MAT 545 --- Complex Geometry

Time:Monday/Wednesday 10:00am-11:20am, Fall 2013
Location: Physics P125

About this course

This course will present an introduction to the theory of complex manifolds. It is aimed at students that are interested in complex geometry or algebraic geometry. Throughout the class, we will emphasize concrete examples and applications of the general theory.

In a general way, we will follow the approach of Griffiths-Harris and Huybrechts' introduction to complex geometry, but we will circulate notes for the class.

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Lecture Notes

- Notes

Syllabus

The tentative syllabus is as follows:

1. Holomorphic functions of several complex variables.
2. Definition and examples of complex manifolds.
3. Holomorphic line bundles and vector bundles. Embeddings into projective space.
4. Complex and Hermitian linear algebra. Differential forms on a complex manifold. Dolbeaut cohomology.
5. Kahler manifolds, Kahler identities.
6. The Hodge theorem.

7. Applications: the Kodaira vanishing and Kodaira embedding theorems.

Note: This class, together with a more algebraically oriented companion course that Sam Grushevsky will teach in the Spring, will serve as a basic introduction to algebraic geometry.

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NOTES ON COMPLEX GEOMETRY FOR MATH 545, FALL 2014

Version of December 3, 2014

Introduction

These are notes on complex geometry intended to accompany Math 545 at SBU in Fall, 2014. As explained in the course syllabus, the overall goal of the course is to present some of the basic material on complex manifolds that are needed to work in complex or algebraic geometry. We will try to emphasize concrete examples and calculations wherever possible.

The present notes generally speaking closely follow the presentations in the texts of Griffiths-Harris [2] and Huybrechts [3] – and in later parts the notes of Schnell [4] – which the reader can consult for more details. In particular, we borrow freely from these sources without explicit attribution.

1. Preliminaries – Local Theory

This section presents some preliminary facts concerning holomorphic functions of several complex variables, following closely the presentations of Griffiths-Harris [2] and especially Huybrechts [3].

Holomorphic Functions of Several Variables. We will be concerned with complex-valued functions defined on some open subset of $\mathbf{C}^n$. In the absence of any distinguished directions in $\mathbf{C}^n$, we generally write

$$z \ = \ (z_1, \ldots, z_n)$$

for the standard linear coordinate functions, with real and imaginary parts x_i and y_i:

$$z_i \ = \ x_i + \sqrt{-1} \cdot y_i.$$

However sometimes it is useful to think of $\mathbf{C}^n$ as a product $\mathbf{C}^n = \mathbf{C}^k \times \mathbf{C}^\ell$, with coordinates

$$(z, w) \ = \ (z_1, \ldots, z_k, w_1, \ldots, w_\ell).$$

As we will be concerned with functions that admit local expressions as Taylor series, it is convenient to use multi-indices $J = (j_1, \ldots, j_n)$ in order to introduce the shorthand

$$z^J \ =_{\text{def}} z_1^{j_1} \cdot \ldots \cdot z_n^{j_n}.$$

We put $|J| = \sum j_i$, so that for example the polynomial $\sum_{|J|=m} z^J$ consists of the sum of all momomials of degree m in $z_1, \ldots, z_n$. Finally, given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i > 0$, and $c = (c_1, \ldots, c_n) \in \mathbf{C}^n$, we denote by

$$\Delta(\varepsilon; c) \ =_{\text{def}} \left\{ z \in \mathbf{C}^n \mid |z_i - c_i| < \varepsilon_i \right\}$$

the polydisk of (mult-)radius ε centered at c. Write $\mathrm{T}(\varepsilon; c)$ for the torus $\{z \in \mathbf{C}^n \mid |z_i - c| = \varepsilon_i\}$.

As in the case of functions of one variable, there are various equivalent ways to characterize holomorphic functions of several variables. For an open subset $U \subseteq \mathbf{C}^n$, write $\mathcal{C}^\infty(U)$ for the collection of all $\mathcal{C}^\infty$ complex-valued functions on U.

Theorem 1.1. *Let $U \subset \mathbf{C}^n$ be an open set, and let $f \in \mathcal{C}^\infty(U)$ be a smooth $\mathbf{C}$-valued function on U. Then the following three conditions are equivalent:*

(i). *f satisfies the Cauchy-Riemann equations on U, i.e. writing*

$$f(z) \ = \ f(x, y) = u(x, y) + \sqrt{-1} \cdot v(x, y),$$

one has:

$$\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial y_i} = 0 \quad , \quad \frac{\partial u}{\partial y_i} + \frac{\partial v}{\partial x_i} = 0 \quad \textit{for} \ \ i = 1, \ldots, n.$$

(ii). *f is holomorphic separately in each of the variables z_i, i.e. for each $c = (c_1, \ldots, c_n) \in U$, the function*

$$f(c_1, \ldots, c_{i-1}, z_i, c_{i+1}, \ldots, c_n)$$

is analytic in z_i in a neighborhood of c_i.

(iii). *Given any point $c = (c_1, \ldots, c_n)$ in U, there is an open neighborhood $V = V(c)$ of c in U in which $f(z)$ can be represented by a convergent power series:*

$$f(z) \ = \ \sum_{|J|=0}^{\infty} a_J \cdot (z - c)^J.$$

Definition 1.2. (Holomorphic functions). One says that f is *holomorphic* on U if the conditions of Theorem 1.1 are satisfied. We write $\mathcal{O}(U)$ for the ring of all such functions.

Sketch of Proof of Theorem 1.1. The equivalences (i) $\Longleftrightarrow$ (ii) and (iii) $\Longrightarrow$ (i) follow from the one-variable theory. For the implication (ii) $\Longrightarrow$ (iii), one chooses $0 < \varepsilon \ll 1$ and uses the Cauchy integral formula in each variable separately to write

$$\begin{aligned} f(z_1, \ldots, z_n) \ &= \ \frac{1}{(2\pi\sqrt{-1})^n} \cdot \int_{|\xi_1 - c_1| = \varepsilon} \cdots \int_{|\xi_n - c_n| = \varepsilon} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \cdot \ldots \cdot (\xi_n - z_n)} \cdot d\xi_1 \cdots d\xi_n \\ &= \ \frac{1}{(2\pi\sqrt{-1})^n} \cdot \int_{\mathrm{T}(\varepsilon; c)} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \cdot \ldots \cdot (\xi_n - z_n)} \cdot d\xi_1 \cdots d\xi_n. \end{aligned}$$

Now use the identity

$$\frac{1}{(\xi_i - z_i)} \;=\; \frac{1}{(\xi_i - c_i)} \cdot \sum_{j_i=0}^{\infty} \frac{(z_i - c_i)^{j_i}}{(\xi_i - c_i)^{j_i}}$$

to replace the integrand by a power series, and then integrate term by term. □

By reduction to the one-variable case, it is elemenatry that the maximum modulus principle holds in higher dimensions, as well as the fact that a holomorphic function is determined by its values on any small open set (Exercise 1.1). However unlike the situation in one dimension, when $n \geq 2$ domains of holomorphicity are quite special. For example, one has a famous result of Hartogs:

Theorem 1.3. (Hartogs' Theorem) *Let $\Delta(\varepsilon) = \Delta(\varepsilon; 0)$ be a polydisk centered at the origin of $\mathbf{C}^n$, and let*

$$\Delta(\varepsilon') \;\subsetneq\; \Delta(\varepsilon)$$

be a smaller polydisk, i.e. a polydisk centered at 0 with $\varepsilon'_i < \varepsilon_i$ for every i. Assume that f is a holomorphic function on

$$U \;=\; \Delta(\varepsilon) - \overline{\Delta(\varepsilon')}.$$

If $n \geq 2$, then f extends to a holomorphic function on all of $\Delta(\varepsilon)$.

In particular, if f is holomorphic away from a point in a domain of dimension ≥ 2, then it extends holomporphically to the whole domain. Of course when $n = 1$ this as well as the statement of Hartogs' Theorem fails.

Sketch of Proof of Hartogs' Theorem. We assume for simplicity of exposition that

$$\varepsilon = (1, \ldots, 1) \quad , \quad \varepsilon' = (\tfrac{1}{3}, \ldots, \tfrac{1}{3}).$$

Note that then U contains the union of the two "shells"

$$\left\{ \tfrac{1}{2} < |z_1| < 1 \right\} \quad , \quad \left\{ \tfrac{1}{2} < |z_2| < 1 \right\}.$$

Now for fixed $(w_2, \ldots, w_n)$ with $|w_i| < 1$, the function

$$\phi_w(z_1) \;=\; f(z_1, w)$$

is analytic in the annulus $\frac{1}{2} < |z_1| < 1$, and hence is given by a Laurent series

$$f(z_1, w) \;=\; \sum_{m=-\infty}^{\infty} a_m(w) \cdot z_1^m. \tag{*}$$

Using the expression of the $a_m(w)$ as Cauchy-type integrals, one sees that the $a_m(w)$ are holomprphic in w. On the other hand, if $|w_2| > \frac{1}{2}$ then $\phi_w(z_1)$ is analytic on the whole unit disk $|z_1| < 1$, and therefore $a_m(w) = 0$ for $m < 0$ when $\frac{1}{2} < |w_2| < 1$. It follows from the Identity principle (Exercise 1.1 (ii)) implies that $a_m(w) = 0$ when $m < 0$ for every w with $|w_i| < 1$. But then (*) expresses f as a holomorphic function on all of $\Delta(\varepsilon)$. We refer to [3, Proposition 1.1.4] for details. □

Concerning the Cauchy-Riemann equations, it is convenient to introduce the differential operators

$$\frac{\partial}{\partial z_i} \ = \ \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \ \ , \ \ \frac{\partial}{\partial \bar{z}_i} \ = \ \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right). \tag{1.1}$$

(The conceptual meaning of these operators will emerge later.) Then one has:

Proposition 1.4. *A $\mathcal{C}^\infty$ function f on U satisfies the Cauchy-Riemann equations – and hence is holomorphic on U – if and only if*

$$\frac{\partial f}{\partial \bar{z}_i} \ = \ 0 \ \ for \ \ i = 1, \ldots, n. \qquad \square$$

.

We conclude this subsection with a Lemma that will be useful on several occasions.

Lemma 1.5. *Working with variables $(z, w_1, \ldots, w_{n-1})$ on $\mathbf{C}^n$, fix $r > \varepsilon > 0$ and $\varepsilon_i > 0$. Let $g(z, w_1, \ldots, w_{n-1})$ be a function which is analytic on the cylindrical shell*

$$\mathrm{S} \ = \ \{ r - \varepsilon < |z| < r + \varepsilon \} \ \times \ \{ |w_i| < \varepsilon_i \}.$$

Then the function

$$f(w_1, \ldots, w_{n-1}) \ =_{def} \ \int_{\{|z|=r\}} g(\xi, w_1, \ldots, w_{n-1}) \cdot d\xi$$

is holomorphic on the polydisk $\{ |w_i| \leq \varepsilon_i \}$.

The reader is asked to prove this as Exercise 1.5. (Or see [3, Lemma 1.1.3].)

Weierstrass Preparation Theorem. An elementary fact from the one-variable theory is that the zeroes of an analytic function consist of isolated points. In particular, one has a complete picture of the (quite trivial) local geometry of these zeroes.

In higher dimensions the situation is of course vastly more complicated and interesting: the zeroes of an analytic function form (by definition) a hypersurface in its domain of definition. The most one can say – which however is already something – is that such a hypersurface is described locally by the vanishing of a monic polynomial in one variable whose coeffients are holomorphic functions of the remaining ones. This is the content of the Weierstrass preparation theorem.

Turning to details, let f be a holomorphic function defined in a polydisk centered at the origin $0 \in \mathbf{C}^n$, with $f(0) = 0$. We single out the first variable z_1, and write

$$g_w(z_1) \ = \ f(z_1, w) \ = \ f(z_1, w_1, \ldots, w_{n-1}),$$

which we view as an analytic function of z_1 depending holomorphically on w. We assume that

$$g_0(z_1) \not\equiv 0 \text{ in a neighborhood of } 0. \tag{1.2}$$

The idea is to study the zero-locus $\mathrm{Zeroes}(f) = \{ f^{-1}(0) \}$ by projecting to $\mathbf{C}^{n-1}$.

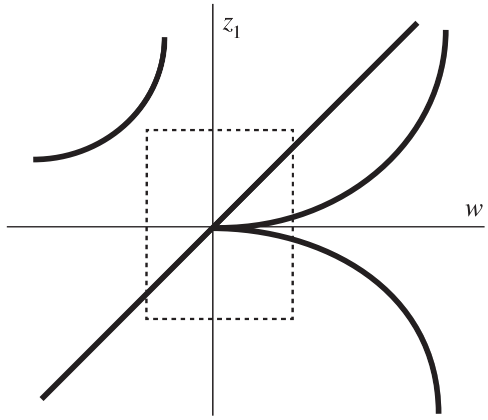

FIGURE 1. Zeroes of a function with Weierstrass polynomial $p = (z_1 - w)(z_1^2 - w^3)$.

Theorem 1.6. (Weierstrass Preparation Theorem) *In a suitable polydisk Δ centered at 0 one can write f uniquely as a product*

$$f \;=\; p \cdot h, \tag{1.3}$$

where h is a holomorphic function that does not vanish at any point of Δ, and p is a is Weierstrass polynomial *in z_1, i.e. a function of the form*

$$p \;=\; z_1^d + a_1(w) \cdot z^{d-1} + \ldots + a_{d-1}(w) \cdot z_1 + a_d(w),$$

where the $a_i(w)$ are holomorphic functions vanishing at the origin.

In other words, the zeroes of f in Δ are given by those of its Weierstrass polynomial. See Figure 1: the zeroes of the Weierstrass polynomial are shown in the indicated polydisk around the origin.

Proof. We start with some preliminary remarks. Fix $0 < \varepsilon \ll r$, and consider for the moment a function $g(z)$ in a single complex variable $z = z_1$ that is analytic in the disk $\{|z| < r + \varepsilon\}$. We assume that g has no zeroes on the circle $\{|z| = r\}$. Denote by $\alpha_1, \ldots, \alpha_d$ the zeroes of g in $\Delta = \{|z| < r\}$ (listed with multiplicity), and put

$$\sigma_k \;=\; \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} \xi^k \cdot \frac{g'(\xi)}{g(\xi)} \cdot d\xi.$$

A standard residue calculation shows that

$$\sigma_k \;=\; \sum_{j=1}^{d} \alpha_i^k.$$

On the other hand, consider the polynomial $p(z) = z^d + a_1 z^{d-1} + \ldots + a_{d-1} z + a_d$ defined by

$$p(z) \;=\; (z - \alpha_1) \cdot \ldots \cdot (z - \alpha_d).$$

By the theory of symmetric functions, the coefficients a_i of p are given by (universal) polynomials in the σ_k. On the other hand, we can write $g = p \cdot h$, where – thanks to the Riemann singularity theorem – h is an analytic function that is nowhere vanishing on Δ. The idea of the proof of Theorem 1.6 is to repeat this discussion with $g(z)$ replaced by $g_w(z_1)$, and to show that the resulting constructions vary holomorphically with w.

Specifically, choose $0 < \varepsilon \ll r \ll 1$ and $0 < \varepsilon_i \ll 1$ so that the cylindrical shall

$$\mathrm{S} \ = \ \{r - \varepsilon \leq |z_1| \leq r + \varepsilon\} \times \{|w_i| \leq \varepsilon_i\}$$

does not meet the zero locus of f, and so that 0 is the only zero of $g_0(z_1)$. Put

$$\sigma_k(w) \ = \ \frac{1}{2\pi\sqrt{-1}} \int_{|z_1|=r} \xi^k \cdot \frac{g_w'(\xi)}{g_w(\xi)} \cdot d\xi.$$

These are holomorphic in w (Exercise 1.5), and since for fixed w the integral $\sigma_0(w)$ counts the number of zeroes (with multiplicities) of $g_w(z_1)$ in the disk of radius r in the z_1-plane, it follows in particular that the number of such zeroes – say d – is independent of w. Denote these zeroes by

$$\alpha_1(w), \ldots, \alpha_d(w).$$

These are not single-valued functions of w, but as above the functions $a_i(w)$ defined by the equation

$$\begin{aligned} p_w(z_1) \ &= \ (z_1 - \alpha_1(w)) \cdot \ldots \cdot (z_1 - \alpha_d(w)) \\ &= \ z_1^d + a_1(w) \cdot z^{d-1} + \ldots + a_{d-1}(w) \cdot z_1 + a_d(w) \end{aligned}$$

are polynomials in the $\sigma_k(w)$, and hence holomorphic in w. Moreover by construction

$$\alpha_1(0) \ = \ \ldots \ = \ \alpha_d(0) \ = \ 0,$$

and hence the $a_i(w)$ vanish at the origin. This constructs the required Weierstrass polynomial for f. It remains to show that we can find a holomorphic function $h(z_1, w)$ on $\Delta = \Delta(\varepsilon; 0)$ so that (1.3) holds.

To this end, consider the function

$$h_0(z_1, w) \ =_{\mathrm{def}} \ \frac{g_w(z_1)}{p_w(z_1)}.$$

For fixed w, $h_0(z_1, w)$ is bounded on $\{|z_1| \leq r + \varepsilon\}$ for suitable $0 < \varepsilon \ll 1$, and hence extends to an analytic function on this disk thanks to the one-variable Riemann extension theorem. Moreover, for fixed w this extension is given inside $\{|z_1| < r\}$ by

$$h(z_1, w) \ = \ \frac{1}{2\pi\sqrt{-1}} \int_{|z_1|=r} \frac{h_0(\xi, w)}{(\xi - z_1)} \cdot d\xi.$$

Now the integrand on the right is analytic in a neighborhood of the cylinder S, and it follows that h is holomorphic in the w variables as well (Exercise 1.5 again). This produces the required function satisfying (1.3). □

The last step in the argument just completed also yields the multivariable analogue of the Riemann extension theorem.

