MAT 683 "Applications of J-holomorphic curves" Spring 2005

General Information

I plan to post exercises and notes from time to time.
Notes on the first few lectures, posted February 15 and available in pdf.

Here are some more notes posted Feb 27 (last revised Mar 1), available in pdf. They are on SO(3) actions.
Here are some more notes on the pentagon space (and a little on SO(3) actions) posted Mar 2.

Here is a list of problems in symplectic topology. If you become interested in any of them, talk to me so I can give you more pointers as to where to begin: posted April 8.

Announcements

- (April 8) I will start running a weekly workshop on questions arising from the Hofer lectures. The lectures in this course will continue with Seidel's paper. We should get to the details of Floer theory very soon.

- I will be away from campus on Thursday March 10 and Tuesday April 5. I hope that some of you will speak those days, on topics to be determined.

- Paul Seidel is visiting Sullivan's seminar at CUNY on March 8th. Class will be cancelled that day; as many of us as possible will go there. Seminar runs approx 2--7pm. (It used to start earlier, I am not quite sure of the details now.)

MAT 683 notes

1. Existence of single holomorphic objects

So far, we have proved the following results.

Theorem 1.1. *Let (M, ω) be a closed symplectic manifold that is aspherical (i.e. $\omega = 0$ on $\pi_2(M)$. Then any 1-periodic Hamiltonian $H_t, t \in \mathbb{R}$, has a contractible 1-periodic orbit.*

Because of the asphericity assumption this is an easy case of Thm 9.1.1 in JHOL (the new version of my book with Dietmar).

Theorem 1.2. *Let L be a closed Lagrangian submanifold in $\mathbb{C}^n, \omega_0$. Then L is not exact, i.e. any primitive λ for ω restricts to a nonexact (but closed) form on L.*

This is Thm 9.2.1 in JHOL. I discussed very briefly the corresponding thm (9.2.14) on Lagrangian intersections.

There is a geometric point here that I didn't have time to explain and hope to come back to later. This is the geometric intepretation of an inhomogeneous equation

$$\partial_s u + J\partial_t u = \nabla H (= JX_H) \tag{1}$$

in terms of the flow ϕ_t^H generated by H. Roughly speaking (this is not quite right) a map $u : (B, \partial B) \to (M, L)$ satisfying this equation corresponds to a J-holomorphic strip $u : \mathbb{R} \times [0, 1] \to M$ satisfying the boundary conditions $u(\mathbb{R} \times \{0\}) \subset L$ and $u(\mathbb{R} \times \{1\}) \subset \phi(L)$ where ϕ is the time 1-map of H. (To make this correct — apart from signs which I haven't checked – you need to consider different J in the two equations.) Such a strip (if it has bounded energy) must converge at its ends to intersection points $L \cap \phi(L)$ and so can exist only if this intersection is nonempty.

There is one thing I would like to point out here. In our proof of Theorem 1.2 we showed by calculation that there is no solution of the equation (1) where $H(z) := a \cdot z$ for $|a| \geq 2c$ where $L \subset \{|z| < c\}$. Note that the flow of such H displaces L in time 1, i.e. if ϕ is the time 1 map then $\phi(L) \cap L = \emptyset$. (and because the diameter of L could be $2c - \varepsilon$ this inequality is sharp.) Hence the remarks in the previous paragraph could be used to give another proof that there is no solution.

Exercise 2. Work out the details of this second proof. **Hint:** cf. Ex 8.1.5 in JHOL. Also suppose you have a J-hol strip $u : \mathbb{R} \times [0, 1] \to M$ such that $u(\mathbb{R} \times \{0\} \subset L$ and $u(\mathbb{R} \times \{1\} \subset \phi_1^H(L)$ and work out the equation satisfied by v given by $v(s, t) = (\phi_t^H)^{-1} u(s, t)$. What would you need to assume about J in order that v be J_0-hol?

In our proof of Thm 1.2 we just produced one J-hol disc. People have worked to get more information on such discs. What can one say about its Maslov index? (The construction above shows that its Maslov index is $\leq n + 1$, but doesn't give much more info.) Can one get a J-hol disc through every point of a displaceable Lagrangian?

Exercise 3. Dietmar claims that in our proof of Thm 1.2 the bubbling must occur at the marked point $z = 1$ on the boundary of the domain for reasons of transersality. (i.e. you can see this from the Fredholm theory.) If so, since one can define the moduli space so that $u(z)$ goes through any fixed point $x_0 \in L$ this would imply that there is a disc through every point in L. Prove this claim.

