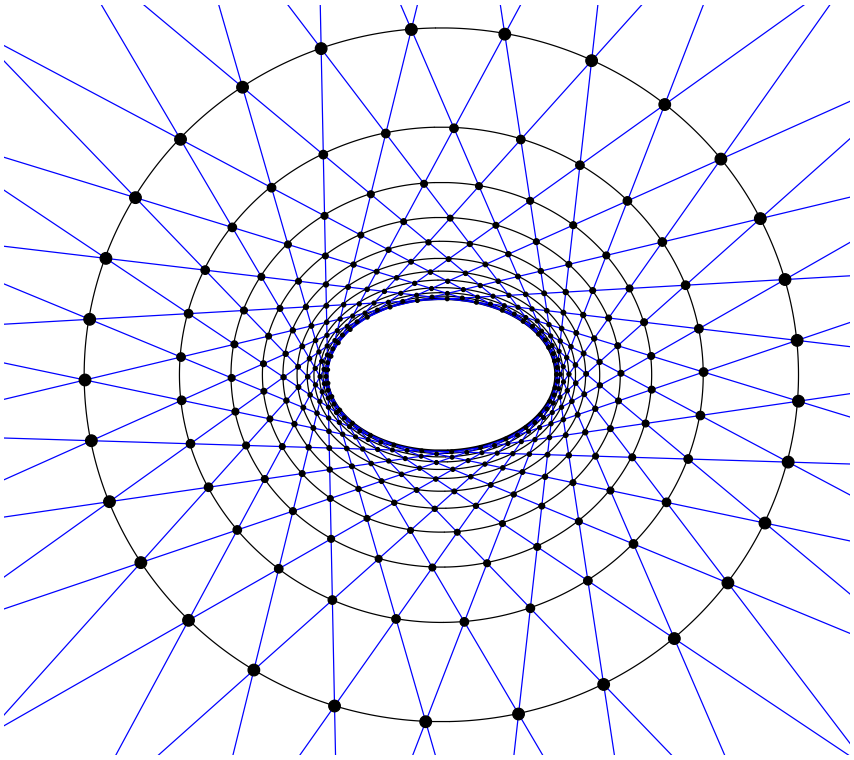
$$f(D) - f(A) - g(A) + g(C) + f(B) - f(C) - g(D) + g(B) = 0,$$

or

$$|AD| + |BC| = |AC| + |BD|,$$

and the quadrilateral is circumscribed. QED

Poncelet grid



Lines $\ell_1, \ldots, \ell_n$, concentric and radial sets

$$P_k = \cup_{i-j=k} \ell_i \cap \ell_j, \quad Q_k = \cup_{i+j=k} \ell_i \cap \ell_j.$$

Theorem [Darboux–R. Schwartz]: *Let n be odd. Then*
(i) The concentric sets lie on nested ellipses, and the radial sets lie on disjoint hyperbolas.
(ii) The complexified versions of these conics have four common tangent lines.
(iii) All the concentric sets are projectively equivalent to each other, and so are all the radial sets.

Proof: WLOG, the coordinates of the tangency points of the sides of the polygon with the inner ellipse are

$$0,\ \frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{n-1}{n}.$$

Then the string coordinates of the points of the concentric set P_k are

$$\left(0, \frac{k}{n}\right), \left(\frac{1}{n}, \frac{k+1}{n}\right), \left(\frac{2}{n}, \frac{k+2}{n}\right), \ldots$$

The difference is equal to k/n, a constant. It follows that P_k lies on a confocal ellipse. Likewise for the radial sets Q_k.

Lemma [Ivory]: *Let γ and Γ be confocal ellipses, centered at the origin and symmetric with respect to the coordinate axes, and let A be the diagonal matrix with positive entries that takes γ to Γ. Then for every point $P \in \gamma$, the points P and $A(P)$ lie on a confocal hyperbola.*

Claim: the map A takes P_k to P_m or to its centrally symmetric set (the argument for the radial sets is similar).