Theorem 1.7. (Riemann Extension Theorem) *Let f be a holomorphic function on an open subset U of $\mathbf{C}^n$, and consider a holomorphic function*

$$h_0 : U - \mathrm{Zeroes}(f) \longrightarrow \mathbf{C}.$$

Assume that h_0 is locally bounded near Zeroes(f). *Then h_0 extends to a holomorphic function h on all of U.* □

We conclude with a variant of the preparation theorem, concerning divisibility by Weierstrass polynomials.

Theorem 1.8. (Weierstrass Division Theorem) *In the setting of Theorem 1.6, suppose that f is holomorphic in a neighborhood of the origin, and let $p = p(z_1, w)$ be a Weierstrass polynomial in z_1 of degree d. Then in a suitable polydisk Δ about the origin, one can uniquely write*

$$f \ = \ q \cdot p + s,$$

where q is holomorphic on Δ and $s = s(z_1, w)$ is a polynomial in z_1 of degree $< d$ whose coefficients are holomorphic functions of w.

Sketch of Proof. Keeping notation as in the proof of the Weierstrass preparation theorem, define

$$q(z_1, w) \ = \ \frac{1}{2\pi\sqrt{-1}} \cdot \int_{|z_1|=r} \frac{f(\xi, w)}{p(\xi, w)\cdot(\xi - z_1)} \cdot d\xi.$$

We assume that $p(z_1, w)$ has no zeroes on the shell

$$\mathrm{S} \ = \ \{r - \varepsilon \le |z_1| \le r + \varepsilon\} \times \{|w_i| \le \varepsilon_i\},$$

and then q is holomorphic in a neighborhood of 0. So the issue is to show that

$$s(z_1, w) \ =_{\text{def}} f(z_1, w) - q(z_1, w) \cdot p(z_1, w)$$

is a polynomial in z_1 of degree $< d$. To this end, write

$$\begin{aligned} s(z_1, w) \ &= \ \frac{1}{2\pi\sqrt{-1}} \cdot \int_{|z_1|=r} \frac{f(\xi, w)}{\xi - z_1} \cdot d\xi \ - \ \frac{1}{2\pi\sqrt{-1}} \cdot \int_{|z_1|=r} \frac{f(\xi, w)\cdot p(z_1, w)}{p(\xi, w)\cdot(\xi - z_1)} \cdot d\xi \\ &= \ \frac{1}{2\pi\sqrt{-1}} \cdot \int_{|z_1|=r} \frac{f(\xi, w)}{p(\xi, w)} \cdot \left(\frac{p(\xi, w) - p(z_1, w)}{(\xi - z_1)} \right) \cdot d\xi. \end{aligned}$$

But since p is a Weierstrass polynomial of degree d in z_1, the second term in the integral is a polynomial of degree $\le d - 1$ in z_1, as required. For the uniqueness statement, see [3, p. 15]. □

Algebraic Properties of the Ring of Holomorphic Germs. We now apply the Weierstrass theorems to study the algebraic properties of the ring of germs of holomorphic functions in a neighborhood of the origin $0 \in \mathbf{C}^n$.

For local questions, it is useful to identify two functions if they agree in a neighborhood of a fixed point, say $0 \in \mathbf{C}^n$: this leads to the notion of the *germ* of a holomorphic function. More formally, consider pairs (U, f) where U is a neighborhood of $0 \in \mathbf{C}^n$, and f is a holomorphic function on U. Define an equivalence relation $\sim$ on such pairs by declaring that $(U_1, f_1) \sim (U_2, f_2)$ if there is a neighborhood $V \subseteq U_1 \cap U_2$ such that

$$f_1|V \ \equiv \ f_2|V.$$

The resulting equivalence classes are called germs of holomorphic functions about 0, and form a ring

$$\mathcal{O}_n \ = \ \mathcal{O}_{\mathbf{C}^n,0}. \tag{1.4}$$

Via Taylor expansions, there is an isomorphism

$$\mathcal{O}_n \ \cong \ \mathbf{C}\{z_1, \ldots, z_n\}$$

of $\mathcal{O}_n$ with the ring of convergent power series in $z_1, \ldots, z_n$. This is a local integral domain: $\mathcal{O}_n$ has a unique maximal ideal, consisting of (germs of) functions vanishing at the origin. Of course one can discuss in a similar manner germs of holomorphic functions at any fixed point $a \in \mathbf{C}^n$, or germs of smooth (or continuous, or ...) functions at a point.

Remark 1.9. As we shall see later, the notation $\mathcal{O}_{\mathbf{C}^n,0}$ indicates the fact that $\mathcal{O}_n$ is the stalk at 0 of the sheaf $\mathcal{O}_{\mathbf{C}^n}$ of holomorphic functions on $\mathbf{C}^n$.

The ring $\mathcal{O}_n$ is very well-behaved algebraically. To begin with:

Proposition 1.10. *The ring $\mathcal{O}_n$ is a unique factorization domain.*

In other words, every non-unit $f \in \mathcal{O}_n$ can be written as a finite product of irreducible elements, and this expression is unique up to multiplication by units and the order of factors.

Sketch of Proof. Induction on n, the case $n = 1$ being clear.[1] Given $f \in \mathcal{O}_n$ vanishing at the origin, we can assume after a linear change of coordinates that the hypothesis (1.2) of the Weierstrass preparation theorem are satisfied. Thus we can write

$$f \ = \ p \cdot u,$$

where $p \in \mathcal{O}_{n-1}[z_1]$ is (the germ of) a Weierstrass polynomial, and $u \in \mathcal{O}_n$ is a unit. Since a polynomial ring over a UFD is itself a UFD, it follows by the induction hypothesis that $\mathcal{O}_{n-1}[z_1]$ is a UFD. Thus p can be written essentially uniquely as a finite product of irreducible elements of $\mathcal{O}_{n-1}[z_1]$, and it is elementary that any factor of a Weierstrass polynomial is again a Weierstrass polynomial. The essential point is then to show that if p is irreducible in $\mathcal{O}_{n-1}[z_1]$, then p remains irreducible considered as an element of $\mathcal{O}_n$. To this end, suppose

[1] In fact, any $f \in \mathbf{C}\{z\}$ can be written uniquely in the form $f = z^m \cdot u$ for some unit $u \in \mathbf{C}z$.

that $p = f_1 \cdot f_2$ is a product of two germs vanishing at 0. Then f_1, f_2 are represented by functions satisfying (1.2), and hence we can write

$$f_1 \;=\; u_1 \cdot q_1 \quad , \quad f_2 \;=\; u_2 \cdot q_2,$$

where $q_1, q_2 \in \mathcal{O}_{n-1}[z_1]$ are germs of Weierstrass polynomials, and $u_1, u_2 \in \mathcal{O}_n$ are units. Then

$$p \;=\; (q_1 q_2) \cdot (u_1 u_2),$$

and by the uniqueness of the Weiestrass expression it follows that $u_1 u_2 = 1$. Thus $p = q_1 \cdot q_2$ is itself reducible in $\mathcal{O}_{n-1}[z_1]$. □

Even more is true:

Theorem 1.11. *The ring $\mathcal{O}_n$ is Noetherian, i.e. every ideal in $\mathcal{O}_n$ is finitely generated.*

Proof. By induction on n, we may suppose that $\mathcal{O}_{n-1}$ is Noetherian, and then it follows by the Hilbert basis theorem that the polynomial ring $\mathcal{O}_{n-1}[z_1]$ is also Noetherian. Taking $z_1, w_2, \ldots, w_n$ as the coordinates on $\mathbf{C}^n$, we may view

$$\mathcal{O}_{n-1}[z_1] \;\subseteq\; \mathcal{O}_n$$

as a subring of $\mathcal{O}_n$. Now let $I \subseteq \mathcal{O}_n$ be any non-trivial ideal. Note to begin with that – possibly after a linear change of coodinates – I contains a Weierstrass polynomial $p(z_1, w)$ in z_1. In fact, if $f \in I$ is any element satisfying (1.2), then $f = p \cdot h$, for some Weierstrass polynomial p and some unit $h \in \mathcal{O}_n$, and hence $p \in I$. On the other hand, the ideal

$$I' \;=\; I \cap \mathcal{O}_{n-1}[z_1]$$

is finitely generated, and we choose generators $p_1, \ldots, p_t \in I'$. We assert that $p, p_1, \ldots, p_t$ generate I. In fact, consider any $f \in I$. By the Weierstrass division theorem, we can write

$$f \;=\; q \cdot p + s,$$

where $s \in \mathcal{O}_{n-1}[z_1]$. But evidently $s \in I$, so s is an $\mathcal{O}_n$ linear combination of $p_1, \ldots, p_t$, and we are done. □

We conclude this subsection by discussing ideals in $\mathcal{O}_n$. These are related to (germs of) analytic subvarieties of $\mathbf{C}^n$. We start with a definition:

Definition 1.12. (Analytic subset or variety) Let $U \subseteq \mathbf{C}^n$ be an open set. An *analytic subset* or *subvariety*[2] of U is a subset $X \subseteq U$ with the property that for each point $x \in X$, there is a neighborhood $V \ni x$ of x in U such that

$$V \cap X \;=\; \text{Zeroes}\{f_1, \ldots, f_k\}$$

for some holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(V)$.

[2]In algebraic geometry, the term "variety" is sometimes reserved for an irreducible set.

For example, the zero-locus $\mathrm{Zeroes}(f) \subseteq U$ of any holomorphic function $f \in \mathcal{O}(U)$ is an analytic variety.

We will be concerned with germs at $0 \in \mathbf{C}^n$ of analytic subvarieties. Such a germ is given by a subvariety $0 \in X \subseteq \mathbf{C}^n$, where X_1, X_2 determine the same germ if

$$X_1 \cap V \ = \ X_2 \cap V$$

for some open neighborhood of 0. Every (germ of a) subvariety X determines an ideal in $\mathcal{O}_{\mathbf{C}^n,0}$, namely

$$I(X) \ = \ \{ f \in \mathcal{O}_{\mathbf{C}^n,0} \mid f \equiv 0 \text{ on } X \text{ in a neighborhood of } 0 \}.$$

Conversely, any ideal $J \subseteq \mathcal{O}_{\mathbf{C}^n,0}$ determines the germ of a subvariety $X = \mathrm{Zeroes}(J)$. In fact, since J is finitely generated (Theorem 1.11), we can choose finitely many generators $f_1, \ldots, f_k \in J$, and we then take X to be the germ of their common zeroes.

It is elementary that given the germ of an analytic subvariety X of $\mathbf{C}^n$,

$$\mathrm{Zeroes}(I(X)) \ = \ X \tag{1.5}$$

(Exercise 1.10). Furthermore, starting with an ideal $J \subseteq \mathcal{O}_{\mathbf{C}^n,0}$, one has

$$J \subseteq I\big(\mathrm{Zeroes}(J)\big). \tag{*}$$

However, in general equality cannot be expected to hold in (*). The reason is that if $f \in \mathcal{O}_{\mathbf{C}^n,0}$ is a function with the property that $f^m \in I\big(\mathrm{Zeroes}(J)\big)$ – in other words, if f^m vanishes on the analytic set $\mathrm{Zeroes}(J)$ – then f itself already vanishes on $\mathrm{Zeroes}(J)$, ie

$$f \ \in \ I\big(\mathrm{Zeroes}(J)\big).$$

This shows that

$$\sqrt{J} \ =_{\mathrm{def}} \big\{ f \mid f^m \in J \text{ for some } m > 0 \big\} \ \subseteq \ I\big(\mathrm{Zeroes}(J)\big). \tag{**}$$

The celebrated Nullstellensatz asserts that equality does hold in (**).

Theorem 1.13. (Nullstellensatz) *For any ideal $J \subseteq \mathcal{O}_{\mathbf{C}^n,0}$, one has*

$$I\big(\mathrm{Zeroes}(J)\big) \ = \ \sqrt{J}.$$

It follows that there is a one-one correspondence between (germs of) analytic subvarieties of $\mathbf{C}^n$ and radical ideals in $\mathcal{O}_{\mathbf{C}^n,0}$ (Exercise 1.11). We will not discuss the proof of the Nullstellensatz here. For a partial proof, see [3].

Finally, we discuss irreducibility of germs.

Definition 1.14. A analytic germ $X \subseteq \mathbf{C}^n$ is *irreducible* if it cannot be written as the union

$$X \ = \ X_1 \cup X_2$$

of two non-trivial germs with $X_1, X_2 \subsetneq X$.

Proposition 1.15. (i). *A germ X is irreducible if and only if its ideal $I(X)$ is a prime ideal in $\mathcal{O}_{\mathbf{C}^n,0}$.*

(ii) *Any germ X can be written as a union*

$$X = X_1 \cup \ldots \cup X_k$$

of irreducible germs with no inclusions among the X_α. This decomposition is unique up to ordering of the factors. □

Holomorphic Mappings. In this subsection we consider holomorphic mappings between domains in $\mathbf{C}^n$ and $\mathbf{C}^m$. These will be the local models for holomorphic mappings between complex manifolds. We follow here Huybrechts' presentation [3, pages 10 - 13] fairly closely.

Consider open sets

$$U \subseteq \mathbf{C}^n \quad , \quad V \subseteq \mathbf{C}^m.$$

Definition 1.16. A *holomorphic mapping*

$$f : U \longrightarrow V \tag{1.6}$$

is a mapping given by holomorphic coordinate functions:

$$f(z) = \big(f_1(z), \ldots, f_m(z)\big) \quad , \quad f_j \in \mathcal{O}(U). \tag{1.7}$$

One says that f is *biholomorphic* if f has a holomorphic inverse (in which case of course $n = m$).

As a matter of notation, we take $w_j = u_j + \sqrt{-1} \cdot v_j$ to be the coordinate functions on $\mathbf{C}^m$, and we write

$$w = f(z) = u(z) + \sqrt{-1} \cdot v(z).$$

The first point is to discuss the derivatives of such a mapping. Suppose for the moment that f is merely $\mathcal{C}^\infty$. We have natural identifications

$$\mathbf{C}^n = \mathbf{R}^{2n} \quad , \quad \mathbf{C}^m = \mathbf{R}^{2m},$$

so we can view f as a mapping between domains in $\mathbf{R}^{2n}$ and $\mathbf{R}^{2m}$. In particular, for each $a \in U$ we can form the derivative

$$D_{\mathbf{R}} f = (D_{\mathbf{R}} f)_a : T_a \mathbf{R}^{2n} \longrightarrow T_{f(a)} \mathbf{R}^{2m}. \tag{1.8}$$

This derivative is given by the usual real Jacobian matrix $J_{\mathbf{R}}(f)$ of f: with respect to the standard real bases of the spaces in question, $J_{\mathbf{R}} f$ is represented by the block matrix:

$$J_{\mathbf{R}}(f) = \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}\right) \\ & \\ \left(\frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) \end{pmatrix}, \tag{1.9}$$

where eg $\left(\frac{\partial u}{\partial x}\right)$ denotes the matrix $\left(\frac{\partial u_i}{\partial x_j}\right)$, and we have supressed evaluation of the derivatives at a.

Now we want to bring the complex structures back into the picture. Namely, there are natural identifications

$$T_a\mathbf{C}^n \;=\; \mathbf{C}^n \quad , \quad T_{f(a)}\mathbf{C}^m \;=\; \mathbf{C}^m.$$

In particular, the source and target of (1.8) are actually complex vector spaces, with multiplication by $\sqrt{-1}$ given by

$$\sqrt{-1}\cdot\frac{\partial}{\partial x_j} \;=\; \frac{\partial}{\partial y_j} \quad , \quad \sqrt{-1}\cdot\frac{\partial}{\partial u_i} \;=\; \frac{\partial}{\partial v_i}. \tag{*}$$

We may provisionally take $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial u_i}$ as complex bases of the tangent spaces in question.[3]

The basic fact is that f is holomorphic if and only if its derivative is complex linear:

Proposition 1.17. *The $\mathcal{C}^\infty$-mapping $f : U \longrightarrow V$ is holomorphic if and only if the derivative*

$$(D_{\mathbf{R}}f)_a : T_a\mathbf{C}^n \longrightarrow T_{f(a)}\mathbf{C}^m$$

is $\mathbf{C}$-linear for every $a \in U$. In this case, the derivative is represented with respect to the standard bases by the complex Jacobian matrix

$$J(f)_a \;=_{def}\; \left(\frac{\partial f_i}{\partial z_j}(a)\right)$$

of f, where as above

$$\frac{\partial f_i}{\partial z_j} \;=\; \frac{1}{2}\left(\frac{\partial f_i}{\partial x_j} - \sqrt{-1}\cdot\frac{\partial f_i}{\partial y_j}\right).$$

When f is holomprphic, we will write

$$Df_a : T_a\mathbf{C}^n \longrightarrow T_{f(a)}\mathbf{C}^m$$

for the corresponding $\mathbf{C}$-linear transformation

Proof. This is a restatement of the Cauchy-Riemann equations using (1.9). Omitting again evaluation of the derivatives, $J_{\mathbf{R}}(f)$ is $\mathbf{C}$-linear if and only if one has the matrix identities

$$\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial v}{\partial y}\right) \quad , \quad \left(\frac{\partial u}{\partial y}\right) = -\left(\frac{\partial v}{\partial x}\right),$$

which are the Cauchy–Riemann equations. Moreover if these hold, then the $\mathbf{C}$-linear mapping determined by $D_{\mathbf{R}}f$ is given by the $m \times n$ complex matrix

$$\left(\frac{\partial u}{\partial x}\right) + \sqrt{-1}\cdot\left(\frac{\partial v}{\partial x}\right) \;=\; \left(\frac{\partial f}{\partial z}\right). \qquad \square$$

As a consequence, we find:

[3] Later we will prefer to work with the complex tagent vectors $\frac{\partial}{\partial z_j}$.