Also there is a lot of recent work about the question of which Lagrangian submanifolds are displaceable. eg Polterovich et al have just shown that the Clifford torus in $\mathbb{C}P^n$ is NOT displaceable. If $\mathbb{C}P^n$ is identified with a compactification of the unit ball in $\mathbb{C}^n$ then the Clifford torus is $\{z = (z_i) : |z_i| = \frac{1}{\sqrt{n+1}}$, the inverse image under the moment map of the barycenter of the n-simplex.

4. Dehn twists

Let $T^*S^2 = \{(u, v) \,|\, u \cot v = 0, \|v\| = 1\}$ with the s. form $\omega = du \wedge dv$. The function $h(u, v) := \|u\|$ induces an S^1 action on $T^*(S^2) \setminus S^2$ given by

$$\sigma_t(u, v) = R^t_{u\times v}(u, v) = \big(\cos tu - \sin t\|u\|v, \cos tv + \sin t\frac{u}{\|u\|}\big).$$

Here R^t_x denotes the rotation by angle t about the axis $x/\|x\|$ and is defined as long as $x \neq 0$. To see this, notice that this action is generated by the vector field $X := \dot\sigma|_{t=0}$ where

$$X = -\|u\|v \cdot \partial_u + \frac{u}{\|u\|} \cdot \partial_v.$$

(Here I am thinking of ∂_u as a vector; thus the components of the tangent vector $u \cdot \partial_u$ are (u_1, u_2, u_3).) Then X is ω-dual in $\mathbb{R}^6$ to the 1-form

$$-\iota(X)\omega = +\|u\|v \cdot dv + \frac{u}{\|u\|} \cdot du.$$

But $v \cdot dv = 0$ on T^*S^2. Hence when restricted to $T^*(S^2)$ this equals $dh = \frac{u}{\|u\|} \cdot u$.

Exercise 5. (i) Consider the S^1 action

$$\sigma^s_t := R^t_{sv+(1-s)u\times v}.$$

When $s \neq 0$ this is well defined on the whole of $T^*(S^2)$. Calculate its generating vector field X^s.

(ii) Let $\omega^s := \omega + s\beta$ where β is the pullback of the 2-form on the base S^2 given by $\beta_v(X, Y) = v \cdot X \times Y$ for $X, Y \in T_v(S^2)$. Thus β is the pullback of the 2-form defined on $\mathbb{R}^3 \setminus \{0\}$ given by $\beta_x(X, Y) = \frac{x}{\|x\|} \cdot X \times Y$. Define the function h^s by

$$h^s(u, v) = \sqrt{s^2 + (1-s)^2\|u\|^2}.$$

Show that σ^s is the Hamiltonian flow of h^s with respect to ω^s.

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2. The cotangent bundle of S^2

Let $T^*S^2 = \{(u,v) \,|\, u \cot v = 0, \|v\| = 1\}$ with the s. form $\omega = du \wedge dv$. The function $h(u,v) := \|u\|$ induces an S^1 action on $T^*(S^2) \setminus S^2$ given by

$$\sigma_t(u,v) = \big(R^t_{u\times v}(u), R^t_{-u\times v}(v)\big) = \big(\cos t\, u - \sin t\, \|u\| v, \cos t\, v + \sin t\, \frac{u}{\|u\|}\big),$$

where R^t_x denotes the rotation by angle t about the axis $x/\|x\|$ and so is defined as long as $x \neq 0$. To see this, notice that this action is generated by the vector field $X := \dot\sigma|_{t=0}$ where

$$X = -\|u\| v \cdot \partial_u + \frac{u}{\|u\|} \cdot \partial_v.$$

(Here I am thinking of ∂_u as a vector; thus the components of the tangent vector $u \cdot \partial_u$ are (u_1, u_2, u_3).) Then X is ω-dual in $\mathbb{R}^6$ to the 1-form

$$-\iota(X)\omega = +\|u\| v \cdot dv + \frac{u}{\|u\|} \cdot du.$$

But $v \cdot dv = 0$ on T^*S^2. Hence when restricted to $T^*(S^2)$ this equals $dh = \frac{u}{\|u\|} \cdot u$.

This flow $(u,v) \mapsto \sigma_t(u,v)$ is the geodesic flow.