Set $u = (x+y)/2, v = (y-x)/2$. The (u,v)-coordinates of the points of the sets P_k and P_m are

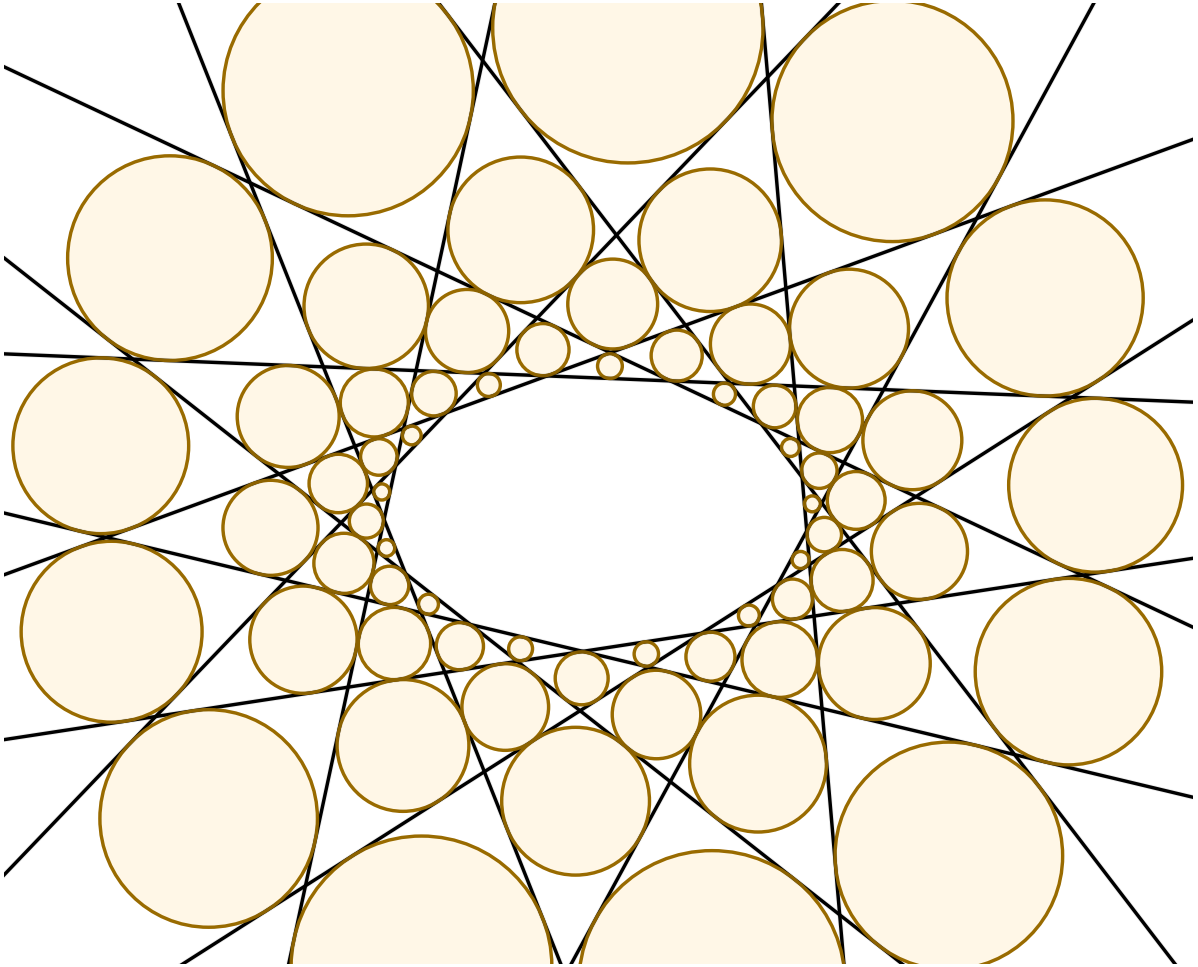
$$\left(\frac{k}{2n} + \frac{j}{n}, \frac{k}{2n}\right), \ \left(\frac{m}{2n} + \frac{j}{n}, \frac{m}{2n}\right), \quad (j = 0, 1, \ldots, n-1).$$