Corollary 1.18. *Assuming that f is holomorphic, the $\mathbf{R}$-linear derivative*

$$D_{\mathbf{R}} f_a : T_a\mathbf{R}^{2n} \longrightarrow T_{f(a)}\mathbf{R}^{2m}$$

is injective or surjective if and only if the complex linear mapping

$$J(f)_a : \mathbf{C}^n \longrightarrow \mathbf{C}^m$$

defined by the complex Jacobian is so. □

For future reference, it will be useful to record the relation between the real and complex derivatives in the case when f is merely $\mathcal{C}^\infty$. To this end, it is convenient to complexify the real tangent spaces in question, i.e. we consider $D_{\mathbf{R}}f$ as defining a $\mathbf{C}$-linear mapping

$$\mathbf{C}^{2n} = T_a\mathbf{R}^{2n} \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow T_{f(a)}\mathbf{R}^{2m} \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}^{2m}. \tag{*}$$

We take the complex tangent vectors

$$\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \quad , \quad \frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_i}$$

as bases for the two sides in (*). Then the matrix representing (*) has the block form

$$\begin{pmatrix} \left(\frac{\partial f}{\partial z}\right) & \left(\frac{\partial f}{\partial \bar{z}}\right) \\ \\ \left(\frac{\partial \bar{f}}{\partial z}\right) & \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) \end{pmatrix}, \tag{1.10}$$

where eg $\left(\frac{\partial f}{\partial z}\right)$ denotes the matrix $\left(\frac{\partial f_i}{\partial z_j}\right)$, and we have again supressed mention of evaluation at a. This yields convenient way of dealing with the chain rule.

The next point is that the inverse and implicit function theorems hold in the complex setting.

Theorem 1.19. (Inverse and Implicit Function Theorems) *Let $U \subseteq \mathbf{C}^n$ and $V \subseteq \mathbf{C}^m$ be open sets, and let*

$$f : U \longrightarrow V$$

be a holomorphic mapping. Assume that

$$Df_a : T_aU \longrightarrow T_{f(a)}V$$

is surjective at some point $a \in U$ (and hence also in a neighborhood of a).

(i). *If $n = m$ then there exist neighborhoods $U' \ni a$, $V' \ni f(a)$ such that f restricts to a biholomorphic isomorphsm*

$$f : U' \xrightarrow{\cong} V'.$$

(ii). *If $n > m$ then one can locally realize the fibre of f through a as the graph of a holomorphic mapping. More precisely, after perhaps reindexing the coordinates so that*

$$\left(\frac{\partial f_i}{\partial z_j}\right)_{1 \leq i,j \leq m}$$

is non-singular, one can find open subsets

$$U_1 \subseteq \mathbf{C}^{n-m}\ , U_2 \subseteq \mathbf{C}^m \quad , \quad \textit{with } U_1 \times U_2 \subseteq U,$$

together with a holomorphic mapping $g : U_1 \longrightarrow U_2$ *such that*

$$f^{-1}(f(a)) \cap (U_1 \times U_2) \ = \ \big\{ (w, g(w)) \mid w \in U_1 \big\}.$$

Sketch of Proof. Consider first the case $n = m$ as in (i). Thanks to Proposition 1.18, it follows from the usual smooth inverse function theorem that a local inverse $g = f^{-1}$ exists as a $\mathcal{C}^\infty$ mapping, and the issue is to show that g is holomorphic. To this end, we differentiate the relation $g \circ f = \mathrm{Id}$. With notation as in (1.10) one has $\left(\frac{\partial(\mathrm{Id})}{\partial \bar{z}}\right) = 0$, and this yields the matrix equation

$$0 \ = \ \left(\frac{\partial g}{\partial w}\right) \cdot \left(\frac{\partial f}{\partial \bar{z}}\right) + \left(\frac{\partial g}{\partial \bar{w}}\right) \cdot \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) \ = \ \left(\frac{\partial g}{\partial \bar{w}}\right) \cdot \overline{J(f)}.$$

Since $J(f)$ is non-singular, it follows that $\left(\frac{\partial g}{\partial \bar{w}}\right) = 0$, i.e. g is holomorphic. The proof of (ii) is similar, and we refer to [3, p. 12] for details. □

Exercises for Section 1.

Exercise 1.1. (Maximum modulus and identity principles) Let $U \subseteq \mathbf{C}^n$ be a connected open set. Reduce to the one-variable case to prove the following:

(i). If $f \in \mathcal{O}(U)$ is a non-constant holomorphic function, then $|f|$ cannot assume a maximum on U. In particular, if U is bounded, and if f extends to a continuous function on the closure $\overline{U}$ of U, then $|f|$ assumes its maximum on the boundary of U.

(ii). Suppose that $V \subseteq U$ is a non-empty open set with the property that f is identically zero on V. Then f is identically zero on U. Hence if $g \in \mathcal{O}(U)$ is a second holomorphic function which agrees with f on V, then $f = g$ on all of U.

Exercise 1.2. Prove Proposition 1.4.

Exercise 1.3. (i). If $f : U \longrightarrow \mathbf{C}$ is a $\mathcal{C}^\infty$ function on an open set $U \subseteq \mathbf{C}^n$, then

$$\frac{\partial \bar{f}}{\partial \bar{z}} \ = \ \overline{\left(\frac{\partial f}{\partial z}\right)} \quad , \quad \frac{\partial \bar{f}}{\partial z} \ = \ \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)}$$

(ii). Let $f : U \longrightarrow \mathbf{C}$, $g : V \longrightarrow \mathbf{C}$ be $\mathcal{C}^\infty$ functions on open subsets of $\mathbf{C}$ such that $f \circ g$ is defined. . Then

$$\begin{aligned} \frac{\partial}{\partial z}(f(g(z)) \ &= \ \frac{\partial f}{\partial w} \cdot \frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{w}} \cdot \frac{\partial \bar{g}}{\partial z} \\ \frac{\partial}{\partial \bar{z}}(f(g(z)) \ &= \ \frac{\partial f}{\partial \bar{w}} \cdot \frac{\partial \bar{g}}{\partial \bar{z}} + \frac{\partial f}{\partial w} \cdot \frac{\partial g}{\partial \bar{z}} \end{aligned}$$

In particular, if f and g are analytic, then so is $f \circ g$ and the usual one-variable chain rule holds for $\frac{\partial}{\partial z}$.

Exercise 1.4. (i). Prove that if f, g are holomorphic on $U \subseteq \mathbf{C}^n$, then the product rule holds for each of their z-derivatives, i.e.

$$\frac{\partial(fg)}{\partial z_i} \;=\; f \cdot \frac{\partial g}{\partial z_i} + g \cdot \frac{\partial f}{\partial z_i}.$$

(ii). Deduce that for a single variable z,

$$\frac{\partial}{\partial z}(z^m) \;=\; m \cdot z^{m-1}.$$

(iii). In the setting of Theorem 1.1 (iii), show that the coefficient of $z_1^{j_1} \ldots z_n^{j_n}$ in the Taylor series of $f(z)$ is given by

$$a_{j_1,\ldots,j_n} \;=\; \frac{1}{j_1! \cdots j_n!} \cdot \frac{\partial^{j_1+\cdots+j_n} f}{\partial z_1^{j_1} \cdots \partial z_n^{j_n}}(c).$$

Exercise 1.5. Prove Lemma 1.5. (Cf. [3, Lemma 1.1.3].)

Exercise 1.6. Show that the Weierstrass polynomial associated to a given holomorphic function $f(z_1, w)$ is unique.

Exercise 1.7. Working in $\mathbf{C}^2$ with variables z, w, find the Weierstrass polynomial for $\sin(w^2 - z^3)$.

Exercise 1.8. Prove the Riemann Extension Theorem 1.7. Show that the conclusion holds more generally if the hypersurface $\{f = 0\}$ is replaced by any analytic subvariety $X \subseteq U$.

Exercise 1.9. Denote by $\mathcal{A}_n$ the ring of germs at $0 \in \mathbf{C}^n$ of smooth functions. In contrast to $\mathcal{O}_n$, the ring $\mathcal{A}_n$ is not well-behaved algebraically. Specifically, prove that it is neither an integral domain nor Noetherian.

Exercise 1.10. Prove the equality (1.5)

Exercise 1.11. Recall that an ideal $J \subseteq \mathcal{O}_{\mathbf{C}^n,0}$ is said to be *radical* if $\sqrt{J} = J$. Granting the Nullstellensatz, prove that there is a one-to-one order-reversing correspondence between germs of analytic subvarieties of $\mathbf{C}^n$, and radical ideals in $\mathcal{O}_{\mathbf{C}^n,0}$.

Exercise 1.12. Prove Proposition 1.15 .

Exercise 1.13. In contrast to the case of holomorphic mappings between domains in $\mathbf{C}$, it is not the case in general that a holomorphic map between domains in higher dimensions is an open mapping. For example, show that

$$f : \mathbf{C}^2 \longrightarrow \mathbf{C}^2 \quad , \quad (z, w) \mapsto (z, zw)$$

is not an open mapping.

Exercise 1.14. Let $f : U \longrightarrow V$ be a holomorphic mapping between domains $U, V \subseteq \mathbf{C}^n$. Then

$$\det J_{\mathbf{R}}(f) \;=\; |\det J(f)|^2.$$

In particular, a biholomorphic mapping is always orientation-preserving.

2. Complex manifolds

In this section we introduce our principle objects of interest, namely complex manifolds.

Definition and First Examples. As one might expect, a complex manifold is simply a differentiable manifold that carries holomorphic coordinate charts.

Definition 2.1. (Holomorphic atlas, complex manifold) Let X be a smooth manifold.

(i). A *holomorphic atlas* $\{(U_i, \varphi_i)\}$ on X consists of a collection of open sets $U_i \subseteq X$ covering X, together with diffeomorphisms

$$\varphi_i : U_i \xrightarrow{\cong} \varphi_i(U_i) \ \subseteq \ \mathbf{C}^n$$

such that the *transition functions*

$$h_{i,j} \ =_{\text{def}} \ \varphi_i \circ \varphi_j^{-1} \ : \ \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j) \tag{2.1}$$

are biholomorphic.

(ii). Two such atlases $\{(U_i, \varphi_i)\}$, $\{(U_i', \varphi_i')\}$ are *equivalent* if all the maps

$$\varphi_i \circ \varphi_j'^{-1} \ : \ \varphi_j'(U_i \cap U_j') \longrightarrow \varphi_i(U_i \cap U_j') \tag{2.2}$$

are biholomorphic.

(iii). A *complex manifold* of of (complex) dimension n is a smooth manifold of (real) dimension $2n$ together with an equivalence class of holomorphic atlases.

Observe that since biholomorphic maps are orientation preserving (Exercise 1.14), a complex manifold X is automatically orientable as a real manifold.

In the usual way, all of the notions studied locally in the previous section make sense on such a manifold.

Definition 2.2. (Holomorphic function, mapping) Let X be a complex manifold.

(i). A holomorphic function on X is a function $f : X \longrightarrow \mathbf{C}$ with the property that there is an atlas $\{(U_i, \varphi_i)\}$ for X for which all the functions

$$f \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \mathbf{C}$$

are holomorphic functions.

(ii). A smooth mapping $f : X \longrightarrow Y$ is *holomorphic* if one can find atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ for X and Y, with $f(U_i) \subseteq V_j(i)$ for some index $j(i)$, with the property that

$$\psi_j \circ f \circ \varphi_i^{-1} \ : \ \varphi_i(U_i) \longrightarrow \psi(V_i)$$

is a holomorphic mapping.

These are independent of the choice of open set or the atlas thanks to the fact that the functions appear in (2.1) and (2.2) are biholomorphic.

In the future, we will abbreviate this sort of definition by saying for example that a function f on a complex manifold is holomorphic if it is given in local coordates by a holomorphic function.

Definition 2.3. (Isomorphism) An isomorphism between complex manifolds is a holomorphic mapping

$$f : X \longrightarrow Y$$

that possesses a holomorphic inverse. In this case one says that X and Y are isomorphic or biholomorphic.

Example 2.4. Any open subset $U \subseteq X$ of a complex manifold is iteself a complex manifold. In particular, any open subset $U \subseteq \mathbf{C}^n$ is a complex manifold of dimension n.

Example 2.5. (Riemann sphere) The standard Riemann sphere $S = \mathbf{C} \cup \{\infty\}$ is a compact complex manifold (of dimension 1). Recall that in the finite part of the plane one takes the usual coordinate function z as the local coordinate, while in a neighborhood of the point at infinity one uses the coordinate w with $w = \frac{1}{z}$. In other words, the transition function associated to this atlas is

$$h : \mathbf{C} - \{0\} \longrightarrow \mathbf{C} - \{0\} \quad , \quad z \mapsto \frac{1}{z}.$$

The Riemann sphere is also the one-dimensional instance of complex projective space $\mathbf{P}^n$: we will study this manifold in detail in the next subsection.

Example 2.6. (Analytic hypersurfaces) Let $U \subseteq \mathbf{C}^{n+1}$ be an open set, let $f : U \longrightarrow \mathbf{C}$ be a holomorphic function, and denote by

$$X \;=\; \mathrm{Zeroes}(f) \subseteq U$$

the zero-locus of f. Assume that for each point $a \in X$ there is at least one index i such that

$$\frac{\partial f}{\partial z_i}(a) \neq 0.$$

Then X is a complex manifold of dimension n. (In fact, it follows from the Implicit Function Theorem 1.19 (ii) that the n remaining coordinate functions on $\mathbf{C}^{n+1}$ give local coordinates in a neighborhood of a.) This construction leads to a vast number of important concrete examples. Note that when $n = 1$ we get further examples of one-dimensional complex manifolds.

Definition 2.7. (Riemann surface, complex curve) A *Riemann surface* is a complex manifold of dimension one. These are also called *smooth complex curves*, especially when they arise as in the previous two examples.

Example 2.8. Generalizing the construction of the previous example, suppose that V is a complex manifold of dimension $n + k$, and that

$$X \;\subseteq\; V$$

is a subset defined as the common zeroes of a collection of k functions $f = (f_1, \ldots, f_k)$ such that the Jacobian $J(f)_a$ has maximal rank at every point $a \in X$. Then again X is a complex manifold of dimension n.

In the situation of the previous example, we say that X is a closed submanifold of V. More generally:

Definition 2.9. (Closed submanifold). Let Y be a complex manifold of dimension $n+k$, and let $X \subseteq Y$ be a closed smooth submanifold of real dimension $2n$. One says that X is a *closed submanifold* of Y if there exists a holomorphic atlas $\{(U_i, \varphi_i)\}$ on Y sich that if $U_i \cap X \neq \varnothing$, then

$$\varphi_i : (U_i \cap X) \xrightarrow{\equiv} \varphi(U_i) \cap \mathbf{C}^n,$$

where $\mathbf{C}^n \subseteq \mathbf{C}^{n+k}$ is a linearly embedded copy of $\mathbf{C}^n$ in $\mathbf{C}^{n+k}$.

It is equivalent to ask that X be locally cut out in Y by a k-tuple of holomorphic functions having maximal rank (Exercise 2.4).

One can also talk about possibly non-closed submanifolds, whose definition we leave to the reader. However since we are mainly interested in closed ones, we adopt the

Convention 2.10. By a “submanifold” $X \subseteq Y$, we always mean unless otherwise stated a closed submanifold.

Example 2.11. (Complex Lie Groups) A complex Lie group is a conplex manifold G, which is simultaneously a group, in such a way that multiplication and inversion are given by holomorphic mappings

$$\mathrm{mult} : G \times G \longrightarrow G \quad , \quad \mathrm{inv} : G \longrightarrow G.$$

For example, there is a natural identification

$$\mathrm{Mat}_{n \times n}(\mathbf{C}) \ = \ \mathbf{C}^{n^2}$$

of the set of all $n \times n$ complex matrices with $\mathbf{C}^{n^2}$: the coordinate functions are the entries of the matrix. Then

$$\mathrm{GL}_n(\mathbf{C}) \ =_{\mathrm{def}} \ \{A \in \mathrm{Mat}_{n \times n} \mid \det(A) \neq 0\}$$

is an open subset of $\mathrm{Mat}_{n \times n}(\mathbf{C})$, hence a complex manifold, and thanks to Cramer's rule multiplication and inversion are holomorphic maps. Thus $\mathrm{GL}_n(\mathbf{C})$ is a complex Lie group. The other standard linear groups – notably the special linear, orthognal and symplectic groups $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{O}_n(\mathbf{C})$, $\mathrm{Sp}_{2n}(\mathbf{C})$ – are closed submanifolds of $\mathrm{GL}_n(\mathbf{C})$, and hence themselves complex Lie groups (Exercise 2.6). On the other hand, observe that the unitary and special unitary groups U_n and SU_n are not complex submanifolds of $\mathrm{GL}_n(\mathbf{C})$. In fact, it turns out that the only compact connexted complex Lie groups are complex tori, which we discuss next.