Exercise 2.1. (i) Consider the S^1 action

$$\sigma^s_t := R^t_{sv+u\times v}.$$

When $s \neq 0$ this is well defined on the whole of $T^*(S^2)$. Show that its generating vector field is

$$X^s = \frac{1}{h}\Big((su \times v - \|u\|^2 v) \cdot \partial_u + u \cdot \partial_v\Big),$$

where $h =: h^s =: \|sv + u \times v\| = \sqrt{s^2 + \|u\|^2}$.

(ii) Let $\omega^s := \omega + s\pi^*\beta$ where $\pi : T^*S^2 \to S^2$ is the obvious projection $(u,v) \mapsto v$ and β is the area form on the base S^2 given by $\beta_v(X,Y) = v \cdot X \times Y$ for $X, Y \in T_v(S^2)$. (Thus β is the restriction to S^2 of the closed form $x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$ on $\mathbb{R}^3$.) Show that σ^s is the Hamiltonian flow of h^s with respect to ω^s. **Hint:** I found this easiest to do using vector notation. eg write $\omega_s = du \cdot dv + \frac{s}{2} v \cdot (dv \times dv)$. Here, dv denotes the 3-vector $(dx_1, dx_2, dx_3) \in \mathbb{R}^3$ (a 1-form with values in $\mathbb{R}^3$. So $dv \times dv$ is a 2-form with values in $\mathbb{R}^3$. Since $\times$ is a combination of the cross product on vectors and the wedge product on forms, $dx_1 \wedge dx_2 = dx_2 \times dx_1$ has a single component in the 3rd direction. This notation works as expected. eg $v \cdot (u \times dv) = u \cdot (dv \times v)$. Note that $(u \times v) \cdot (u \times dv) = 0$ on T^*S^2. If you are confused by this notation, you might prefer to use that in Ex 9.7.5 in JHOL, which is equivalent, but written a little differently.

Note that the orbits of the geodesic flow σ_t lie in the fibers of the projection

$$\rho : T^*(S^2) \setminus S^2 \to S^2, \quad (u,v) \mapsto \frac{u}{\|u\|} \times v.$$

Since these orbits form the leaves of the characteristic foliation on the level set $\|u\| = const$, ω is nondegenerate on the fibers of ρ.

Exercise 2.2. Show that $\rho^*(-\beta) = \omega$ on the level set $\|u\| = 1$ in T^*S^2 (where β is the standard area form on S^2. **Hint:** Because $\rho^*(\beta)$ vanishes on the orbits of the geodesic flow, which are the null directions of ω on $\|u\| = 1$, it must equal ω up to a positive or negative constant. So you just need to check this at one point.

Let $pr : L \to S^2$ be a symplectic line bundle with Chern number -2 with connection 1-form α. Thus $d\alpha = -pr^*(\tau)$, where the curvature 2-form τ satisfies $\int_{S^2} \tau = -2$ the Chern number of L. Consider the form

$$\omega_L := d(r^2\alpha) = 2rdr \wedge \alpha + r^2 d\alpha \tag{1}$$

where $r := \|u\|$ is the radial distance from the zero section.

Exercise 2.3. (i) Show that ω_L is nondegenerate away from the zero section L_0 of L.
(ii) Show that there is a symplectomorphism Ψ that fits into the commutative diagram:

$$\begin{array}{ccc} (T^*S^2 \setminus S^2, \omega) & \stackrel{\Psi}{\to} & (L \setminus L_0, \omega_L) \\ \rho \downarrow & & pr \downarrow \\ S^2 & = & S^2. \end{array}$$

I am not sure if there is a nice explicit formula for $\Psi(u, v)$; in any case, to define this you would have to construct an explicit model for L. It might be better to argue more abstractly, defining Ψ on the unit sphere bundles and then extending using the obvious Liouville vector fields. Note that Ψ does not extend continuously over the zero section S^2 of T^*S^2.

Now consider the λ-disc bundle $(T^*_{\le\lambda}S^2, \omega) = \{(u, v) \in T^2S^2 : \|u\| \le 1\}$. Construct a symplectic manifold $(X, \omega_{\lambda,X})$ from the compact manifold $(T^*_{\le\lambda}S^2, \omega)$ by identifying each null orbit of ω on the boundary $T^*_{=\lambda}S^2$ to a single point and smoothing in the r direction. (Here it is good to think of ω as given near the boundary by formula (1), which is permissible by Exercise 2.3.) This is the symplectic cutting procedure of Lerman and the details are in his paper of that name.