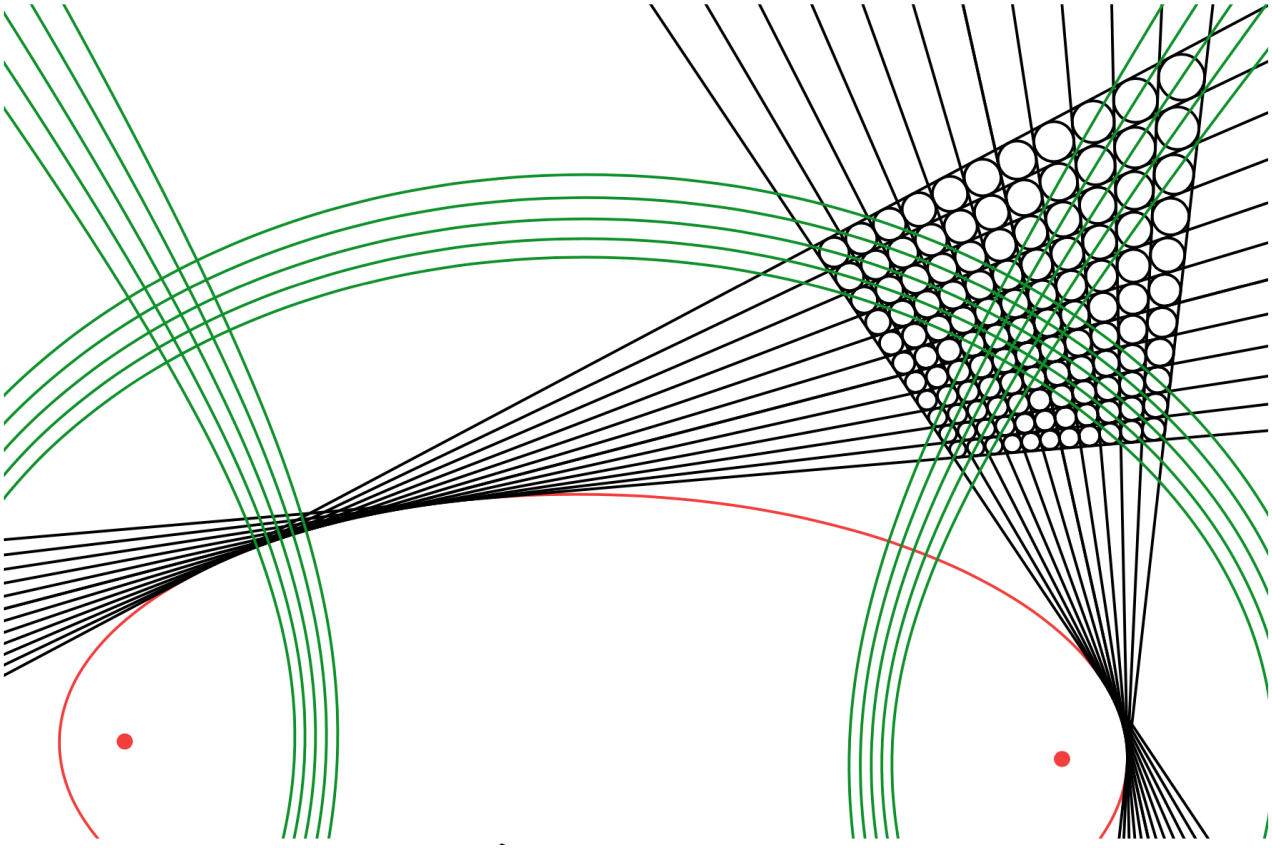
The sets P_k and P_m lie on confocal ellipses γ and Γ. By Lemma, the map A preserves the u-coordinate. Hence the coordinates of the points of the set $A(P_k)$ are

$$\left(\frac{k}{2n}+\frac{j}{n},\frac{m}{2n}\right)\ (j=0,1,\ldots,n-1).$$

If m has the same parity as k, this coincides with the set P_m, and if the parity of m is opposite to that of k, then this set is centrally symmetric to the set P_m. QED