Example 2.12. (Complex tori, I). Let

$$\Lambda \ \subseteq \ \mathbf{C}^n$$

be a lattice, i.e. a discrete subgroup of maximal rank. It is known that $\Lambda \cong \mathbf{Z}^{2n}$, and the embedding $\Lambda \subset \mathbf{C}^n$ is determined by choosing $2n$ complex vectors

$$\lambda_1, \ldots, \lambda_{2n} \ \in \ \mathbf{C}^n$$

that are linearly independent over $\mathbf{R}$, and taking

$$\Lambda \ = \ \mathbf{Z} \cdot \lambda_1 + \ldots + \mathbf{Z} \cdot \lambda_{2n}.$$

Now put

$$X \ = \ X_\Lambda \ = \ \mathbf{C}^n/\Lambda,$$

and denote by $\pi : \mathbf{C}^n \longrightarrow X$ the quotient mapping. Then X is diffeomorphic to a torus (real) dimension $2n$ – i.e. a product ot $2n$ copies of the circle S^1 – and π is its universal covering. A moment's thought shows that there is a unique way to put a complex structure on X in such a way that π is holomorphic: in fact, π is locally a diffeomorphism, and one takes local inverses as coordinate charts. (See Exercise 2.7). Thus X_Λ is a compact complex manifold of dimension n, which is called a *complex torus*. Note however that whereas the diffeomorphism type of X_Λ doesn't depend on the particular lattice $\Lambda \subset \mathbf{C}^n$ one starts with, we will see in the next examples that its complex isomorphism type definitely does.

Example 2.13. (Morphisms of complex tori) Consider two lattices

$$\Lambda \ , \ \Lambda' \ \subseteq \ \mathbf{C}^n,$$

defining complex tori $X_\Lambda, X_{\Lambda'}$. If $A : \mathbf{C}^n \longrightarrow \mathbf{C}^n$ is a linear transformation such that

$$A(\Lambda) \ \subseteq \ \Lambda', \tag{*}$$

then evidently A induces a holomorphic mapping $f_A : X_\Lambda \longrightarrow X_{\Lambda'}$. Conversely, we will show that any holomorphic mapping

$$f : X_\Lambda \longrightarrow X_{\Lambda'} \tag{**}$$

is essentially of this form. Granting this, it follows that for "most" pairs of lattices Λ, Λ', the corresponding tori $X_\Lambda, X_{\Lambda'}$ cannot be biholomorphic.[4] Suppose then given a holomorphic mapping f as in (**). Then f lifts to a holomorphic mapping $F : \mathbf{C}^n \longrightarrow \mathbf{C}^n$ on the universal covering spaces with the property that

$$F(z + \lambda) - F(z) \ \in \ \Lambda' \tag{***}$$

for every $z \in \mathbf{C}^n$, $\lambda \in \Lambda$, and after replacing F by a translate we may suppose that $F(0) = 0$. As Λ' is discrete, the left hand side of (***) is constant as a function of z, which implies that all the holomorphic partials of F are Λ-periodic. These partials are therefore bounded, and hence constant thanks to Liouville. Therefore $F(z) = Az$ for some linear transformation A, and of course A must then satisfy (*) since by construction F decends to a morphism $\mathbf{C}^n/\Lambda \longrightarrow \mathbf{C}^n/\Lambda'$.

[4] Roughly speaking, a lattice $\Lambda \subseteq \mathbf{C}^n$ depends on $2n^2$ complex parameters, whereas an $n \times n$ complex matrix is determined on n^2 complex parameters, so for "most" Λ, Λ' there should be no non-zero matrices satisfying (*). See Exercise 2.8 for some concrete examples.

Remark 2.14. It is an interesting (but somewhat counter-intuitive) fact that if one starts with a "random" lattice $\Lambda \subseteq \mathbf{C}^n$, the resulting torus X_Λ does not contain any proper analytic submanifolds or subvarieties of positive dimension. Equivalently, there are no non-trivial Λ-periodic sub varieties of $\mathbf{C}^n$. On the other hand, if for instance

$$\Lambda \ = \ \mathbf{Z}^n + \sqrt{-1} \cdot \mathbf{Z}^n \ \subseteq \ \mathbf{C}^n,$$

then X_Λ is a product of one-dimensional complex tori, and so carries lots of submanifolds (e.g. sub-products).

Example 2.15. (One-dimensional tori). We consider in more detail one-dimensional complex tori. Denote by $\mathbf{H}$ the upper half plane in $\mathbf{C}$. After a linear change of coordinates, any lattice in $\mathbf{C}$ is spanned by 1 and some complex number $\tau \in \mathbf{H}$: write

$$\Lambda_\tau \ = \ \mathbf{Z} + \mathbf{Z} \cdot \tau \ \subseteq \ \mathbf{C},$$

and let X_τ denote the corresponding one-dimensional torus. We will show that

$$X_\tau \ \cong \ X_{\tau'} \quad \Longleftrightarrow \quad \tau' \ = \ \frac{a\tau + b}{c\tau + d} \tag{*}$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Granting this for the moment, it follows that isomorphism classes of one-dimensional complex tori are parametrized by $\mathbf{H}/\mathrm{SL}_2(\mathbf{Z})$ under the action defined by (*). This quotient space has been much studied: for example, a fundamental domain is given (modulo some identifications on the boundary) by

$$|\tau| \geq 1 \quad , \quad |\mathrm{Re}(\tau)| \leq \tfrac{1}{2}.$$

In fact, it turns out (somewhat non-trivially) that $\mathbf{H}/\mathrm{SL}_2(\mathbf{Z})$ can itself be given the structure of a one-dimensional complex manifold with

$$\mathbf{H}/\mathrm{SL}_2(\mathbf{Z}) \ \cong \ \mathbf{C}.$$

In summary, isomorphism classes of one-dimensional complex tori are parametrized by the complex plane: this is the first example of a *moduli space*, ie a complex manifold (or analytic variety) that parametrizes isomorphism classes of complex manifolds of fixed diffeomorphism type. As for the proof of (*), it follows from Example 2.13 that $X_\tau \cong X_{\tau'}$ if and only if there is a complex number $\mu \in \mathbf{C}^*$ such that multiplication by μ carries $\Lambda_{\tau'}$ onto Λ_τ. This is equivalent to asking that

$$\mu \cdot \tau' \ = \ a\tau + b \quad , \quad \mu \cdot 1 \ = \ c\tau + d$$

for some integers $a, b, c, d \in \mathbf{Z}$. Hence $\tau' = \frac{a\tau+b}{c\tau+d}$, and the resulting matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must have determinant $= 1$ since it is invertible and maps the upper half plane to iteself.

Example 2.16. (Hopf manifolds). Given a real number $0 < \lambda < 1$ and an integer $k \in \mathbf{Z}$, define

$$\sigma_k : \mathbf{C}^n - \{0\} \longrightarrow \mathbf{C}^n - \{0\} \ , \quad \sigma_k(z_1, \ldots, z_n) = (\lambda^k z_1, \ldots, \lambda^k z_n).$$

This defines a fixed point-free action of $\mathbf{Z}$ on $\mathbf{C}^n - \{0\}$, and every point $z \in \mathbf{C}^n - \{0\}$ has an open neighborhood disjoint from all of its images under the action. It follows that the quotient

$$X \ = \ (\mathbf{C}^n - \{0\})\,/\mathbf{Z}$$

carries in the natural way the structure of a complex manifold of dimension n, called the *Hopf manifold*. It is diffeomorphic to the product $S^{2n-1} \times S^1$. (See Exercise 2.9.) The interest in this construction, as we will see later, is that it gives an example of a complex manifold which cannot carry a Käbler metric.

Especially for algebraic geometers – who like to define spaces by means of defining equations – the construction in Example 2.6 and its generalizations are fundamental. But it suffers from one serious drawback, namely that positive-dimensional closed submanifolds of $\mathbf{C}^n$ are never compact. In fact, any submanifold of $\mathbf{C}^n$ carries lots of non-constant holomorphic functions (for instance the coordinate functions). On the other hand, one has the elementary but basic

Proposition 2.17. *Let X be a compact connected complex manifold. Then the only global holomorphic functions on X are constants.*

Proof. If f is a holomorphic function on X, then $|f|$ assumes a maximum at some point of X. By the maximum principle, f must then be constant. □

It is very important to have some ambient complex manifold in which one can "write down" (using equations of some form) many compact submanifolds. We turn next to the construction of projective space, which serves this function.

Projective Space. There are various equivalent definitions of projective space. As our base construction we will take

Definition 2.18. Let $\mathbf{C}^*$ act on $\mathbf{C}^{n+1} - \{0\}$ by scalar multiplication, i.e.

$$\lambda \cdot (a_0, \ldots, a_n) \ = \ (\lambda \cdot a_0, \ldots, \lambda \cdot a_n) \ \text{ for } \lambda \neq 0. \tag{2.3}$$

We define

$$\mathbf{P}^n \ = \ \mathbf{P}^n(\mathbf{C}) \ =_{\text{def}} \ \big(\mathbf{C}^{n+1} - \{0\}\big)/\mathbf{C}^*$$

to be the quotient by this action, and we denote by

$$\pi : \big(\mathbf{C}^{n+1} - \{0\}\big) \longrightarrow \mathbf{P}^n \tag{2.4}$$

the quotient map.

This specifies $\mathbf{P}^n$ as a set and a topological space. We will show momentarily how it carries the structure of a complex manifold. Note that all the fibres of π are copies of $\mathbf{C}^*$ sitting in a one-dimensional complex subspace of $\mathbf{C}^{n+1}$.

A couple of preliminary remarks will be useful. First, note that any non-zero vector $0 \neq a \in \mathbf{C}^{n+1}$ determines a unique one-dimensional complex subspace

$$\mathbf{C} \cong \mathbf{C} \cdot a \ \subseteq \mathbf{C}^{n+1}.$$

Moreover, two non-zero vectors $a, a' \in \mathbf{C}^{n+1}$ determine the same subspace if and only if

$$a' \ = \ \lambda \cdot a \quad \text{for some } \lambda \in \mathbf{C}^*.$$

Hence

Viewpoint 2.19. The projective space $\mathbf{P}^n$ parametrizes all one-dimensional vector subspaces of $\mathbf{C}^{n+1}$.

This being the case, it is sometimes useful to take a coordinate-free approach. Specifically, given any complex vector space V of dimension $n+1$, we denote by $\mathbf{P}(V)$ the n-dimensional projective space of one-dimensional vector subspaces of V.[5]

Next, there is a very useful way to specify points in projective space by so-called homogeneous coordinates. Namely, we can view (2.3) as determining an equivalence relation on all non-zero vectors $0 \neq a \in \mathbf{C}^{n+1}$, where

$$a' \sim a \iff a' = \lambda \cdot a \ \text{ for } \lambda \in \mathbf{C}^*,$$

and then evidently

$$\mathbf{P}^n \ = \Big\{ a \in \big(\mathbf{C}^{n+1} - \{0\}\big) \Big\} / \sim .$$

Given a non-zero vector $a = (a_0, \ldots, a_n)$, denote by

$$[a] \ = \ [a_0, \ldots, a_n]$$

its equivalence class. Thus $[a]$ determines a point $\pi(a) \in \mathbf{P}^n$ with *homogeneous coordinates* $[a_0, \ldots, a_n]$. In other words, points in $\mathbf{P}^n$ are specified by homogeneous coordinates $[a_0, \ldots, a_n]$ with at least one non-zeroi entry, where

$$[a_0', \ldots, a_n'] = [a_0, \ldots, a_n] \iff a_i' = \lambda \cdot a_i \ \text{ for some } \lambda \in \mathbf{C}^*.$$

Remark 2.20. Note that one can replace $\mathbf{C}$ in these definitions and constructions by any field k, leading to

$$\mathbf{P}^n(k) \ = \ \big(k^{n+1} - \{0\}\big)/k^*.$$

Proposition 2.21. *Complex projective space $\mathbf{P}^n$ carries a unique structure of a complex manifold in such a way that the map*

$$\pi : \mathbf{C}^{n+1} - \{0\} \longrightarrow \mathbf{P}^n$$

is a locally trivial holomorphic $\mathbf{C}^*$-bundle.

[5]Following a convention introduced by Grothendieck – which is a more general context is essentially forced on one – algebraic geometers sometimes use $\mathbf{P}(V)$ to mean the projective space of one-dimensional *quotients* of V. So if you are reading a book you have to be careful to figure out which of these two possibilities "$\mathbf{P}(V)$" refers to. We'll try to stick to the subspace convention here.

In other words, $\mathbf{P}^n$ carries the structure of a complex manifold in such a way that first of all π is holomorphic. Moreover we can find an open covering $\{U_i\}$ of $\mathbf{P}^n$ by open sets with the property that

$$\pi^{-1}(U_i) \ \cong \ U_i \times \mathbf{C}^* \tag{2.5}$$

as complex manifolds.

Sketch of Proof. For each index $0 \leq i \leq n$, let

$$\mathbf{P}^n \ \supseteq \ U_i \ = \ \{[a_0, \ldots, a_n] \mid a_i \neq 0.\}$$

This is the image of the open set $\{a_i \neq 0\} \subseteq \mathbf{C}^{n+1}$, hence is open in $\mathbf{P}^n$. By dividing through by a_i, the homogeneous coordinates of any point in U_i can be expressed uniquely as a vector whose i^{th} coordinate is 1:

$$[a_0, \ldots, a_n] \ = \ [\tfrac{a_0}{a_i}, \ldots, 1, \ldots \tfrac{a_n}{a_i}].$$

Thus there is a natural identification

$$U_i \longrightarrow \mathbf{C}^n \quad , \quad [a_0, \ldots, a_n] \mapsto (\tfrac{a_0}{a_i}, \ldots, \tfrac{a_{i-1}}{a_i}, \tfrac{a_{i+1}}{a_i}, \ldots, \tfrac{a_n}{a_i}).$$

We take these maps as local coordinates on U_i. The reader should check that the resulting transition functions are biholomorphic. We likewise leave it to the reader to construct the isomorphism (2.5). □

Example 2.22. ($\mathbf{P}^1$). In the case $n = 1$ we recover the Riemann sphere S, i.e. $\mathbf{P}^1 = S$. In fact, keeping the notation of the previous proof, consider a point $[a_0, a_1] \in \mathbf{P}^1$. On the open set $U_0 = \{a_0 \neq 0\}$ take we use $z = a_1/a_0$ as the local coordinate, while on $U_1 = \{a_1 \neq 0\}$ we use $w = a_0/a_1$. Thus

$$w = \frac{1}{z} \quad \text{on} \quad U_0 \cap U_1 = \mathbf{C}^*,$$

as required.

It is perhaps not yet apparent that $\mathbf{P}^n$ is compact. To see this, let $\| \, \|$ denote the standard Euclidean norm on $\mathbf{C}^n$, and consider the unit sphere

$$S^{2n+1} \ = \ \{\| \, z \, \| = \, 1\} \ \subseteq \ \mathbf{C}^{n+1}.$$

Consider the restriction

$$h : S^{2n+1} \longrightarrow \mathbf{P}^n$$

to S^{2n+1} of the basic quotient mapping π in (2.4). The compactness of $\mathbf{P}^n$ is a consequence of the fact that h is surjective, which in turn follows from the fact that

$$\pi(a) \ = \ \pi\left(\frac{a}{\| \, a \, \|}\right)$$

for any non-zero vector $a \in \mathbf{C}^{n+1}$. To say the same thing a little differently, consider the unit circle $S^1 = \{|z| = 1\} \subseteq \mathbf{C}$ of complex numbers of length 1. Then S^1 acts on S^{2n+1} by scalar multiplication, and h is the quotient map. The mapping h is called the *Hopf vibration*: the reader should check that it expresses S^{2n+1} as the total space of a locally trivial S^1 bundle over $\mathbf{P}^n$.

Example 2.23. The case $n = 1$ of this discussion is already very interesting. Recalling that $\mathbf{P}^1 = S^2$ is the Riemann sphere, the Hopf fibration in this case is a smooth mapping

$$h : S^3 \longrightarrow S^2$$

expressing the three-sphere as a circle bundle over the two-sphere. In particular, for any point $p \in S^2$, the fibre $h^{-1}(p)$ is a circle in S^3. The interesting fact is that for distinct $p, q \in S^2$, these circles are *linked*. (One can verify this pictorially by making a stereographic projection from $S^3 - \{\text{point}\}$ to $\mathbf{R}^3$ and working out explicitly the equations of two fibres.)

We next discuss the paving of $\mathbf{P}^n$ by affine spaces. By construction there is a holomorphic inclusion

$$\mathbf{C}^n \ = \ U_0 \ \subseteq \ \mathbf{P}^n \quad , \quad (z_1, \ldots, z_n) \mapsto [1, z_1, \ldots, z_n].$$

The complement $\mathbf{P}^n - U_0$ consists of the set of points $[0, z_1, \ldots, z_n]$ whose first homogeneous coordinate vanishes. This is in turn isomorphic to a projective space of dimension $n - 1$, with homogeneous coordinates $[z_1, \ldots, z_n]$. In other words:

$$\mathbf{P}^n \ = \ \mathbf{C}^n \sqcup \mathbf{P}^{n-1},$$

and one thinks of $\mathbf{P}^{n-1}$ as the "hyperplane at infinity" that one adds to compactify $\mathbf{C}^n$: there is one point at infinity added for each complex direction in $\mathbf{C}^n$. Repeating the same discussion starting with the hyperplane at infinity, one ends up by expressing $\mathbf{P}^n$ as a disjoint union of affine spaces:

$$\mathbf{P}^n \ = \ \mathbf{C}^n \sqcup \mathbf{C}^{n-1} \sqcup \ldots \ \sqcup \ \mathbf{C}^1 \sqcup \mathbf{C}^0.$$

The analogous thing works for projective space over any field, and the resulting decomposition is the analogue of the fact familiar from topology that $\mathbf{P}^n$ has the structure of a CW complex with one cell in each even (real) dimension $0, 2, \ldots, 2n$.