Lemma 2.4. *$(X, \omega_{\lambda,X})$ is symplectomorphic to $(S^2 \times S^2, \beta \oplus \beta)$ when $\lambda = \sqrt{2}$.*

This follows from the classification of ruled surfaces: (X, ω_X) is diffeomorphic to $S^2 \times S^2$ and contains a symplectically embedded 2-sphere with self-intersection 2 (namely the image of the boundary $T^*_{=\lambda}S^2$) as well as a Lagrangian sphere S. Since $\int_{S^2} \beta = 4\pi$, we need $\omega_{\lambda,X}$ to integrate to 8π on C and so need to take $\lambda = \sqrt{2}$. There is only one symplectic manifold of this kind.

One should be able to construct an explicit symplectomorphism $(X, \omega_X) \to (S^2 \times S^2, \beta \oplus \beta)$ that takes C to the diagonal and the Lagrangian S to the antidiagonal. A very similar problem appears in the proof of Prop 9.7.2 (ii) in JHOL (cf p 340) except that here it concerns the Hirzebruch surface corresponding to the form $\omega + s\pi^*\beta$. It seems to me that the same calculations should yield a fairly direct proof of Lemma **??** (ie. no need for J-holomorphic curves, just the Moser argument) but I haven't checked the details.

Another possible way of getting explicit formula would be to look at the toric picture. Namely, consider the map

$$\Phi : (X, \omega_X) \to \mathbb{R}^2, \quad \Phi(u, v) = (e_1 \cdot (u \times v), \|u\|).$$

Its image is the triangle T with vertices $(0,0),(-1,1),(1,1)$. Also, it is the moment map for a T^2 action on $\Phi^{-1}(T\setminus\{(0,0\}$. In fact, T corresponds to an orbifold, that is smooth except for the one singular point $T^{-1}((0,0)$. If P_ε is the parallelogram obtained by cutting off the vertex $(0,0)$ of T by the cut $y=\varepsilon$ (where (x,y) are the coordinates in $\mathbb{R}^2$) then the corresponding toric manifold M_ε is a Hirzebruch surface which is known to be symplectomorphic to $S^2\times S^2$ with an appropriate symplectic form ω_ε. This is the manifold obtained from $(T^*_{\le 1}S^2\setminus T^*_{\le\varepsilon}S^2,\omega)$ by compactifying both its ends. It is easy enough to relate the limit of M_ε as $\varepsilon\to 0$ with X. But to relate it to $S^2\times S^2$ you still need an explicit symplectomorphism $M_\varepsilon\to(S^2\times S^2,\omega_\varepsilon)$, which is essentially the same problem as before.

As you see, none of these approaches are very useful, i.e. at best they would involve you in significant computations. I think that it might be best to use SO(3) actions: see Exercise 3.3

3. SO(3) actions

The Lie algebra $\mathfrak{so}(3)$ has generators $\partial_i, i=1,2,3$ where ∂_i denotes an infinitesimal rotation about the ith axis. Thus

$$\partial_1 := \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\quad \partial_2 := \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\quad \partial_3 := \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The Lie bracket is $[\partial_1,\partial_2]=\partial_3$ and cyclic permutations of this. Using these as basis, we can identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ and then identify the dual Lie algebra $\mathfrak{so}^*(3)$ with $\mathfrak{so}(3)=\mathbb{R}^3$ via the standard inner product $\langle\cdot,\cdot\rangle$.

Note that any dual Lie algebra $\mathfrak{g}^*$ is a **Poisson manifold**, ie functions on $\mathfrak{g}^*$ have a Poisson bracket that satisfies the Jacobi identity. It is defined as follows, if $f,g:\mathfrak{g}^*\to\mathbb{R}$ then the value of the Poisson bracket at $x\in\mathfrak{g}^*$ is:

$$\{f,g\}(x) := \langle [df(x),dg(x)],x\rangle.$$

To understand this, note that $df(x),dg(x)\in T^*_x\mathfrak{g}^*\equiv\mathfrak{g}$, so that the Lie bracket $[df(x),dg(x)]$ makes sense and can be paired with $x\in\mathfrak{g}^*$.

To say that SO(3) acts in a Hamiltonian way on (M,ω) means that there is a moment map $\Phi: M\to\mathfrak{so}^*(3)$ that is a Poisson map. Thus for any $\xi\in\mathfrak{g}$ the function

$$H_\xi: M\to\mathbb{R},\quad p\mapsto\langle\xi,\Phi(p)\rangle,$$

has Hamiltonian vector field X_ξ, the tangent vector to the flow given by the action of $\exp(t\xi)\in SO(3)$ on M. (This is the usual property of a moment map.) But we also require that if $f,g:\mathfrak{g}^*\to\mathbb{R}$ are any two functions, then

$$\{f\circ\Phi, g\circ\Phi\}_M = \Phi\circ\{f,g\}_{\mathfrak{g}^*}.$$

This is the Poisson property.
NOTE: for consistency with Seidel's signs, we require that $\iota(X_H)\omega=-dH$. Also, if M is noncompact we require that Φ be proper, i.e. the inverse image of compact sets is compact.

