The Atiyah-Singer Index Theorem — MAT644

Blaine Lawson

Place: P127, Physics Building

Time: 1:00 - 2:20 PM, Tuesdays and Thursdays

Office: 5-109, Math Tower

Office Hours: Tuesday and Thursday, 2:30-4:00 PM or by appointment either in my office or by zoom.

Syllabus

1. HISTORY: Riemann-Roch, and Riemann-Roch-Hirzebruch

This will be discussion where undoubtedly most students will not understand everything. However, this course is about one of the very deep theorems in mathematics with enormous implications. It has evolved through a series of wonderful results, and I would like to start by reviewing them. Take this part of the course as culture. Things you do not understand will be clear by the end of the course.

2. Vector Bundles and Classifying Spaces

3. Characteristic Classes

(i) Multiplicative Sequences

(ii) The Chern Character

4. K-Theory

(i) General Theory

(ii) The Atiyah-Bott-Shapiro Construction (Basic for proving establishing the index in K-theory)

(iii) Bott periodicity

(iv) Clifford algebras and their representations – connections to Bott periodicity.

(iii) The Thom Isomorphism

5. Elliptic Differential Operators

(i) Differential Operators

(ii) The Principal Symbol

(iii) Ellipticity

(iv) The Symbol Class in $K_{\text{cpt}}(T^*X)$.

6. Main Analytical Results for Elliptic Operators on Compact manifolds

7. Index theorems

(i) Versions of the general index theorem

(ii) The Families Index Theorem

(iii) The G Index Theorem

(iv) The Cl_n Index Theorem

8. Fundamental Operators in Geometry

(i) Connections; Curvature, Torsion, the Riemannian connection, connections on principal bundles

(ii) The Clifford bundle of a riemannian manfold

(iii) The Dirac Construction

(iv) Spin Manifolds and Spin Cobordism

(v) The Atiyah-Siinger Operator on Spin manifolds

(vi) The Atiyah-Siinger Cl_n Operator on Spin manifolds

(vii) The Lichnerowicz-Singer Theorem

(viii) Fundamental Operators

Euler Characteristic

Signature (with a coefficient bundle)

Dolbeault (with a coefficient bundle)

Dirac (with a coefficient bundle)

(ix) Index Theorems in all these cases

9. Applications.

10. Pseudo-Differential Operators and Analytic Results

11. Proof of the Index Theorem

REFERENCES

1. J. Minor and J. Stasheff, *Characteristic Classes*, Princeton Univ. Press

2. M. Aniyah, *K-Theory*, Benjamin Press.

3. R. Bott, *Lectures on K(X)*, Benjamin Press

4. H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, :Princeton Press

5. N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag

6. P. Gilkey, Invariance Theory, *The Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish Press

7. D. Bleeker, B. Boos-Bavnbek, *Index Theory with Applications to Mathematics and Physics*,

8. F. R. Harvey, *Spinors and Calibrations*, Academic Press

9. M. Taylor, *Pseudodifferential Operators*, Princeton Press

10. W. Fulton and S. Lang, *Riemann-Roch Algebra*, Springer-Verlag

11. J. Schwarz, *Differential Geometry and Topology*, Gordon and Breach.

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