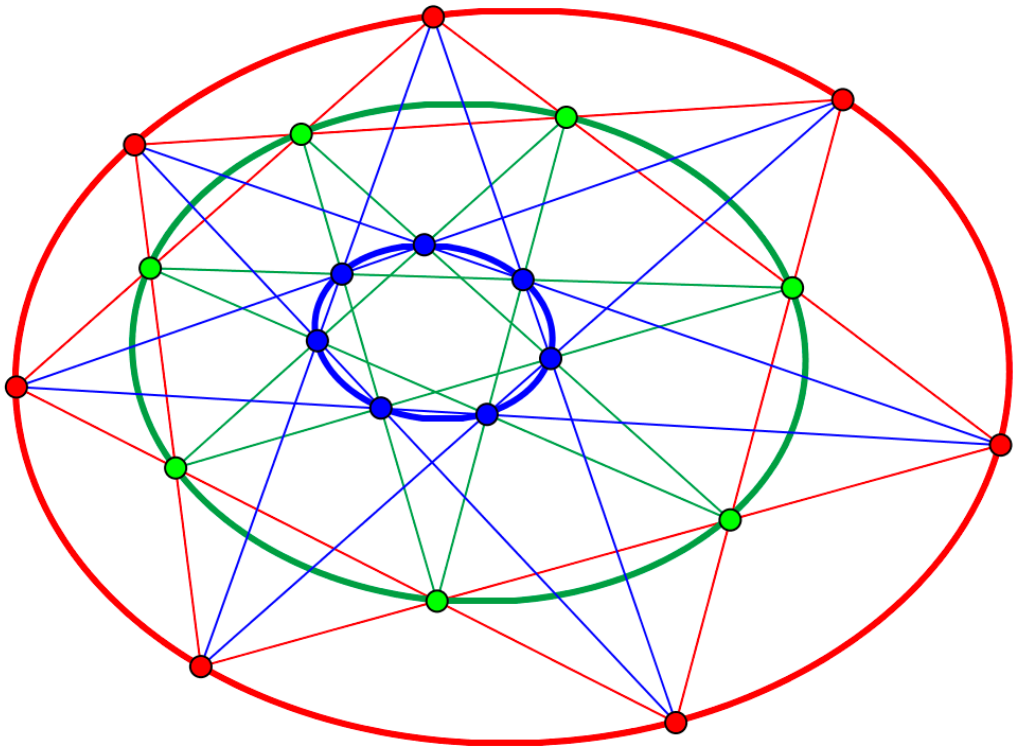
Poncelet grid of circles





(A. Akopyan, A. Bobenko. "Incircular nets and confocal conics". DOI: 10.1090/tran/7292).

Configuration theorems



P is the inner-most Poncelet heptagon, B, G, and R are the three outer ones.

Proof: Let

$$f : P \mapsto B, \;\; g : P \mapsto G$$

be the *commuting* projective transformations from the Poncelet grid theorem, and let

$$\phi : P \mapsto B, \;\; \psi : P \mapsto G$$

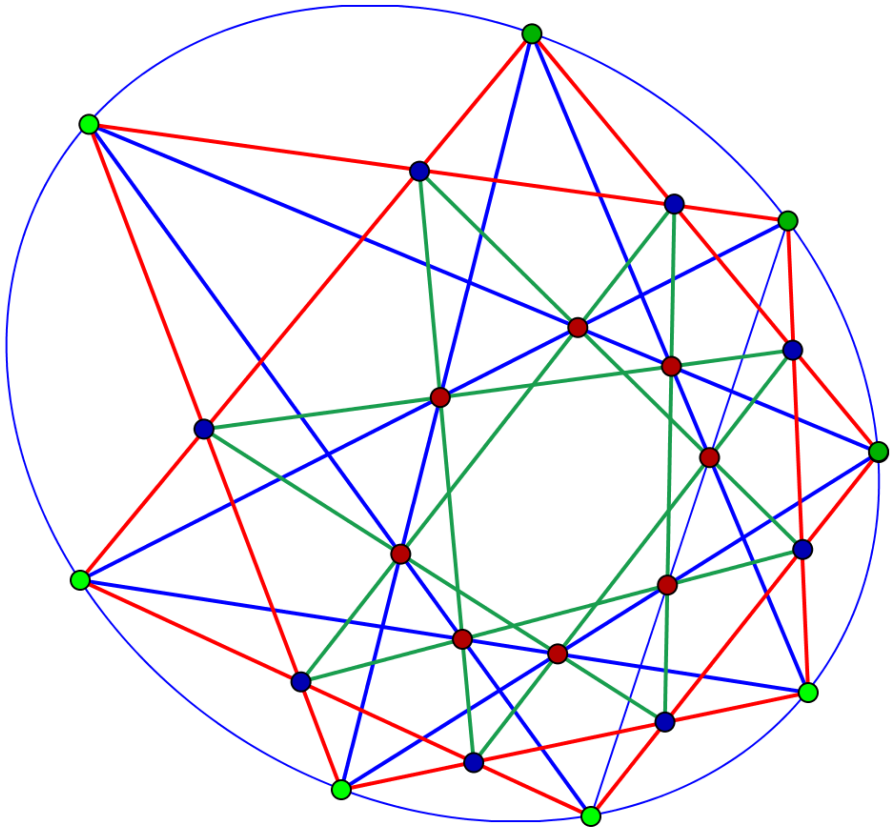
be the operations of intersecting the sides of a polygon, skipping one or two sides, respectively. The maps f and g commute with ϕ and ψ.