The relation between the symplectic form and the Poisson bracket on any symplectic manifold is

$$\omega(X_F,X_G)(p)=\{F,G\}(p).$$

Hence if $F = F_\xi$ (defined by $F_\xi(p) = \Phi \circ \langle \xi, \Phi(p)\rangle$ for $\xi \in \mathfrak{g}$) and $G = F_\eta$, then

$$\omega(X_\xi, X_\eta)(p) = \langle [\xi, \eta], \Phi(p)\rangle.$$

In other words, the symplectic form along the orbits of $G = \mathrm{SO}(3)$ is determined by the image of this orbit under the moment map. Since this is a coadjoint orbit of SO(3) the image is either the single point $\{0\}$ (in which case the orbit is isotropic) or is a 2-sphere with nontrivial area form (in which case the orbit is a quotient of SO(3) by a finite subgroup with symplectic form pulled back from S^2.

NOTE: this is consistent with the situation for Hamiltonian actions of the torus T^k. In this case the Lie algebra is abelian, which means that all Poisson brackets vanish and the orbits are isotropic.

The next exercise shows that if SO(3) acts on M there is an additional S^1 action on most of M. This means that the symplectic form on most of M is entirely determined by the moment image. Presumably, if M is closed ω is determined everywhere. (There is a paper by Iglesias that classifies 4-dimensional manifolds with SO(3) action.) cf. Exercise 3.3.

Exercise 3.1. Check that the function $r = \sqrt{\sum x_i^2}$ Poisson commutes with the coordinate functions on $\mathfrak{so}^*(3) \equiv \mathbb{R}^3$. Deduce that if SO(3) acts on a 4-manifold M with (proper) moment map Φ then the function $|\Phi|$ induces an S^1-action on $\Phi^{-1}(\mathbb{R}^3 \setminus \{0\})$ that rotates the fibers of Φ. **Hint:** Recall that if two functions $F, G : M \to \mathbb{R}$ Poisson commute then they generate commuting flows.

Exercise 3.2. (i) Consider the standard action of SO(3) on the unit sphere S^2 in $\mathbb{R}^3$. Show that the moment map $\Phi : S^2 \to \mathbb{R}^3$ for this action is $-\iota$ where ι is the obvious inclusion. **Hint:** This is equivalent to saying that the Hamiltonian function that generates the rotation about the x_3-axis is the negative of the height function. Check this.

(ii) Check that the moment map for the diagonal action of SO(3) on $(S^2 \times S^2, \beta \oplus \beta)$ is $\mu : (x, y) \mapsto -x - y$. Points on the diagonal and antidiagonal are critical points. Show there no others. Note that the moment image is the ball centered at 0 and radius 2.

(iii) Show that the moment map for the obvious SO(3) action on (T^*S^2, ω) is $(u, v) \mapsto -u \times v$. **Hint:** By symmetry, you just need to check this for the rotation $(u, v) \mapsto (R^t_{-e_1} u, R_{e_1} v)$, which you can do by direct calculation.

(iv) Deduce from (iii) that the moment image for the SO(3) action on (X, ω_X) is the ball of radius 2. Again, what are the critical points?

Exercise 3.3. Consider the map

$$\psi : S^2 \times S^2 \setminus \{x \neq \pm y\} \to T^*_{<2} S^2, (x, y) \mapsto \big(\|x + y\| \frac{x \times y}{\|x \times y\|}, \frac{y - x}{\|y - x\|}\big).$$

(i) Show that this map is well defined, i.e. does have image $T^*_{<2} S^2$, and extends smoothly over the antidiagonal $x + y = 0$.

(ii) Show that it is SO(3)-equivariant.

(iii) Check that it is a symplectomorphism. Instead of doing this directly, it's easiest to use the fact that the symplectic form on (most of) these spaces is determined by the moment map.

(iv) Prove Lemma 2.4.