We now turn to describing loci in projective space $\mathbf{P}^n$ by means of polynomial equations. Note that if $f \in \mathbf{C}[z_0, \ldots, z_n]$ is a polynomial, the value of f at a point $[a] = [a_0, \ldots, a_n] \in \mathbf{P}^n$ is not well-defined, since in general

$$f(a_0, \ldots, a_n) \ \neq \ f(\lambda \cdot a_0, \ldots, \lambda \cdot a_n).$$

On the other hand, suppose that f is *homogeneous* of degree d, i.e. every monomial in f has degree d. Then

$$f(\lambda \cdot a_0, \ldots, \lambda \cdot a_n) \ = \ \lambda^d \cdot f(a_0, \ldots, a_n)$$

for every $\lambda \in \mathbf{C}^*$. Therefore it makes sense to talk about whether or not f vanishes at a point $[a] \in \mathbf{P}^n$.

Definition 2.24. A projective algebraic set $X \subseteq \mathbf{P}^n$ is the locus of common zeroes of a collection

$$F_1, \ldots, F_N \in \mathbf{C}[Z_0, \ldots, Z_n]$$

of homogeneous polynomials.[6] We write

$$X \ = \ \text{Zeroes}(F_\alpha).$$

[6]One often uses capital letters to denote homogeneous polynomials and variables.

Example 2.25. (Linear subspaces). Suppose that

$$L_1(Z_0, \ldots, Z_n), \ldots, L_e(Z_0, \ldots, Z_n)$$

are *linear polynomials* that are $\mathbf{C}$-linearly independent. Then the L_α cut out a sumbanifold

$$\Lambda \ \subseteq \ \mathbf{P}^n$$

that is isomorphic to $\mathbf{P}^{n-e}$. Submanifolds arising in this manner are called linear subspaces of $\mathbf{P}^n$.

Example 2.26. (A smooth conic). Working on $\mathbf{P}^2$ with homogeneous coordinates X, Y, Z, consider the conic curve

$$C \ = \ \{XY - Z^2 = 0\}.$$

This is illustrated schematically in Figure 2, which shows the intersections of C with the three "axes" $X = 0, Y = 0$ and $Z = 0$.[7] It is interesting to consider the intersections of C with the three copies U_Z, U_Y, U_Z of $\mathbf{C}^2$ obtained as complement of these axes. We identify $U_Z = \{Z \neq 0\}$ with CC^2 by taking affine coordinates

$$x \ = \ \frac{X}{Z} \ \ , \ \ y \ = \ \frac{Y}{Z},$$

and then $C_0 \subseteq \mathbf{C}^2$ is defined by the equation

$$xy - 1 = 0.$$

Note that (the real points of) this curve is a hyperbola with the x- and y-axes as asymptotes: this corresponds to the fact that the compact curve $C \subseteq \mathbf{P}^2$ meets the line $Z = 0$ at infinity at two points. On the other hand, consider the intersection $C \cap U_X$. Here the affine coordinates are

$$y \ = \ \frac{Y}{X} \ \ , \ \ z \ = \ \frac{Z}{X},$$

and the defining equation is $y - z^2 = 0$. Here we get a parabola, reflecting the fact that the compact curve C is tangent to the line $X = 0$ at infinity.

Remark 2.27. (Dehomogenization). As the previous example illustrates, if $X \subseteq \mathbf{P}^n$ is defined by homogeneous polynomials

$$F_\alpha(Z_0, \ldots, Z_n) \ \in \ \mathbf{C}[Z_0, \ldots, Z_n],$$

and if as above

$$U_i = \{Z_i \neq 0\} \ = \ \mathbf{C}^n$$

denotes the i^{th} standard copy of $\mathbf{C}^n$ sitting inside $\mathbf{P}^n$, then

$$X \cap U_i \ \subseteq \ U_i \ = \ \mathbf{C}^n$$

is cut out by the (generally inhomogeneous polynomials) obtained by setting $Z_i = 1$ in F_α, and treating the remaining variables as affine coordinates on $\mathbf{C}^n$.[8]

[7] One typically tried to visualize complex varieties by drawing their real parts. This what the Figure shows is the real conic $XY - Z^2 = 0$ in $\mathbf{P}^2(\mathbf{R})$. But in happy situations, with a little practice these real drawings can be quite suggestive.

[8] However, if one starts with an algebraic set

$$X_0 \subseteq U_0 = \mathbf{C}^n$$

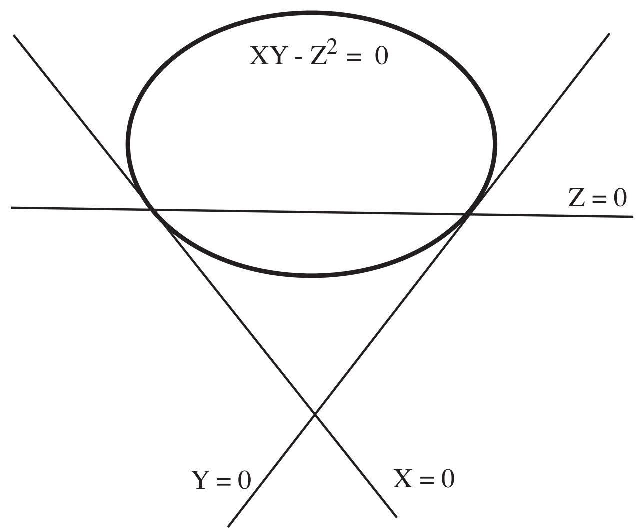

FIGURE 2. The conic $XY = Z^2$ in $\mathbf{P}^2$.

Definition 2.28. (Complex projective manifold). A *complex projective manifold* is a complex manifold X that arises as a closed complex sumbanifold

$$X \subseteq \mathbf{P}^r$$

of some projective space $\mathbf{P}^r$.

Thanks to the compactness of $\mathbf{P}^r$, any projective manifold is compact. It may happen (and in fact always does) that a given complex manifold can be realized in different ways as submanifolds of projective spaces of various dimensions.

Remark 2.29. (Chow's Theorem). A remarkable theorem of Chow states that if $X \subseteq \mathbf{P}^r$ is a complex submanifold, then in fact X is the algebraic set cut out by a collection of homogeneous polynomials. The same statement remains true if one asks merely that X be an analytic subvariety of $\mathbf{P}^r$, ie locally cut out by holomprohic functions. Unfortunately we won't be able to prove this in Math 545, but it's good to keep the fact in mind.

The following statement generalizes Example 2.6:

Proposition 2.30. *Let $F \in \mathbf{C}[Z_0, \ldots, Z_n]$ be a homogeneous polynomial of degree d, and assume that the partial derivatives*

$$\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}$$

defined by a collection of polynomials $\{f_\alpha(z_1, \ldots, z_n)\}$ and looks for the "projective closure" $X \subseteq \mathbf{P}^n$ of X_0 – ie the smallest projective algebraic set restricting to X_0 on U_0 – then one typically needs to consider more equations than the homogenizations of the f_α.

(which are homogeneous polynomials of degree $d-1$) have no common zeroes in $\mathbf{P}^n$. Then

$$X \;=\; \mathrm{Zeroes}(F) \;\subseteq\; \mathbf{P}^n$$

is an $(n-1)$-dimensional sub manifold of $\mathbf{P}^n$.

Proof. This follows from Example 2.6 (ie the implicit function theorem) together with *Euler's Theorem*: if F is homogeneous of degree d, then

$$\frac{\partial F}{\partial Z_0} + \ldots + \frac{\partial F}{\partial Z_n} \;=\; d \cdot F.$$

We leave detais to the reader. $\square$

Given a projective manifold

$$X \subseteq \mathbf{P}^r,$$

we next discuss a convenient way to write down holomorphic mappings

$$f : X \longrightarrow \mathbf{P}^s.$$

For this, consider

$$F_0, \ldots, F_s \;\in\; \mathbf{C}[Z_0, \ldots, Z_n]$$

homogeneous polynomials of the same degree d. Assume that

$$\mathrm{Zeroes}(F_0, \ldots, F_s) \cap X \;=\; \emptyset.$$

Note that given a point $x = [a] \in X$, the homogeneous vector $[F_0(a), \ldots, F_s(a)]$ gives a well-defined point in $\mathbf{P}^s$: the values $F_i(a)$ are determined up to a common scalar multiple, and they don't vanish simultaneously for $x \in X$.

Proposition 2.31. *The mapping $f : X \longrightarrow \mathbf{P}^s$ defined by*

$$f(x) \;=\; [F_0(x), \ldots, F_s(x)]$$

is holomorphic. If $Y \subseteq \mathbf{P}^s$ is a complex submanifold, and if $\mathrm{Im}(f) \subseteq Y$, then f gives rise to a holomorphic mapping

$$f : X \longrightarrow Y. \qquad \square$$

Example 2.32. Define

$$\nu : \mathbf{P}^1 \longrightarrow \mathbf{P}^2 \;\text{ via }\; [s,t] \mapsto [s^2, t^2, st].$$

The image of ν is the conic $C = \{XY - Z^2 = 0\}$ discussed above, and in fact ν establishes an isomorphism $\mathbf{P}^1 \cong C$. It is a nice exercise to write down the inverse of ν.

Example 2.33. (PGL action on $\mathbf{P}^n$) Using Example 2.31 (or directly) one sees that

$$\mathrm{PGL}_{n+1}(\mathbf{C}) \;=_{\mathrm{def}}\; \frac{\mathrm{SL}_{n+1}(\mathbf{C})}{\mathbf{C}^* \cdot \mathrm{Id}}$$

acts on $\mathbf{P}^n$ by linear automorphisms. In fact, it turns out that $\mathrm{Aut}(\mathbf{P}^n) = \mathrm{PGL}_{n+1}(\mathbf{C})$.

Finally, we discuss products. Given projective manifolds

$$X \subseteq \mathbf{P}^r \quad , \quad Y \subseteq \mathbf{P}^s,$$

it is natural to ask how one can realize the product $X \times Y$ as a projective manifold. The essential point is to realize $\mathbf{P}^r \times \mathbf{P}^s$ projectively, and this is accomplished by the Segre embedding

$$\mathbf{P}^r \times \mathbf{P}^s \subseteq \mathbf{P}^{rs+r+s},$$

defined as follows. View $\mathbf{P}^{rs+r+s}$ as the projective space of all $(r+1) \times (s+1)$ matrices, so that a point in $\mathbf{P}^{rs+r+s}$ is described by giving a non-zero $(r+1) \times (s+1)$ matrix $[c_{i,j}]$ defined up to scalars. Then put

$$\sigma = \sigma_{r,s} : \mathbf{P}^r \times \mathbf{P}^s \longrightarrow \mathbf{P}^{rs+r+s} \quad , \quad [a_0, \ldots, a_r] \times [b_0, \ldots, b_s] \mapsto \big[\, a_i b_j \big]. \tag{2.6}$$

For example, when $r = s = 1$ the Segre mapping becomes

$$\mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^3 \quad , \quad [a_0, a_1] \times [b_0, b_1] \mapsto \begin{pmatrix} a_0 b_0 & a_0 b_1 \\ a_1 b_0 & a_1 b_1 \end{pmatrix}, \tag{2.7}$$

whose image is the (smooth) quadric in $\mathbf{P}^3$ arising as the set of all 2×2 matrices of rank 1.

Proposition 2.34. (Segre embedding). *The Segre mapping $\sigma_{r,s}$ defines a holomorphic embedding*

$$\mathbf{P}^r \times \mathbf{P}^s \subseteq \mathbf{P}^{rs+r+s}$$

whose image is the (projectivized) set of matrices of rank ≤ 1.

Intrinsically, given vector spaces V and W, one can describe the Segre embedding as the map

$$\mathbf{P}(V) \times \mathbf{P}(W) \hookrightarrow \mathbf{P}(V \otimes W) \quad , \quad [v] \times [w] \mapsto [v \otimes w].$$

(Recall that $V \otimes W$ consists of *linear combinations* of pure tensors $v \otimes w$; the Segre variety consists of the projectivization of the set of all elements of $V \otimes W$ that happen to be of the form $v \otimes w$ for some $v \in V$ and $w \in W$.)

Having defined an embedding $\mathbf{P}^r \times \mathbf{P}^s \subseteq \mathbf{P}^{rs+r+s}$, one can define an algebraic subset of $\mathbf{P}^r \times \mathbf{P}^s$ to be the intersection of $\mathbf{P}^r \times \mathbf{P}^s$ with an algebraic subset of $\mathbf{P}^{rs+r+s}$. However there is an easier "internal" way to work with these. Specifically, one says that a polynomial

$$P(Z, W) \;=\; p(Z_0, \ldots, Z_r, W_0, \ldots, W_s)$$

is bihomogeneous of bidegree (d, e) if P is homogeneous of degree d in the Z_i and homogeneous of degree e in the W_j. For example

$$Z_0^3 W_0 W_1 + Z_1^2 Z_2 W_2^2$$

is bihomogeneous of degree $(3, 2)$, but $Z_1 - W_1$ is not bihomogeneous. If P is bihomogeneous of bidegree (d, e), then

$$P(\lambda Z, \mu W) \;=\; \lambda^d \mu^e P(Z, W),$$

so the zero-locus of a bihomogeneous polynomial is a well-defined subseteq of $\mathbf{P}^r \times \mathbf{P}^s$. We can then define an algebraic subset of $\mathbf{P}^r \times \mathbf{P}^s$ to be the common zero-locus of a collection of bihomogeneous polynomials; the reader is asked to check in Exercise 2.17 that this is the same thing as intersecting the Segre variety with an algebraic subset of $\mathbf{P}^{rs+r+s}$.

There is a similar "internal" description of algebraic subsets of $\mathbf{P}^r \times \mathbf{C}^s$: these are described by polynomials $p(Z, w)$ that are homogeneous in the Z-variables, but arbitrary in the w-variables. Equivalently, $p(Z, w)$ is a homogeneous polynomial in Z whose coefficients are polynomials in w. (One could allow the coefficients to be holomorphic functions in the w.) The following example gives an important special case of this construction.

Example 2.35. (The blowing-up of $\mathbf{C}^{n+1}$ at the origin.) Recalling that $\mathbf{P}^n$ parametrizes one-dimensional subspaces of $\mathbf{C}^{n+1}$ we can form the *incidence correspondence* in $\mathbf{P}^n \times \mathbf{C}^{n+1}$ formed by pairs consisting of a point in $[a] \in \mathbf{P}^n$ and a vector $v \in \mathbf{C} \cdot a$ lying on the line $\mathbf{C} \cdot a$ corresponding to $[a]$:

$$Z \;=\; \big\{ ([a], v) \mid v \in \mathbf{C} \cdot a \big\} \;\subseteq \mathbf{P}^n \times \mathbf{C}^{n+1}.$$

This is an algebraic subset: in fact, $v = (v_0, \ldots, v_n)$ lies on the line spanned by $[a] = [a_0, \ldots, a_n]$ if and only if the vectors v and a are linearly dependent, so B is cut out in $\mathbf{P}^n \times \mathbf{C}^{n+1}$ by the 2×2 minors of the matrix

$$\begin{pmatrix} Z_0 & \ldots & Z_n \\ w_0 & \ldots & w_n \end{pmatrix}.$$

It is very interesting to consider the two projections

$$p : Z \longrightarrow \mathbf{P}^n \quad , \quad b : Z \longrightarrow \mathbf{C}^{n+1}.$$

For any point $[a] \in \mathbf{P}^n$, the fibre $p^{-1}([a])$ is the one-dimensional subspace $\mathbf{C} \cdot a$ spanned by a, and in fact Z is locally a product of $\mathbf{P}^n$ and $\mathbf{C}$. On the other hand, consider the fibres of b. If $0 \neq v \in \mathbf{C}^{n+1}$, then v lies on a unique line $\mathbf{C} \cdot v \subseteq \mathbf{C}^{n+1}$, and $b^{-1}(v)$ consists of the corresponding point of $\mathbf{P}^n$; in other words, b is an isomorphism away from 0. On the other hand, $b^{-1}(0)$ is all of $\mathbf{P}^n$. The mapping $b : Z \longrightarrow \mathbf{C}^{n+1}$ is called the *blowing-up* of $\mathbf{C}^{n+1}$ at the origin. Homotopically, Z is obtained from $\mathbf{C}^{n+1}$ by removing a small disk about 0, with boundary S^{2n+1}, and gluing in a copy of $\mathbf{P}^n$ via the Hopf map $h : S^{2n+1} \longrightarrow \mathbf{P}^n$. See Figure 3.

Holomorphic Line and Vector Bundles. Given a projective manifold, we have seen (Proposition 2.31) that one can use homogeneous polynomials to define holomorphic mappings to $\mathbf{P}^s$. But suppose one starts with an abstract compact complex manifold X. How then can one attempt to construct maps – and ideally embeddings – $\phi : X \longrightarrow \mathbf{P}^r$? The answer is that such morphisms are defined via sections of holomorphic line bundles; ϕ necessarily has the form

$$\phi(x) \;=\; [s_0(x), \ldots, s_r(x)]$$

where $s_i(x) \in \Gamma(X, L)$ are sections of a holomorphic line bundle without no common zeroes. In this subsection we give the basic definitions and examples of line and vector bundles.