Now

$$f \circ g(P) = f(G) = f \circ \psi(P) = \psi \circ f(P) = \psi(B),$$

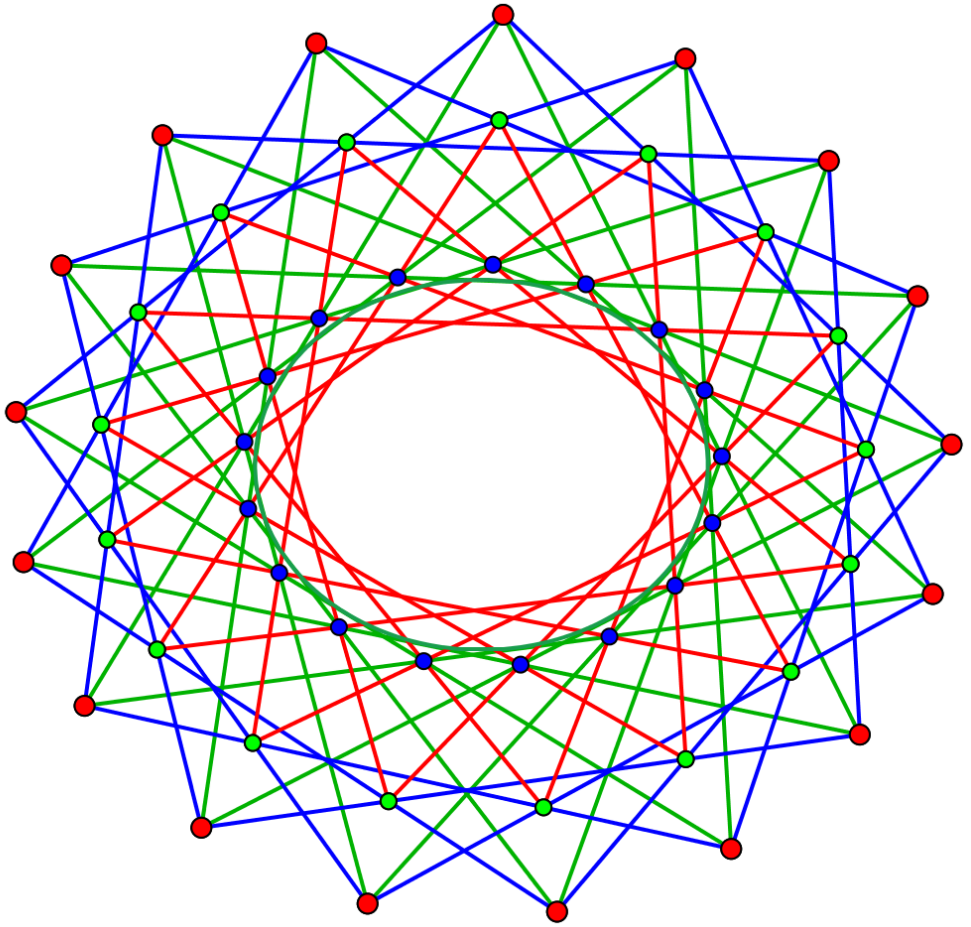
$$g \circ f(P) = g(B) = g \circ \phi(P) = \phi \circ g(P) = \phi(G),$$