3.1. **The pentagon space.** Consider the diagonal action of SO(3) on $(S^2)^5$. It has moment map $\Phi((x_i)) = -\sum x_i$. Its critical points are those configurations (x_i) in which all the points x_i lie in the same direction. Therefore if all the spheres have the same radius there are no critical points in $\Phi^{-1}(\{0\})$. The pentagon space is the corresponding reduced space

$$P := (S^2)^5//\mathrm{SO}(3) := \Phi^{-1}(\{0\})/\mathrm{SO}(3).$$

Each point in P can be thought of as a closed configuration of 5 rods of length 1 in $\mathbb{R}^3$, modulo the obvious action of SO(3). Varying the sizes of the spheres (or lengths of the rods) gives different spaces.

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Hint for Exercise 3.1. Let X_r be the Hamiltonian vector field on $M\backslash\Phi^{-1}(\{0\})$ corresponding to the function $\|\Phi\|$ and let ϕ_t be its flow. We want to see ϕ_t is an S^1-action.

First note that the moment map is SO(3) equivariant, i.e. $\Phi(g(p)) = Ad^*(g)\cdot\Phi(p)$ for all $g \in G = \mathrm{SO}(3)$ and $p \in M$. Here Ad^* denotes the coadjoint action of G on $\mathfrak{g}^*$, which in this case is by $Ad^*(g)(x) = gx$ where g acts on x by the standard action. (This property is probably equivalent to saying that Φ is a Poisson map; I haven't checked.)

We first claim that ϕ_t preserves each of the fibers $\Phi^{-1}(x)$, $x \neq 0$, of Φ. To see this, it suffices by equivariance to show this for points of the form $x = (\lambda, 0, 0) \in \mathbb{R}^3 = \mathfrak{g}^*$. Such a point is fixed by the rotation $R_1 \in G$ about the first axis. The flow ϕ_t fixes the level set $\|\Phi\|^{-1}(\lambda)$ and commutes with the action of R_1. Since the only points in the sphere in $\mathbb{R}^3$ of radius λ that are fixed by R_1 are $(\pm\lambda, 0, 0)$ ϕ_t preserve the set $\Phi^{-1}(\pm\lambda, 0, 0)$, and hence must preserve each of the two components of this set.

Second, we need to check that ϕ_t is a circle action. Again it suffices to check this on sets of the form $\Phi^{-1}(x)$, $x = (\lambda, 0, 0)$. But the restriction of the function $\|\Phi\|$ to the x_1-axis is the first coordinate function x_1, i.e. the Hamiltonian function for the rotation R_1. So you just need to check that Φ and x_1 generate the same flow on $\Phi^{-1}((\lambda, 0, 0))$. Similarly, the restriction of ϕ_t to $\Phi^{-1}(x)$ is the rotation about the axis $x/\|x\|$. (The Killing vector fields that Seidel mentions are just the infinitesimal generators of these rotations.)

4. The pentagon space

Consider the diagonal action of SO(3) on $(S^2)^5$. It has moment map $\Phi((x_i)) = -\sum x_i$. Its critical points are those configurations (x_i) in which all the points x_i lie in the same direction. Therefore if all the spheres have the same radius there are no critical points in $\Phi^{-1}(0)$. The pentagon space is the corresponding reduced space

$$P := (S^2)^5//\mathrm{SO}(3) := \Phi^{-1}(0)/\mathrm{SO}(3).$$

Each point in $P = P_{11111}$ can be thought of as a closed configuration of 5 rods of length 1 in $\mathbb{R}^3$ put end to end in the obvious circular order, modulo the obvious action of SO(3). Varying the sizes of the spheres (or lengths of the rods) gives different spaces.

As we saw in class the elements $\mathbf{x} = (x_i)$ with $x_1 + x_2 = 0$ form a Lagrangian 2-sphere L_1, the image of the antidiagonal in the the first two factors. There are five of these Lagrangians L_i, (where L_5 is given by $x_5 + x_1 = 0$.)

The Dehn twists τ_i around these Lagrangians satisfy $\tau_i^2 = id$ in $\pi_0(\mathrm{Symp}(P))$, because there is a circle action γ_1 on $P \setminus L_i$, obtained by rotating the first two rods x_1, x_2 about the axis $x_1 + x_2$ (and keeping the other rods fixed.)

Exercise 4.1. Check that $L_i \cap L_j \neq \emptyset$ iff $i = j$ or i and j are adjacent in the circular order. Deduce that these spheres generate a subspace of dimension 4 in $H_2(M)$.