Let X be a connected complex manifold. There are various equivalent definition of holomorphic line (or vector) bundles on X. We start with the most geometric, although it is not always the most useful in practice.

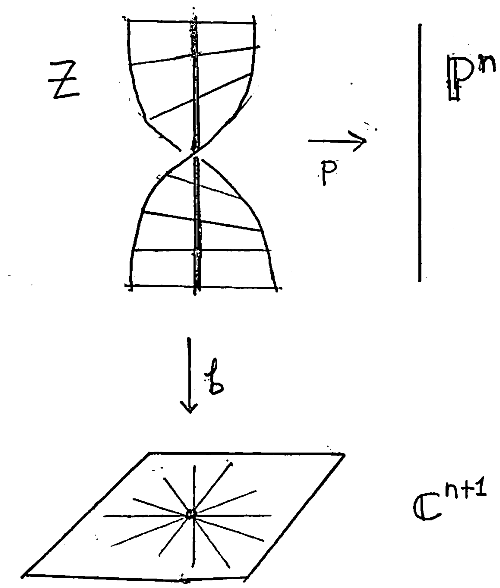

FIGURE 3. Blowing up the origin in $\mathbf{C}^{n+1}$

FIRST VIEWPOINT (TOTAL SPACE). Intuitively, we can define a holomorphic vector bundle on X to be a holomorphic family of vector spaces parametrized by X.

Definition 2.36. A holomorphic vector bundle on X of rank e is a complex manifold $\mathbf{E}$, together with a holomorphic mapping

$$p : \mathbf{E} \longrightarrow X,$$

with the property that p realizes $\mathbf{E}$ a locally the product of X with an e-dimension vector space in such a way that the transition functions are given by linear automorphisms.

The definition requires some elaboration. To begin with, we ask that there be given an open covering $\{U_i\}$ of X together with biholomorphic isomorphisms

$$\phi_i : p^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbf{C}^e$$

commuting with the projections to U_i:

$$\begin{tikzcd}
\phi_i : p^{-1}(U_i) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U_i \times \mathbf{C}^e \arrow[dl, "pr_1"] \\
& U_i. &
\end{tikzcd}$$

Now set

$$U_{ij} \ = \ U_i \cap U_j,$$

and consider the "j-to-i" transition map

$$\phi_{ij} \ = \ \phi_i \circ \phi_j^{-1} : U_{ij} \times \mathbf{C}^e \longrightarrow U_{ij} \times \mathbf{C}^e.$$

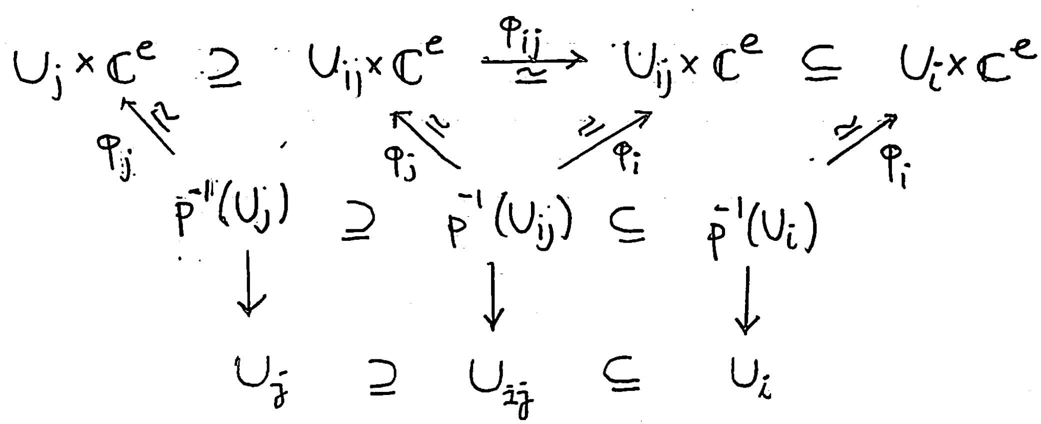

FIGURE 4. Diagram of Transition Functions

The second requirement is that ϕ_{ij} be given by

$$\phi_{ij}(x, v) \ = \ \big(x \,,\, g_{ij}(x) \cdot v\big),$$

where

$$g_{ij} : U_{ij} \longrightarrow \mathrm{GL}_e(\mathbf{C})$$

is a holomorphic mapping. In other words, the transition functions are given by a family of linear automorphisms

$$g_{ij}(x) : \mathbf{C}^e \longrightarrow \mathbf{C}^e$$

varying holomophically with x. The situation is summarized in a big commutative diagram shown in Figure 4. Note that it follows from the definitions that on the triple overlaps $U_{ijk} = U_i \cap U_j \cap U_k$, the transition matrices satisfy the *cocycle condition*:

$$g_{ij} \cdot g_{jk} \ = \ g_{ik}. \tag{2.8}$$

Definition 2.37. A vector bundle of rank $e = 1$ is called a *line bundle.* Note that in this case the $g_{ij}(x)$ are simply nowhere-vanishing holomorphic functions on U_{ij}, i.e.

$$g_{ij} \in \mathcal{O}^*(U_i \cap U_j).$$

Remark 2.38. These definitions make sense in any fixed geometric setting: one requires that the local product isomorphisms ϕ_i and the transition matrices g_{ij} be morphisms in the appropriate category. So for example one can discuss continuous real or complex vector bundles on a topological space, or $\mathcal{C}^\infty$ vector bundles (real or complex) on a smooth manifold. So for example, if M is a smooth manifold of dimension n, its tangent bundle TM is a real vector bundle of rank n. (See Exercise 2.18.)

Example 2.39. The global product $\mathbf{1}^e = \mathbf{1}^e_X = X \times \mathbf{C}^e$ is called the trivial vector bundle of rank e. As its name implies, this is not a very exciting example.

Example 2.40. **($\mathcal{O}_{\mathbf{P}^n}(-1)$).** As in Example 2.35, consider the incidence correspondence

$$\mathbf{P}^n \times \mathbf{C}^{n+1} \supseteq \mathbf{L} = \big\{([a], v) \mid v \in \mathbf{C} \cdot a\big\}.$$

Then the first projection $p : \mathbf{L} \longrightarrow \mathbf{P}^n$ realizes $\mathbf{L}$ a line bundle on $\mathbf{P}^n$, which is called (for reasons that will become apparent later) $\mathcal{O}_{\mathbf{P}^n}(-1)$.[9] We work out the transition functions for this bundle, assuming for ease of notation that $n = 1$. On $\mathbf{P}^1$ with homogeneous coordinates $[Z_0, Z_1]$ we consider as usual the open sets

$$U_0 = \big\{[1, \tfrac{Z_1}{Z_0}]\big\} \ \ , \ \ U_1 = \big\{[\tfrac{Z_0}{Z_1}, 1]\big\}$$

. We can trivialize p over these sets. In fact

$$\begin{aligned} U_0 \times \mathbf{C}^2 \supseteq p^{-1}(U_0) &= \Big\{\big(\,[1, \tfrac{Z_1}{Z_0}]\,,\, t \cdot (1, \tfrac{Z_1}{Z_0})\,\big)\Big\} \\ U_1 \times \mathbf{C}^2 \supseteq p^{-1}(U_1) &= \Big\{\big(\,[\tfrac{Z_0}{Z_1}, 1]\,,\, s \cdot (\tfrac{Z_0}{Z_1}, 1)\,\big)\Big\}, \end{aligned}$$

and then

$$\phi_0 : p^{-1}(U_0) \longrightarrow U_0 \times \mathbf{C} \ \ , \ \ \phi_1 : p^{-1}(U_1) \longrightarrow U_1 \times \mathbf{C}$$

are defined by

$$\begin{aligned} \phi_0\Big(\,[1, \tfrac{Z_1}{Z_0}]\,,\, t \cdot (1, \tfrac{Z_1}{Z_0})\Big) &= \big([1, \tfrac{Z_1}{Z_0}], t\big) \\ \phi_1\Big(\,[\tfrac{Z_0}{Z_1}, 1]\,,\, s \cdot (\tfrac{Z_0}{Z_1}, 1)\Big) &= \big([\tfrac{Z_0}{Z_1}, 1], s\big) \end{aligned}$$

In order to determine the transition function g_{10}, we compute the composition $\phi_1 \circ \phi_0^{-1}$:

$$\big([1, \tfrac{Z_1}{Z_0}], t\big) \mapsto \big([Z_0, Z_1], t \cdot (1, \tfrac{Z_1}{Z_0})\big) = \big([Z_0, Z_1], (\tfrac{Z_1}{Z_0} t) \cdot (\tfrac{Z_0}{Z_1}, 1)\big) \mapsto \big([\tfrac{Z_0}{Z_1}, 1], (\tfrac{Z_1}{Z_0} t)\big).$$

In other words,

$$g_{10}\big([Z_0, Z_1]\big) = \left(\frac{Z_1}{Z_0}\right).$$

Similarly, with respect to the standard open covering $U_i = \{Z_i \neq 0\}$ of $\mathbf{P}^n$, the transition functions of $\mathcal{O}_{\mathbf{P}^n}(-1)$ are given by

$$g_{ij} = \left(\frac{Z_i}{Z_j}\right). \tag{2.9}$$

Example 2.41. **($\mathcal{O}_{\mathbf{P}^n}(1)$).** In this example, we describe the *hyperplane line bundle* on $\mathbf{P}^n$, which is also called $\mathcal{O}_{\mathbf{P}^n}(1)$. Embed $\mathbf{P}^n$ into $\mathbf{P}^{n+1}$ as the hyperplane $Z_{n+1} = 0$, and set $O = [0, \ldots, 0, 1] \in \mathbf{P}^{n+1}$. Linear projection from O defines a mapping

$$p : \mathbf{P}^{n+1} - \{O\} \longrightarrow \mathbf{P}^n \ \ , \ \ [Z_0, \ldots, Z_{n+1}] \mapsto [Z_0, \ldots, Z_n].$$

Geometrically, p sends a point $x \in \mathbf{P}^{n+1} - \{O\}$ to the unique point of intersection of the line joining O and x with the hyperplane $\mathbf{P}^n \subseteq \mathbf{P}^{n+1}$. (See Figure 5.) The reader should check that p realizes $\mathbf{P}^{n+1} - \{O\}$ as the total space of a line bundle on $\mathbf{P}^n$, whose transition functions with respect to the standard open covering $\{U_i\}$ of $\mathbf{P}^n$ are given by

$$g_{ij} = \left(\frac{Z_j}{Z_i}\right). \tag{2.10}$$

[9] This is sometimes also called the Hopf line bundle, or the tautological line bundle (since its fibre over a point in $\mathbf{P}^n$ is "the line that that point is").

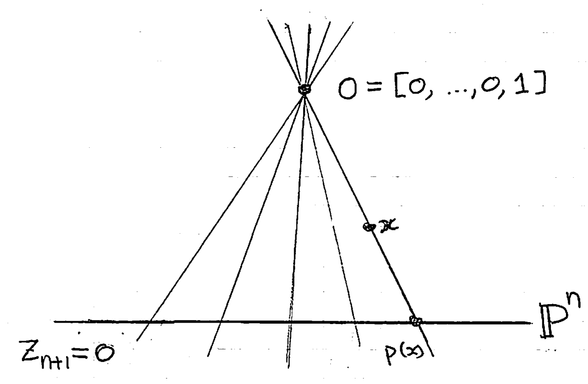

FIGURE 5. The hyperplane line bundle

Note that if $p : \mathbf{E} \longrightarrow X$ is a vector bundle, and $x \in X$ is any point, then the fibre $p^{-1}(x)$ has the structrue of a vector space (defined up to isomophism): this comes from any of the local trivializations of $\mathbf{E}$ and the fact that the comparison maps are linear isomorphisms.[10] The general yoga is that any intrinsically defined notion or operation on vector spaces makes sense for vector bundles. For example, if

$$p : \mathbf{E} \longrightarrow X \quad , \quad q : \mathbf{F} \longrightarrow X$$

are two vector bundles, then a *homomorphism* u from $\mathbf{E}$ to $\mathbf{F}$ is a holomorphic mapping

$$u : \mathbf{E} \longrightarrow \mathbf{F},$$

commuting with p and q, such that u is linear on each fibre. (The reader should convince him/herself that the linearity condition is well-defined.) Similarly, one can associate to $p : \mathbf{E} \longrightarrow X$ the dual bundle $\mathbf{E}^* \longrightarrow X$ whose fibre over a point $x \in X$ is the vector space of linear functionals on $p^{-1}(x)$. (See Exercise 2.22.)

SECOND VIEWPOINT (TRANSITION DATA): Another viewpoint of the theory – which is often more convenient for concrete calculations – is to take as basic objects the transition data $\{U_i, g_{ij}\}$ coming from local trivializations of $\mathbf{E}$ over an open covering $\{U_i\}$ o9f X. Note that if we're given such an open covering, together with invertible matrices

$$g_{ij} \in \mathrm{GL}_e(\mathcal{O}(U_{ij}))$$

satisfying the cocycle condition (2.8), then we can build a vector bundle $p : \mathbf{E} \longrightarrow X$ described by the given data. In fact, one starts with the disjoint union of the products $U_i \times \mathbf{C}^e$, and then uses the g_{ij} to glue these along the common open sets $U_{ij} \times \mathbf{C}^e$. Working with transition data offers a very concrete way of dealing with vector bundles, and we will make it a point to explicate all the constructions in these terms. Of course we will then also have to understand when two data $\{U_i, g_{ij}\}$ and $\{U_i, g'_{ij}\}$ give rise to isomorphic bundles.

[10]Compare Exercise 2.20.

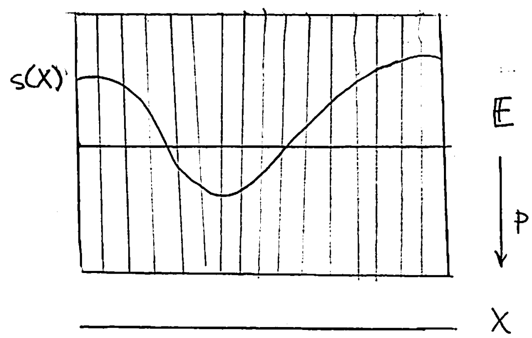

FIGURE 6. Section of a vector bundle

For our purposes, the most important feature of a bundle are its global sections:

Definition 2.42. Let $p : \mathbf{E} \longrightarrow X$ be a holomorphic vector bundle. A *global section* of $\mathbf{E}$ is a holomorphic mapping

$$s : X \longrightarrow \mathbf{E} \ \text{ such that } \ p \circ s = \mathrm{id}_X.$$

The condition on $p \circ s = \mathrm{id}_X$ guarantees that $s(x) \in p^{-1}(x)$ for every $x \in X$. Thus a section picks out a vector in each fibre of $\mathbf{E}$ over X. We denote by

$$\Gamma(X, \mathbf{E}) \ \text{ or } \ \Gamma(X, \mathcal{O}(\mathbf{E}))$$

the set of all global sections of $\mathbf{E}$. Thanks to the vector space structure on the fibres of p, we can add sections and multiply them by functions. Thus $\Gamma(X, \mathcal{O}(\mathbf{E}))$ is a module over the ring of $\mathcal{O}(X)$ of holomorphic functions on X; in particular, $\Gamma(X, \mathbf{E})$ is naturally a complex vector space. The zero section $0 = 0_{\mathbf{E}} \in \Gamma(X, \mathcal{O}(\mathbf{E}))$ is the section that picks out the zero vector in each fibre. See Figure 6.

Example 2.43. A section of the trivial line bundle $\mathbf{1}$ is just a holomorphic function on X.

Example 2.44. Working in the C^∞ setting, a section of the tangent bundle of a smooth manifold M is a vector field on M.

It is important to work out the meaning of sections in terms of transition data. Suppose that $p : \mathbf{E} \longrightarrow X$ is described by data $\{U_i, g_{ij}\}$: in other words, $\{U_i\}$ is an open cover of X on which p trivializes, and $g_{ij} \in \mathrm{GL}_e(\mathcal{O}(U_{ij}))$ are the corresponding transition matrices. Then the composition

$$\phi_i \circ s : U_i \longrightarrow p^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbf{C}^e$$

is the graph of a vector-valued holomorphic function

$$f_i \, : \, U_i \longrightarrow \mathbf{C}^e \, :$$

equivalently, f_i is a vector of holomorphic functions. A moment's thought shows that on the overlaps U_{ij}, these satisfy the basic compatibility relation

$$f_i \;=\; g_{ij} \cdot f_j. \tag{2.11}$$

It is very suggestive to think of the data $\{U_i, f_i\}$ as a sort of "twisted" vector-valued holomorphic function on X. If $\mathbf{E} = \mathbf{1}^e$ – so that one may take each g_{ij} to be the identity matrix – then (2.11) simply means that the f_j patch together to a global vector-valued holomorphic function on X, and of course when X is compact the f_i must therefore be constant. On the other hand, as we shall see shortly, the presence of non-trivial transition matrices g_{ij} raises the possibility of having many non-trivial holomorphic sections even in the compact setting.