and since $f \circ g = g \circ f$, one has $\psi(B) = \phi(G)$. QED

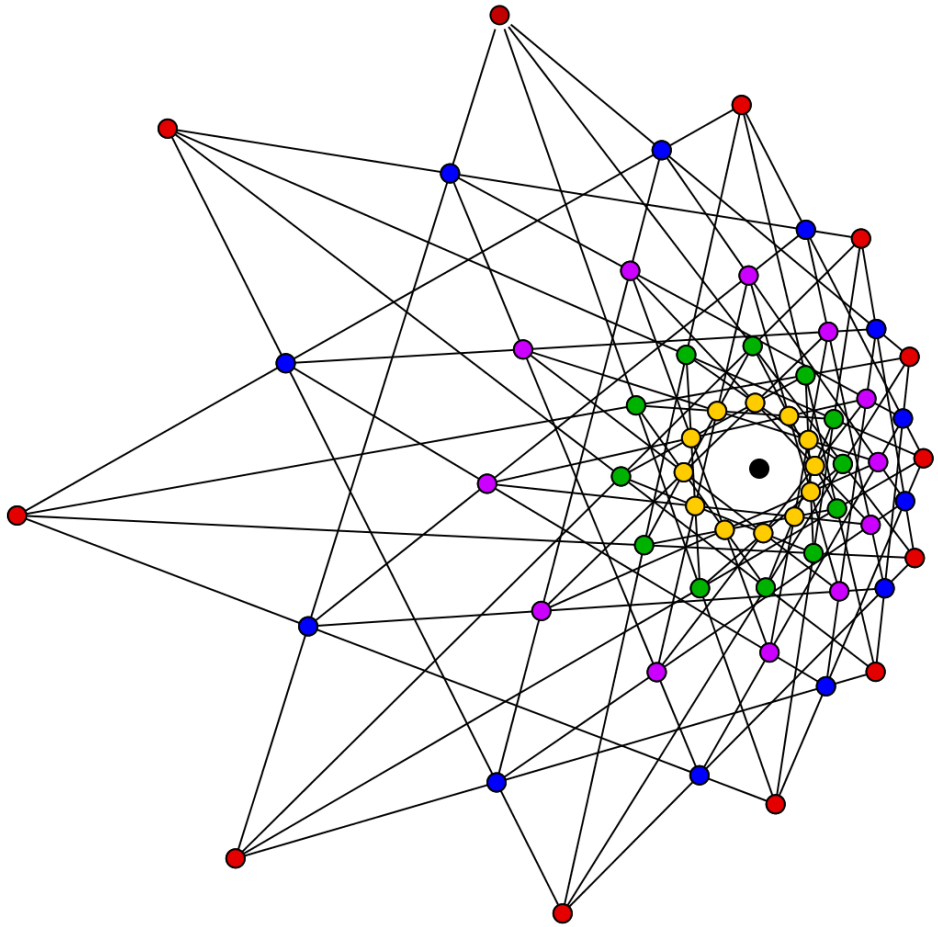


The same result holds for even n, but the proof is more involved; work in progress with L. Berman, G. Gévay, J. Richter-Gebert.

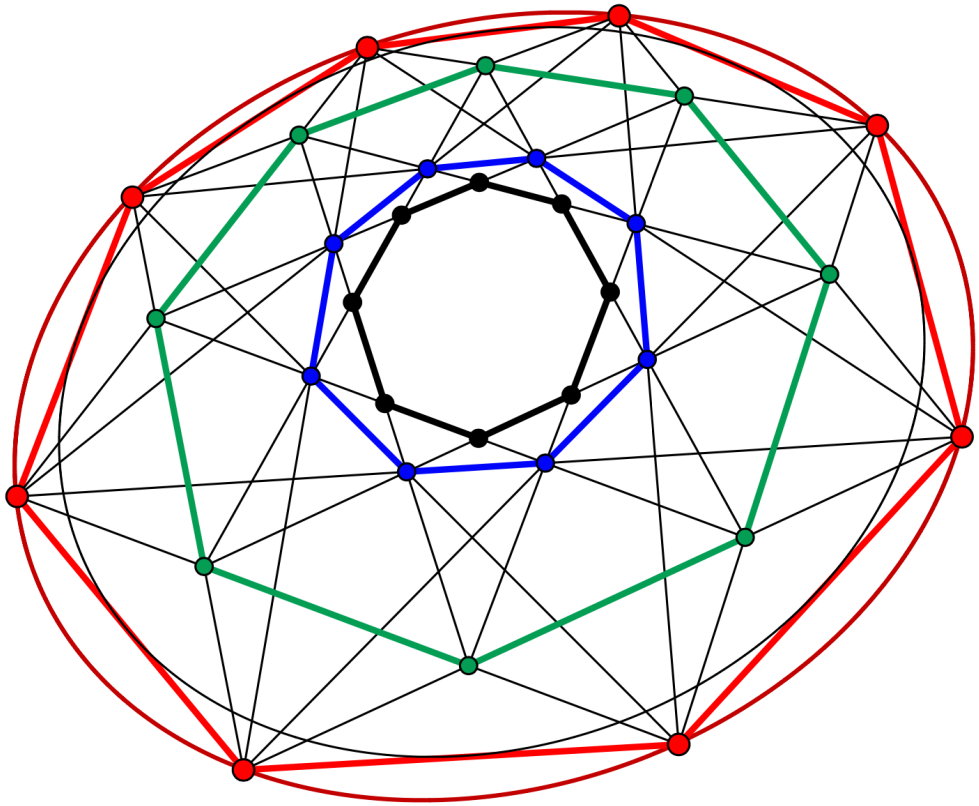
More points, other skip parameters (17-gon):