To see a nonLagrangian sphere C_1, consider configurations in P with $x_1 = x_2$. This is equivalent to having four spheres, but where the first one has radius 2, and the others radius 1 as before. The corresponding moment map

$$\Phi' : (S^2)^4 \to \mathbb{R}^3, \quad (y_i) \to -\sum y_i,$$

has no critical points with $\sum y_i = 0$. Hence we can form the reduced space $(S^2)^4//\mathrm{SO}(3)$ which now has dimension 2 and so must be a 2-sphere with positive symplectic area.

It follows that $H_2(P)$ must have rank at least 5. On the other hand, I claim that the Euler characteristic of P is 7. To see this, note that the points of C_1 are all fixed by the circle action γ_1 on $P \setminus L_1$. There is a similar sphere C_3 (where $x_3 = x_4$) which is pointwise fixed by the commuting circle action γ_3 on $P \setminus L_3$. Hence there is a T^2 action on $P \setminus (L_1 \cup L_3)$. This action has three fixed points: there is one point in $C_1 \cap C_3$ (given by an isosceles triangle) and two others, one with $x_1 = x_2 = -x_3$ and the other with $-x_2 = x_3 = x_4$. Hence the Euler characteristic of $P \setminus (L_1 \cup L_3)$ is 3. And so, adding two disjoint 2-spheres each of self-intersection -2 gives 7. (Check this via the Mayer-Vietoris sequence or via a simplicial decomposition – note that the Euler characteristic of $\mathbb{R}P^3$, the boundary of a nbhd of the Lagrangian spheres, is zero.)

Next, one checks that P is simply connected by analysing its decomposition into the simply connected pieces $P \setminus (L_1 \cup L_3)$ plus neighborhoods of the spheres L_1, L_3. It follows that $H_2(P)$ has rank 5. It is spanned by the spheres $L_1, \dots, L_4$ and C_1 (though I haven't checked whether they form a $\mathbb{Z}$-basis).

Lemma 4.2. *P is symplectomorphic to $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P}^2$ with its monotone symplectic form.*

NOTE: Monotone means that $[\omega] = \lambda c_1$; so the line A has $\omega = 3\lambda$ and the exceptional divisors have $\omega = \lambda$ for some $\lambda > 0$. Thus up to scaling there is only one monotone symplectic form on a blow up of $\mathbb{C}P^2$. And you cannot blow up more than 8 points: since any 9 points lie on an elliptic curve (ie torus of degree 3), the class $3A - \sum_{i+1}^{9} E_i$ has a J-hol representative and so must have positive area. Note that $(S^2)^5$ is monotone if all the spheres have radius 1. Also $c_1(\mathit{sphere})$ is always even.

Proof. It is hard to find a completely elementary proof, and I won't try. (But Exercise 4.5 does just use the properties of circle actions.) First of all let us show that P is monotone.

Consider the fibration

$$\mathrm{SO}(3) \to \Phi^{-1}(0) \xrightarrow{\pi} P.$$

Since $G = SO(3)$ acts freely on $Y := \Phi^{-1}(0)$ the tangent bundle PY splits as the sum $TG \oplus \pi^*(TP)$. Notice that TG is trivial, with trivialization given by the vector fields $X_i, i = 1, 2, 3$ generating the rotations around the three axes. Thus $TG = G \times \mathfrak{g}$. Because we are reducing at 0 the orbits of G are isotropic. In fact Y is a coisotropic manifold with trivial normal bundle which can be identified with T^*G where the pairing is given by the symp. form ω on TW, where $W := (S^2)^5$. It follows that the restriction of the full tangent bundle TW to Y can be identified (as a *symplectic* bundle) with the sum $TG \oplus T^*G \oplus \pi^*(TP)$. (Note that any bundle of the form $E \oplus E^*$ has a natural symplectic structure.) Moreover $TG \oplus T^*G \to Y$ is trivial as a symplectic bundle since $TG \to Y$ is a trivial real bundle. Hence the pullback of $c_1(TP)$ by π equals the restriction of $c_1(TW)$ to Y. Since a similar statement is true for the symplectic forms, the identity $c_1(TW)(A) = \lambda\omega_W(A)$ for $A \in \pi_2(Y)$ implies that $c_1(TP)(\overline{A}) = \lambda\omega_P(\overline{A})$, where $\overline{A} = \pi_*(A)$. But the image of $\pi_2(Y)$ under the projection π has finite index in $\pi_2(P)$. Therefore this identity must hold for all classes in $\pi_2(P) = H_2(P)$ (recall P is simply connected.)