The most important case of this principle occurs when $e = \text{rank } \mathbf{E} = 1$, in which case sections of suitable line bundles serve as substitutes for global holomorphic functions. This is illustrated by the basic

Proposition 2.45. *Global sections of $\mathcal{O}_{\mathbf{P}^n}(1)$ are identified with linear forms on $\mathbf{P}^n$, ie*

$$\Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big) \;\cong\; \big\{\text{linear forms in } Z_0, \ldots, Z_n\big\}.$$

Proof. For simplicity we will treat the case $n = 1$. Recall that in general the transtion functions of $\mathcal{O}_{\mathbf{P}^n}(1)$ with respect to the standard open covering $\{U_i\}$ of $\mathbf{P}^n$ are given by $g_{ij} = \frac{Z_j}{Z_i}$. In the case $n = 1$ this gives the one transition function

$$g_{10} \;=\; \frac{Z_0}{Z_1}.$$

Now say $s \in \Gamma(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$. Then $s|U_0$ and $s|U_1$ are given by entire functions $s_0(\frac{Z_1}{Z_0})$ and $s_1(\frac{Z_0}{Z_1})$ satisfying

$$s_1\left(\tfrac{Z_0}{Z_1}\right) \;=\; \left(\tfrac{Z_0}{Z_1}\right) \cdot s_0\left(\tfrac{Z_1}{Z_0}\right). \tag{*}$$

Writing

$$\begin{aligned} s_0 \;&=\; a_0 + a_1\left(\tfrac{Z_1}{Z_0}\right) + a_2\left(\tfrac{Z_1}{Z_0}\right)^2 + \ldots \\ s_1 \;&=\; b_0 + b_1\left(\tfrac{Z_0}{Z_1}\right) + b_2\left(\tfrac{Z_0}{Z_1}\right)^2 + \ldots, \end{aligned}$$

(*) immediately implies that

$$a_2 = a_3 = \ldots = 0 \quad , \quad b_2 = b_3 = \ldots = 0,$$

and

$$a_0 = b_1 \quad , \quad a_1 = b_0.$$

Thus s may be identified with the linear form

$$s \;=\; a_0 Z_0 + a_1 Z_1,$$

with $s_0 = \frac{s}{Z_0}$ and $s_1 = \frac{s}{Z_1}$. $\square$

Remark 2.46. (Finite dimensionality). Note that in particular $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ is a finite-dimensional vector space. This is a special case of a non-trivial theorem to the effect that if X is a compact complex manifold, then the space $\Gamma(X, \mathcal{O}(\mathbf{E}))$ of sections of any holomorphic vector bundle is finite-dimensional.

Suppose that $\mathbf{E} \longrightarrow X$ and $\mathbf{L} \longrightarrow X$ are respectively a vector bundle and a line bundle given by transition data $\{U_i, g_{ij}\}$ and $\{U_i, h_{ij}\}$. Then we may form the vector bundle

$$\mathbf{E} \otimes \mathbf{L} :$$

this is a bundle having the same rank as E, and transition matrices $h_{ij} \cdot g_{ij}$.[11] In particular for $k \geq 1$, the k-fold tensor product $\mathbf{L}^{\otimes k}$ is described by the transition functions $(h_{ij})^k$. For projective space, the computation of Proposition 2.45 generalizes to yield:

Proposition 2.47. *Set*

$$\mathcal{O}_{\mathbf{P}^n}(k) \ = \ \big(\mathcal{O}_{\mathbf{P}^n}(1)\big)^{\otimes k}.$$

Then

$$\Gamma\Big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k)\Big) \ \cong \ \big\{\text{homogeneous polynomials of degree } k \text{ in } Z_0, \ldots, Z_n\big\}.$$

If $\mathbf{L}$ is a line bundle given by data $\{U_i, g_{ij}\}$, put

$$\mathbf{L}^{-1} \ = \ \mathbf{L}^* \ \ , \ \ \mathbf{L}^{-k} \ = \ (\mathbf{L}^{-1})^{\otimes k}.$$

Then $\mathbf{L}^{-k}$ has transition functions g_{ij}^{-k}. Noting in particular that

$$\mathbf{L} \otimes \mathbf{L}^* \ \cong \ \mathbf{1},$$

we see that the set of all isomorphism classes of complex line bundles is an abelian group under tensor product: this is the *Picard group* $\mathrm{Pic}(X)$ of X.

We next discuss homomorphisms and isomorphisms from the viewpoint of transition data. Suppose that $p : \mathbf{E} \longrightarrow X$ and $q : \mathbf{F} \longrightarrow X$ are two bundles of ranks e and f, described by data $\{U_i, g_{ij}\}$ and $\{U_i, h_{ij}\}$, and consider a homomorphism $\nu : \mathbf{E} \longrightarrow \mathbf{F}$. Via the identifications

$$p^{-1}(U_i) \ = \ U_i \times \mathbf{C}^e \ \ , \ \ p^{-1}(V_i) \ = \ U_i \times \mathbf{C}^f,$$

we see that $\nu | U_i$ is described by a mapping

$$U_i \times \mathbf{C}^e \longrightarrow U_i \times \mathbf{C}^f \ \ , \ \ (x, v) \mapsto \big(x, \nu_i(x) \cdot v\big),$$

where $\nu_i(x)$ is an $e \times f$ matrix of holomorphic functions on U_i, or equivalently a family

$$\nu_i(x) : \mathbf{C}^e \longrightarrow \mathbf{C}^f$$

of linear transformations varying holomorphically with $x \in U_i$. On the overlaps $U_{ij} = U_i \cap U_j$, these must satisfy the compatibility condition

$$\nu_i \circ g_{ij} \ = \ h_{ij} \circ \nu_j. \tag{2.12}$$

[11] This follows from the fact that if E and L are vector spaces of ranks e and 1 respectively, and if

$$g : E \longrightarrow E \ \ , \ \ h : L \longrightarrow L$$

are linear transformations, then the induced linear transformation $E \otimes L \longrightarrow E \otimes L$ is given by $h \cdot g$.

Conversely, a collection of matrices $\{\nu_i(x)\}$ satisfying (2.12) determines a homorphism $\nu : \mathbf{E} \longrightarrow \mathbf{F}$.

We can now explicate when data $\{U_i, g_{ij}\}$ and $\{U_i, g'_{ij}\}$ determine isomorphic bundles. By the discussion of the previous paragraph, this happens if and only if there exist invertible matrices

$$\nu_i \in \mathrm{GL}_e(\mathcal{O}(U_{ij}))$$

of holomorphic functions such that

$$g_{ij} \ = \ \nu_i^{-1} \circ g'_{ij} \circ \nu_j$$

on U_{ij}. In particular, the bundle determined by data $\{U_i, g_{ij}\}$ is isomorphic to the trivial bundle if and only if

$$g_{ij} \ = \ \nu_i^{-1} \circ \nu_j.$$

Finally suppose that $f : Y \longrightarrow X$ is a holomorphic mapping. Then there is a natural way to pull back a holomorphic vector bundle $p : \mathbf{E} \longrightarrow X$ to arrive at a holomorphic bundle $f^*\mathbf{E} \longrightarrow Y$ on Y. For example, if $\mathbf{E}$ is described by transition data $\{U_i, g_{ij}\}$, we may describe $f^*\mathbf{E}$ by taking the open cover $\{f^{-1}(U_i)\}$, and using as transition matrices the pull-backs $f^*g_{ij} \in \mathrm{GL}_e(\mathcal{O}_Y(f^{-1}(U_{ij}))$. Geometrically, if $\mathbf{E}_x = p^{-1}(x)$ denotes the fibre of $\mathbf{E}$ over x, then $f^*(\mathbf{E})_y = \mathbf{E}_{f(y)}$.

Mappings to projective space. We close this section by discussing mappings to projective space.

Let X be a complex manifold, and $\mathbf{L}$ a holomorphic lind bundle on X. Suppose given global sections

$$s_0, \ldots, s_r \ \in \ \Gamma\big(X, \mathcal{O}(\mathbf{L})\big).$$

We ask that the s_i do not simultaneously vanish at any point $x \in X$. In other words

$$\text{Given any } x \in X, \text{ assume that there is at least one index } \alpha \text{ such that } s_\alpha(x) \neq 0. \tag{2.13}$$

We claim that then for any $x \in X$ the expression

$$[s_0(x), \ldots, s_r(x)]$$

determines a well-defined point in $\mathbf{P}^r$. In brief, we locally represent s_α by a function f_α in a neighborhood of x on which $\mathbf{L}$ trivializes, and set

$$[s_0(x), \ldots, s_r(x)] \ =_{\text{def}} \ [f_0(x), \ldots, f_r(x)]. \tag{*}$$

The f_α are not canonically defined, but thanks to (2.11) the vector arising from another choice of local representatives differs from the given one by multiplication by a non-zero scalar, so as a *homogeneous* vector the right-hand side of (*) is well-defined. With this understood, it is immediate that the mapping

$$\phi : X \longrightarrow \mathbf{P}^r \quad , \quad x \mapsto [s_0(x), \ldots, s_r(x)]$$

is holomorphic. Conversely, any holomorphic mapping $\phi : X \longrightarrow \mathbf{P}^r$ arises in this manner. In fact, writing $Z_0, \ldots, Z_r$ for the homogeneous coordinates on $\mathbf{P}^r$, one simply takes

$$\mathbf{L} \ = \ \phi^* \mathcal{O}_{\mathbf{P}^r}(1) \quad , \quad s_\alpha = \phi^*(Z_\alpha).$$

Given a compact complex manifold X, we can now begin to approach the question whether X admits a holomorphic embedding into some projective space. In order to produce such an embedding one needs to produce a holomorphic line bundle $\mathbf{L}$ on X that in the first place carries enough sections $s_\alpha \in \Gamma\big(X, \mathcal{O}(\mathbf{L})\big)$ so that (2.13) holds, and so that moreover the resulting holomorphic mapping

$$\phi : X \longrightarrow \mathbf{P}^r$$

actually be an embedding. It is a remarkable fact that in many ways the most appealing characterization of when this happens is differentio-geometric in nature. Specifically, one starts by choosing a Hermitian metric on the fibres of $\mathbf{L}$, which gives rise to a curvature form Θ, which is a closed two-form representing the so-called first Chern class of $\mathbf{L}$:[12]

$$[\Theta] \ = \ c_1(\mathbf{L}) \ \in \ H^2(X, \mathbf{Z}).$$

The first basic theorem – the Lefschetz $(1,1)$ Theorem (Theorem 6.29) – allows one under suitable hypotheses to identifiy which cohomology classes arise in this fashion. The second fundamental result – the Kodaira Embedding Theorem (Theorem 7.24) – states that if $\Theta = \Theta(\mathbf{L})$ satisfies a certain positivity hypothesis, then some large tensor power of $\mathbf{L}$ carries enough holomorphic sections to define an embedding. Putting these two theorems together, one arrives at a complete characterization of when a compact complex manifold admits a projective embedding: it should carry a Kähler metric with rational periods (Exercise 7.7).

Our main goal in the rest of the class is to prove these results. This requires that we study calculus on complex manifolds, to which we turn in the following sections.

Exercises for Section 2.

Exercise 2.1. Given complex manifolds X and Y, the product $X \times Y$ is a complex manifold in such a way that the two projections

$$X \times Y \longrightarrow X \quad , \quad X \times Y \longrightarrow Y$$

are holomorphic.

Exercise 2.2. Let X be a complex manifold of dimension $n \geq 2$, and fix a point $x \in X$. Then any holomorphic function on $X - \{x\}$ extends to a holomorphic function on all of X.

Exercise 2.3. Let $U \subseteq \mathbf{C}^n$ be an open set, and $f = f(z) \in \mathcal{O}(U)$ a non-zero holomorphic function. Denote by $X \subseteq U$ the complement of the zero-locus of f:

$$X \ = \ U \ - \ \{f = 0\} \ \subseteq \ U.$$

Show that X is isomorphic to a closed submanifold of $U \times \mathbf{C}$. (Hint: consider the function $f(z) \cdot t - 1 \in \mathcal{O}(U)[t]$.)

[12] We're ignoring some normalizing constants here

Exercise 2.4. Let Y be a complex manifold of dimension $n+k$. A subset $X \subseteq Y$ is a closed submanifold if and only if one can find an open covering $\{U_\alpha\}$ of Y, together with a k-tuple $\{f_\alpha\}$ of holomorphic functions on each U_α, such that

$$U_\alpha \cap X \;=\; \text{Zeroes}(f_\alpha),$$

and $J(f_\alpha)$ has maximal rank on $U_\alpha \cap X$.

Exercise 2.5. (Hyperelliptic Riemann surfaces) Working in $\mathbf{C}^2$ with coordinates z, w, let

$$X \;=\; \{w^2 - p_{2g+2}(z) = 0\}$$

be the curve defined as the zero-locus of a polynomial $w^2 - p_{2g+2}(z)$ where $p_{2g+2}(z)$ is a polynomial of degree $2g+2$ in z with distinct roots.

(i). Prove that the condition of Example 2.6 is met, and hence that X is a complex manifold of dimension 1.

(ii). By studying the projection

$$\pi : X \longrightarrow \mathbf{C} \quad , \quad (z, w) \mapsto z,$$

show that one can compactify X by adding two points "at infinity" to obtain a compact Riemann surface $\overline{X}$ in such a way that π extends to a holomorphic map

$$\overline{\pi} : \overline{X} \longrightarrow S = \mathbf{C} \cup \{\infty\}.$$

(Hint: Because p has even degree, the inverse image under π of the complement D^* of a disk of very large radius consists of the disjoint union of two copies of D^*. But D^* is a punctured neighborhood of the point at infinity, and therefore we need to add two points to compactify X.)

It is instrucitve to think through why $\overline{X}$ is diffeomorphic to a compact surface of genus g. When $g \geq 2$, such a curve or Riemann surface is called hyperelliptic.

Exercise 2.6. (Linear groups) Keeping the notation of Example 2.11, let

$$\det : \mathrm{GL}_n(\mathbf{C}) \longrightarrow \mathbf{C}^*$$

be the holomorphic function that takes a matrix to its determinant. Prove that the complex Jacobian of det is non-zero at any invertible matrix A. (Reduce to the case $A = \mathrm{Id}$.) Conclude that

$$\mathrm{SL}_n(\mathbf{C}) \;=_{\text{def}}\; \{A \mid \det A = 1\}$$

$\mathrm{GL}_n(\mathbf{C})$, and hence a complex Lie group. The complex orthogonal and symplectic groups can be treated similarly.

Exercise 2.7. (Covering spaces). Let

$$f : Y \longrightarrow X$$

be a covering space in the sense of topology.

(i), If X is a complex manifold, then Y carries a unique structure of a complex manifold such f is holomorphic and the deck transformations of f are holomorphic isomorphisms.

(ii). Conversely, if Y is a complex manifold, and the deck transformations of f are holomorphic, then there is a unique complex structure on X that renders f a holomorphic mapping.

Exercise 2.8. (Non-isomorphic tori). Let

$$\Lambda_0 \ = \ \mathbf{Z}^n + \sqrt{-1} \cdot \mathbf{Z}^n \ \subseteq \ \mathbf{C}^n,$$

and let $\Lambda \subseteq \mathbf{C}^n$ be a lattice spanned by $2n$ vectors $\lambda_j \in \mathbf{C}^n$. Assume that the $2n^2$ complex numbers appearing as coordinates of the λ_j are algebraically independent over $\mathbf{Q}$. Then Λ_0 and Λ determine non-isomorphic complex tori.

Exercise 2.9. Verify the assertions made in Example 2.16 concerning the construction of the Hopf manifold. I.e. prove that

$$X \ = \ \big(\mathbf{C}^n - \{0\}\big)/\mathbf{Z}$$

is indeed a complex manifold in such a way that the quotient mapping

$$\mathbf{C}^n - \{0\} \longrightarrow X$$

is holomorphic, and that $X \approx S^{2n-1} \times S^1$.

Exercise 2.10. Fill in the details in the proof of Proposition 2.21.

Exercise 2.11. Let $K = \mathbf{F}_q$ be a finite field with q elements. How many points does the finite set $\mathbf{P}^n(K)$ contain?

Exercise 2.12. Verify pictorially that the fibres of the Hopf fibration $h : S^3 \longrightarrow S^2$ are simply linked circles in S^3.

Exercise 2.13. Let $C \subseteq \mathbf{P}^2$ be the plane curve

$$Y^2Z - X^3 - X^2Z \ = \ 0.$$

As in Example 2.26, draw the restrictions of C to each of the affine coordinate charts U_Z, U_Y, U_X, and explain how the pictures fit together.

Exercise 2.14. Fill in the details of the proof of Proposition 2.30, including a proof of Euler's theorem. (For the latter, note that it suffices to check it for a single monomial.)

Exercise 2.15. Prove Proposition 2.31.

Exercise 2.16. Prove Proposition 2.34, that the Segre embedding realizes $\mathbf{P}^r \times \mathbf{P}^s$ as the sub manifold of $\mathbf{P}^{rs+r+s}$ arising as matrices of rank 1. (Write out what $\sigma_{r,s}$ looks like on the product of the standard coordinate patches in $\mathbf{P}^r$ and $\mathbf{P}^s$.)

Exercise 2.17. (Algebraic subsets of $\mathbf{P}^r \times \mathbf{P}^s$).

(i). Prove that a subset $Z \subseteq \mathbf{P}^r \times \mathbf{P}^s$ is the common zero-locus of a collection of bihomogeneous polynomials if and only if it is the intersection of the Segre variety with an algebraic subset of $\mathbf{P}^{rs+r+s}$. (Note that a homogenous polynomial of degree d

on $\mathbf{P}^{rs+r+s}$ pulls back under the Segre mapping to a bihomogeneous polynomial of bidegree (d, d), so one direction is clear.)

(ii). Find bihomogeneous polynomials defining the diagonal $\Delta \subseteq \mathbf{P}^r \times \mathbf{P}^r$.

Exercise 2.18. Let M be a smooth manifold of dimension n, and TM the tangent bundle of M. What are the transition matrices g_{ij} of TM?