And more (dodecagon, iterated):



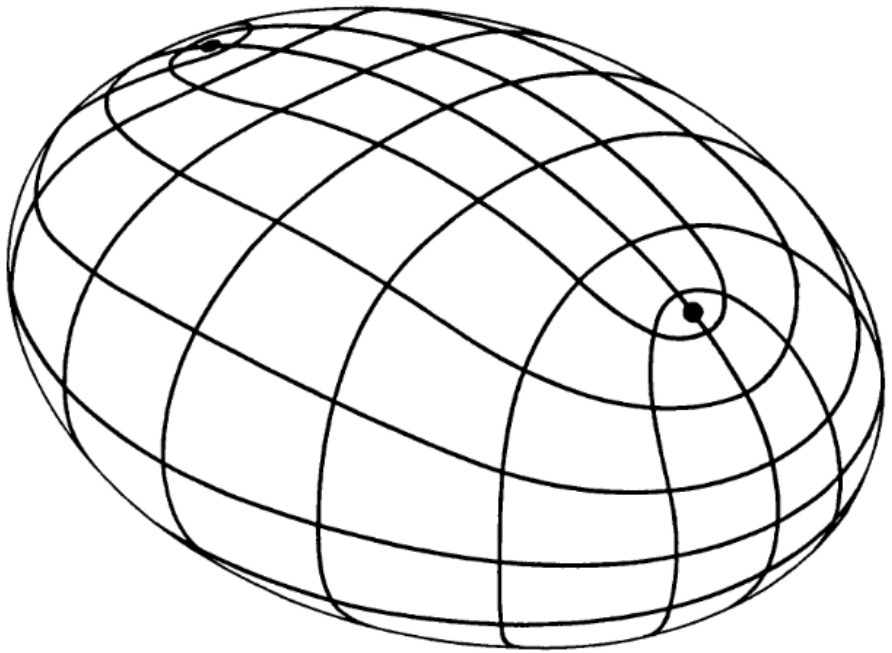
This result can be interpreted in terms of the *pentagram map*: for a Poncelet polygon, the deep-diagonal pentagram maps commute, e.g., $T_2 \circ T_3 = T_3 \circ T_2$:



Much of what I have said extends to Liouville metrics

$$ds^2 = [f_1(x_1) + f_2(x_2)](dx_1^2 + dx_2^2).$$

For example,



Dan Reznik's experiments



D. Reznik, R. Garcia, J. Koiller. Eighty New Invariants of N-Periodics in the Elliptic Billiard. arXiv:2004.12497.

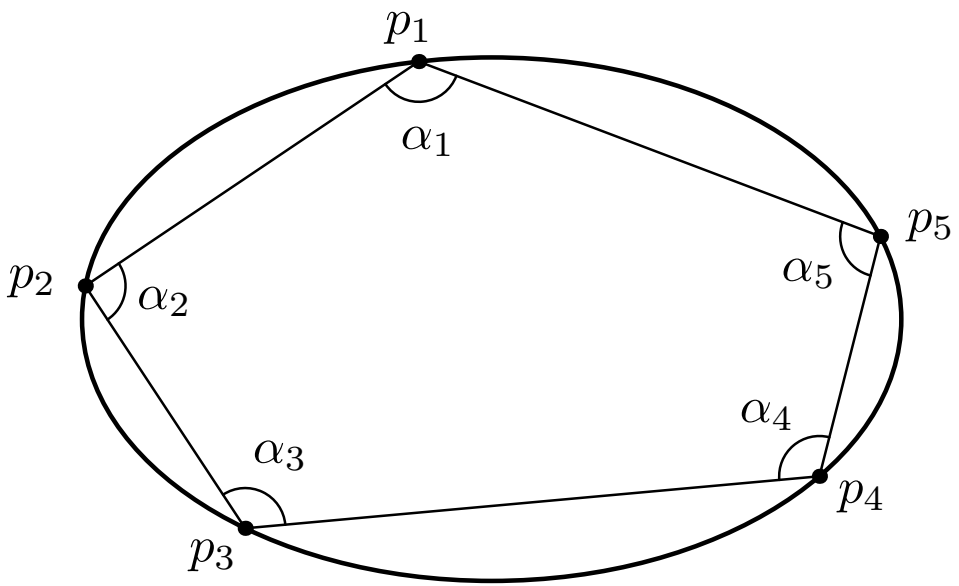
A. Akopyan, R. Schwartz, S. Tabachnikov. Billiards in ellipses revisited. European J. Math., 8 (2022), 1313–1327;