I'm going to use the classification theorem for rational sympl 4-manifolds which says that any simply connected manifold that contains a symplectically embedded 2-sphere in classes

A such that $A^2 \geq 0$ must be $S^2 \times S^2$ or a blow up of $\mathbb{C}P^2$. P has two symplectically embedded spheres namely C_1 and C_3 that intersect once transversally and positively. It follows that the class $A = [C_1] + [C_3]$ has a symplectically embedded representative. (You just need to resolve the singularity of the nodal curve, keeping it symplectic, not holomorphic. That's easy to do.) By monotonicity $c_1(C_i) > 0$, $i = 1, 3$. Hence $c_1(A) \geq 2$, which means that $A^2 \geq 0$. (For a symplectically embedded sphere S, the bundle $TP|_S$ splits as the sum of the complex line bundles $TS \oplus \nu_S$. Thus $c_1(TP)(S) = c_1(TS)(S) + c_1(\nu_S)(S) = 2 + S \cdot S$.) Then since $H_2(P)$ has rank 5 the manifold must the 4-fold blow up of $\mathbb{C}P^2$. □

Note that this means there are spheres in P with $c_1(P) = 1$, namely the exceptional divisors. These spheres cannot lift as spheres to Y since c_1 is even on Y. Hence the map $\pi_2(P) \to \pi_1(\mathrm{SO}(3))$ is nonzero. You can check that the spheres C_1 and C_3 lift to embedded discs in Y, but not spheres.

Exercise 4.3. Let $X = \mathbb{C}P^2 \# 4\overline{\mathbb{C}P}^2$. Denote the class of the line by A and the class of the exceptional divisors by E_i. What classes do the Lagrangians L_i lie in? Note that the classes $E_i - E_j$ and $A - E_1 - E_2 - E_3$ can all be represented by Lagrangian spheres. Can you pick out 5 of these with the right intersection patterns? (I haven't done this, but I can't see what other classes the L_i could be in, so I think this must be possible.) **Hint** It might help to use the fact that the monotone manifolds $S^2 \times S^2 \# \overline{\mathbb{C}P}^2$ (which does have a Lagrangian sphere) and $X = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ are symplectomorphic. If B, C are the two spheres in $S^2 \times S^2$ and the excpetional divisor is F then $B + C - F$, the 1-point blow up of the diagonal corresponds to the line A and the new exceptional divisors are $E_1 = B - F, E_2 = C - F$. Thus the antidiag in class $B - C$ corresponds to a Lagrangian in class $E_1 - E_2$.

Exercise 4.4. Try to find spheres representing A and the four E_i in Y. I think the spheres C_1 and C_3 are exceptional spheres. But how do you find four disjoint spheres?? You might be able to see them using Exercise 4.5.

You can understand how the topology of these spaces changes as the size of the spheres varies by the following trick. When one reduces a symplectic manifold by a G action, one always must reduce at a coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$, i.e. take $\Phi^{-1}(\mathcal{O})/G$. Therefore, if $G = \mathrm{SO}(3)$ and we don't reduce at 0, we must reduce at a sphere S_r of some radius $r < 5$. The corresponding reduced space $X_r := \Phi^{-1}(S_r)/\mathrm{SO}(3)$ is made of configurations of *six* rods, 5 of length 1 and one of length r. (So we might write $X_r = P_{r11111}$.) Note also that S_r is the level set of the function $\|\Phi\|$, which generates an S^1 action. It follows from Ex 3.1 that X_r is also the reduced space of $\Phi^{-1}(x)$ by this S^1 action where $\|x\| = r$. But it is known what happens to reduced spaces when you pass critical levels of an S^1 action (in general, a combination of blow up and blow down operations, but in low dimensions there's no room for more than blowing up/down at a point.)

Exercise 4.5. You can understand P_{11111} by thinking of it as lying in the family P_{r1111} gotten by suitably reducing $(S^2)^4$, where all spheres have size 1. Let $\Phi : (S^2)^4 \to \mathbb{R}^3$ be the moment map. As above P_{r1111} is the reduction of the 6 manifold $\Phi^{-1}(r, 0, 0)$ by the rotation in the first coordinate direction. When $r = 4$ the reduced space is a point. This means $P_{r1111} = \mathbb{C}P^2$ for r near 4. Show that $r = 2$ is the only critical value and that there are 4 critical points on this level. Can you see that when you pass this level each of these points blows up?

The pentagon space can be identified with the space of stable genus zero curves with 5 marked points: see JHOL p 597/8.