M. Bialy, S. Tabachnikov. Dan Reznik's identities and more. European J. Math., 8 (2022), 1341–1354.

Theorem: *For an n-periodic billiard trajectory P in an ellipse, one has*

$$\sum_{i=1}^{n} \cos \alpha_i = JL - n,$$

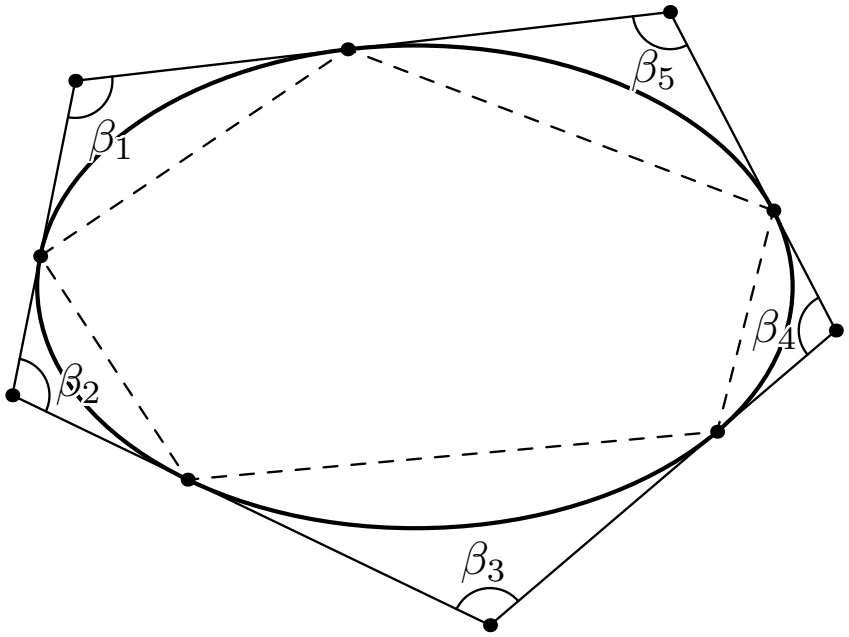
where L and J are the perimeter and the value of the Joachimsthal integral of P.

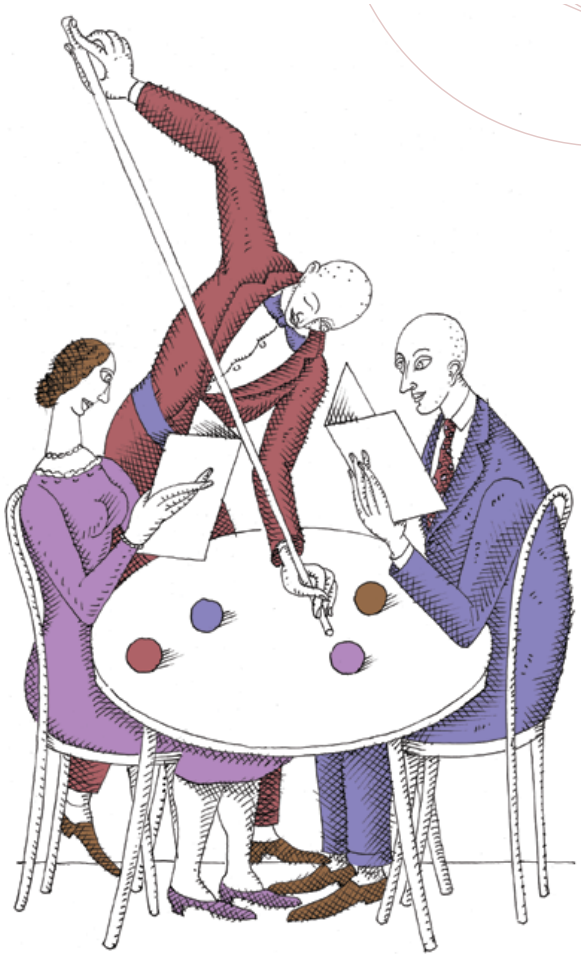


Theorem: *The product*

$$\prod_{i=1}^{n} \cos \beta_i$$

remains constant as P varies in a 1-parameter family of periodic billiard paths in an ellipse.





Thank you!