Exercise 2.19. Verify that the transition functions for the hyperplane line bundle on $\mathbf{P}^n$ constructed in Example 2.41 are given as stated in (2.10)

Exercise 2.20. (A locally trivial affine bundle). Working in $\mathbf{P}^1 \times \mathbf{P}^1$, denote by $B \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ the complement of the diagonal:

$$B \ = \ \left(\mathbf{P}^1 \times \mathbf{P}^1\right) - \Delta,$$

and consider (either of) the projections

$$p : B \longrightarrow \mathbf{P}^1.$$

Taking the standard open subsets $U_0, U_1 \subseteq \mathbf{P}^1$, show that there are isomorphisms

$$\phi_i : p^{-1}(U_i) \ \overset{\cong}{\longrightarrow} \ U_i \times \mathbf{C}$$

commuting with the projections to U_i, but that p cannot be the total space of a line bundle over $\mathbf{P}^1$, i.e. one cannot arrange for the comparison maps to be given by linear isomorphisms.[13] (In fact, observe that B is isomorphic via the Segre to a sumanifold of $\mathbf{C}^3$, and deduce that p does not admit any sections.)

Exercise 2.21. Prove that there is a surjective map of vector bundles

$$\mathbf{1}^{n+1} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(1)$$

of vector bundles on $\mathbf{P}^n$.

Exercise 2.22. If $\mathbf{E} \longrightarrow X$ is a vector bundle described by transition data $\{U_i, g_{ij}\}$, find the transition matrices of the dual bundle $\mathbf{E}^* \longrightarrow X$.

Exercise 2.23. Prove Proposition 2.47, that sections of $\mathcal{O}_{\mathbf{P}^n}(k)$ are given by homogeneous polynomials of degree k.

Exercise 2.24. (Projective bundles). Let $p : \mathbf{E} \longrightarrow X$ be a vector bundle, say described by transition data $\{U_i, g_{ij}\}$. Then one can form the *projective bundle*

$$\pi : \mathbf{P}(\mathbf{E}) \longrightarrow X,$$

whose fibre over x is the projective space of one-dimensional subspaces of $E_x = p^{-1}(x)$. This may be constructed by starting with the products $U_i \times \mathbf{P}^{e-1}$, and using the g_{ij} – viewed as defining automorphisms of $\mathbf{P}^{e-1}$ – to glue them along $U_{ij} \times \mathbf{P}^{e-1}$.

(i). Show that if $\mathbf{L}$ is a line bundle, then

$$\mathbf{P}(\mathbf{E} \otimes \mathbf{L}) \cong \mathbf{P}(\mathbf{E}).$$

[13]Rather, they are given by affine isomorphisms $\mathbf{C} \longrightarrow \mathbf{C}$.

(ii). Show that there is a "tautological" line sub-bundle

$$\mathcal{O}_{\mathbf{P}(\mathbf{E})}(-1) \subseteq \pi^*(\mathbf{E}) :$$

if we view a point in $\mathbf{P}(E)$ as the choice of a point $x \in X$ together with a one-dimension subspace Λ of the fibre $\mathbf{E}_x = p^{-1}(x)$ then the fibre of $\mathcal{O}_{\mathbf{P}(\mathbf{E})}(-1)$ at (x, Λ) is Λ. What are the transition functions of this line bundle?

3. Complex and Hermitian Linear Algebra

In this section we discuss the linear algebra underlying calculus on complex manifolds. We follow closely the presentation of Huybrechts in [3, Chapter 1.2].

Almost complex structures on a vector space. Let V be a finite-dimensional real vector space.

Definition 3.1. An *almost complex structure* on V is an $\mathbf{R}$-linear endomorphism

$$J : V \longrightarrow V \quad \text{such that} \quad J^2 = \text{Id}.$$

Note that giving such a J is equivalent to equipping V with the structure of a complex vector space by taking

$$\sqrt{-1} \cdot v \ = \ J(v).$$

In particular, the existence of an almost complex structure on V implies that $\dim_{\mathbf{R}} V$ is even, say

$$\dim_{\mathbf{R}} V \ = \ 2n.$$

Remark 3.2. Suppose that V carries an almost complex structure J. Then one can find vectors $x_i \in V$ such that the vectors

$$\big\{ x_i \ , \ y_i =_{\text{def}} J(x_i) \, \big\}$$

form a basis for V (Exercise 3.1). With respect to this basis J is given by the block matrix

$$\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Given a real vector space V with $\dim_{\mathbf{R}} V = m$, set

$$V_{\mathbf{C}} \ =_{\text{def}} \ V \otimes_{\mathbf{R}} \mathbf{C}.$$

This is naturally a complex vector space of complex dimension m, which contains V as a (real) subspace. It is useful (if slightly abusive) to think of elements of $V_{\mathbf{C}}$ as having the form

$$v + \sqrt{-1} \cdot w \quad \text{for} \quad v, w \in V, \tag{3.1}$$

with the obvious expression for scalar multiplication by a complex number. Note that there is an operation of complex conjugation on $V_{\mathbf{C}}$, given by the rule

$$\overline{v + \sqrt{-1} \cdot w} \;=\; v - \sqrt{-1} \cdot w.$$

The real subspace $V \subseteq V_{\mathbf{C}}$ consists of those vectors invariant under complex conjugation.

Now suppose that $J : V \longrightarrow V$ is an almost complex structure. Then J extends to a $\mathbf{C}$-linear map

$$J_{\mathbf{C}} : V_{\mathbf{C}} \longrightarrow V_{\mathbf{C}} \quad , \quad J_{\mathbf{C}}(v + \sqrt{-1} \cdot w) = J(v) + \sqrt{-1} \cdot J(w).$$

By abuse of notation we will sometimes write simply J in place of $J_{\mathbf{C}}$.

The eigenvalues of this transformation are $\pm\sqrt{-1}$, and the corresponding eigenspaces play a crucial role in the story:

Definition 3.3. We set

$$\begin{aligned} V^{1,0} \;&=\; \big\{ \mu \in V_{\mathbf{C}} \mid J\mu = \sqrt{-1}\mu \big\} \\ V^{0,1} \;&=\; \big\{ \mu \in V_{\mathbf{C}} \mid J\mu = -\sqrt{-1}\mu \big\} \end{aligned}$$

These are complex subspaces of $V_{\mathbf{C}}$.

Lemma 3.4. *There is a canonical direct sum decomposition:*

$$V_{\mathbf{C}} \;=\; V^{1,0} \oplus V^{0,1}$$

of complex vector spaces. The corresponding projections

$$\pi^{1,0} : V_{\mathbf{C}} \longrightarrow V^{1,0} \quad , \quad \pi^{0,1} : V_{\mathbf{C}} \longrightarrow V^{0,1}$$

are given by

$$\begin{aligned} \pi^{1,0}(\mu) \;&=\; \tfrac{1}{2}\big(\mu - \sqrt{-1} \cdot J(\mu)\big) \\ \pi^{0,1}(\mu) \;&=\; \tfrac{1}{2}\big(\mu + \sqrt{-1} \cdot J(\mu)\big). \end{aligned}$$

Finally, complex conjugation defines an $\mathbf{R}$-linear isomorphism $V^{1,0} \cong_{\mathbf{R}} V^{0,1}$, ie

$$\overline{V^{1,0}} \;=\; V^{0,1}.$$

Again we leave the proof for the reader (Exercise 3.2).

Note that $J_{\mathbf{C}} : V_{\mathbf{C}} \longrightarrow V_{\mathbf{C}}$ satisfies $J_{\mathbf{C}}^2 = \mathrm{Id}$, and hence defines a complex structure on $J_{\mathbf{C}}$. However this is *not* the complex structure defined via (3.1), which is the one we always use. In fact, by definition $J_{\mathbf{C}}$ acts by multiplication by $\sqrt{-1}$ on the subspace $V^{1,0} \subseteq V_{\mathbf{C}}$, but it acts by multiplication by $-\sqrt{-1}$ on $V^{0,1} \subseteq V_{\mathbf{C}}$. In fact:

Proposition 3.5. *The composition*

$$V \;\subseteq\; V_{\mathbf{C}} \xrightarrow{\pi^{1,0}} V^{1,0}$$

gives rise to a $\mathbf{C}$-linear isomorphism

$$(V, J) \;\cong\; (V^{1,0}, \cdot\sqrt{-1})$$

of complex vector spaces. On the other hand, the composition

$$V \ \subseteq \ V_{\mathbf{C}} \xrightarrow{\pi^{0,1}} V^{0,1}$$

determines a complex antilinear *isomorphism between* (V, J) *and* $(V^{0,1}, \cdot\sqrt{-1})$.

It is useful to have explicit descriptions of bases. As in Remark 3.2, consider a basis of V of the form $\{x_i, y_i = J(x_i)\}$. Then

$$\begin{aligned} z_i \ &=_{\text{def}} \ \tfrac{1}{2}\big(x_i - \sqrt{-1}\cdot y_i\big) \ \in \ V^{1,0} \\ \overline{z}_i \ &=_{\text{def}} \ \tfrac{1}{2}\big(x_i + \sqrt{-1}\cdot y_i\big) \ \in \ V^{0,1} \end{aligned} \tag{3.2}$$

are $\mathbf{C}$-bases of the two spaces in question, and in particular $\{z_i, \overline{z}_i\}$ is a complex basis for $V_{\mathbf{C}}$.[14]

We now turn to forms. Given a real vector space V, write as usual

$$V^* \ = \ \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{R})$$

for the dual space of linear functionals on V. Put $(V^*)_{\mathbf{C}} = V^* \otimes_{\mathbf{R}} \mathbf{C}$. We may view elements of this space as functionals of the form $f + \sqrt{-1}\cdot g$ for $f, g \in V^*$, and in particular one has identifications

$$(V^*)_{\mathbf{C}} \ = \ \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) \ = \ \mathrm{Hom}_{\mathbf{C}}(V_{\mathbf{C}}, \mathbf{C}) \ = \ V_{\mathbf{C}}^*.$$

Now suppose that $J : V \longrightarrow V$ is an almost complex structure on V. then J induces an almost complex structure

$$J^* : V^* \longrightarrow V^* \quad , \quad J^*(f)(v) \ = \ f\big(J(v)\big).$$

Then as before we get an eigenspace decomposition $(V^*)_{\mathbf{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$, where

$$\begin{aligned} (V^*)^{1,0} \ &= \ \big\{ f \in \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) \mid f(Jv) = \sqrt{-1}\cdot f(v) \big\} \\ (V^*)^{0,1} \ &= \ \big\{ f \in \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) \mid f(Jv) = -\sqrt{-1}\cdot f(v) \big\} \end{aligned} \tag{3.3}$$

Note that

$$(V^*)^{1,0} \ \cong \ \mathrm{Hom}_{\mathbf{C}-\mathrm{lin}}\big((V, J), \mathbf{C}\big) \quad , \quad (V^*)^{0,1} \ \cong \ \mathrm{Hom}_{\mathbf{C}-\mathrm{antilin}}\big((V, J), \mathbf{C}\big).$$

If we choose a basis $\{x_i, y_i = J(y_i)\}$ for V as above, and denot by x^i, y^i the corresponding dual basis of V^*, then

$$z^i \ = \ x^i + \sqrt{-1}y^i \quad , \quad \overline{z}^i \ = \ x^i - \sqrt{-1}y^i \tag{3.4}$$

are bases for $(V^*)^{1,0}$ and $(V^*)^{0,1}$ respectively, dual to the bases $\{z_i\}$ and $\{\overline{z}_i\}$ appearing in (3.2).

[14]In effect we have seen these expressions before. If we think of V as the (real) tangent space to a complex manifold, then (3.2) corresponds to the definitions

$$\frac{\partial}{\partial z_i} \ = \ \frac{1}{2}\Big(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\Big) \quad , \quad \frac{\partial}{\partial \overline{z}_i} \ = \ \frac{1}{2}\Big(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\Big).$$

Next we turn to exterior products. Goven as before a real vector space V with $\dim_{\mathbf{R}} V = 2n$, we consider the real and complex vector spaces

$$\Lambda^k V \quad , \quad \Lambda^k V_{\mathbf{C}},$$

of real and complex dimensions $\binom{2n}{k}$ respectively. Now suppose that $J : V \longrightarrow V$ is an almost complex structure, giving rise to the decomposition

$$V_{\mathbf{C}} \ = \ V^{1.0} \oplus V^{0,1}.$$

Then there is a canonical isomorphism:

$$\begin{aligned} \Lambda^k V_{\mathbf{C}} \ &= \ \Lambda^k\big(V^{1,0} \oplus V^{0,1}\big) \\ &= \ \bigoplus_{p+q=k} \big(\Lambda^p(V^{1,0}) \otimes \ \Lambda^q(V^{0,1})\big) \end{aligned}$$

as complex vector spaces. We define

$$\Lambda^{p,q} V \ = \ V^{p,q} \ = \ \Lambda^p(V^{1,0}) \otimes \Lambda^q(V^{0,1}), \tag{3.5}$$

so that

$$\Lambda^k V \ = \ \bigoplus_{p+q=k} V^{p,q}. \tag{3.6}$$

Explicitly, if

$$\alpha_1, \ldots, \alpha_p \in V^{1,0} \ \text{ and } \ \beta_1, \ldots, \beta_q \in V^{0,1},$$

then

$$\alpha_1 \wedge \ldots \wedge \alpha_p \wedge \beta_1 \wedge \ldots \wedge \beta_q \ \in \ V^{p,q},$$

and an arbitrary element of $V^{p,q}$ is a $\mathbf{C}$-linear combination of terms of this form. As for the decomposition (3.6), starting with $\mu_1, \ldots, \mu_k \in V_{\mathbf{C}}$, start by writing

$$\mu_i \ = \ \mu_i^{1,0} \ + \ \mu_i^{0,1}$$

as a sum of vectors of types $(1,0)$ and $(0,1)$. Then expand out the wedge product

$$\mu_1 \wedge \ldots \wedge \mu_k \ = \ (\mu_1^{1,0} + \mu_1^{0,1}) \wedge \ldots \wedge (\mu_k^{1,0} + \mu_k^{0,1})$$

via linearity, and for each (p,q) with $p+q=k$, collect together those terms that are wedges of p vectors of type $(1,0)$ and q vectors of type $(0,1)$. Note that

$$\overline{V^{p,q}} \ = \ V^{q,p}.$$

In terms of the basis $\{z^i, \overline{z}^i\}$ appearing in (3.4), $(V^*)^{pq}$ has as basis the elements

$$z^{i_1} \wedge \ldots \wedge z^{i_p} \wedge \overline{z}^{j_1} \wedge \ldots \wedge \overline{z}^{j_q}$$

with $i_1 < \ldots < i_p$, $j_1 < \ldots < j_q$. Finally, consider the endomorphism

$$\Lambda^k J_{\mathbf{C}} : \Lambda^k V_{\mathbf{C}} \longrightarrow \Lambda^k V_{\mathbf{C}}$$

determined by J. Then the reader should verify that:

$$\Lambda^k J_{\mathbf{C}} \ \text{ acts on } \Lambda^{p,q} V \text{ by multiplication by } (\sqrt{-1})^{p-q}. \tag{3.7}$$

Metrics and the fundamental form. We now introduce metrics. Focusing still on a finite-dimensional real vector space V, assume now that we are given a positive definite inner product $<\,,\,>$ on V.

Definition 3.6. An almost complex structure J on V is *compatible* with $<\,,\,>$ if

$$< Jv, Jw > = < v, w > \quad \text{for all} \quad v, w \in V.$$

In other words, we ask that $J \in \mathrm{O}(V, <\,,\,>)$.

Note that a compatible complex structure J on $(V, <\,,\,>)$ determines an orientation on V: choosing vectors $x_1, \ldots, x_n \in V$ such that

$$\{x_1, Jx_1, \ldots, x_n, Jx_n\} \tag{*}$$

is a basis of V, we require that the ordered basis (*) determine the positive orientation of V.

Example 3.7. It is instructive to consider the case $\dim_{\mathbf{R}}(V) = 2$. If we fix an orientation on V, then a Euclidean structure $<\,,\,>$ determines a unique compatible almost complex structure. In fact, choose any non-zero vector $v \in V$, and define J by requiring that

$$< v, Jv >= 0 \quad , \quad ||v|| = ||Jv|| = 1 \quad , \quad J^2(v) = -v$$

and that $\{v, Jv\}$ be a positively oriented basis of V. Conversely, if J is an almost complex structure on V, then up to scalar multiples there is a unique Euclidean inner product $<\,,\,>$ on V with respect to which J is compatible and $\{v, Jv\}$ gives a positive orientation. See also Exercise 3.3.

Fix a Euclidean space $(V, <\,,\,>, J)$ with a compatible almost complex structure. These data are combined in the crucial

Definition 3.8. The *fundamental form* associated to $<\,,\,>$ and J is the two-form $\omega \in \Lambda^2 V^*$ defined by

$$\omega(v, w) \ = < Jv, w > = \ - < v, Jw > .$$

Proposition 3.9. *The form ω is in fact alternating, and under the inclusion*

$$\Lambda^2 V^* \ \subseteq \ \Lambda^2 V^*_{\mathbf{C}},$$

ω has type $(1,1)$, ie:

$$\omega \ \in \ \Lambda^2 V^* \cap \Lambda^{1,1} V^*.$$

Proof. One has

$$\omega(w, v) \ = < Jw, v > = < J^2 w, Jv > = < -w, Jv > = \ - < Jv, w > = \ \omega(v, w),$$

so ω is alternating. For the second statement, observe that ω has type $(1,1)$ if and only if $\omega(Jv, Jw) = \omega(v, w)$ (Exercise 3.4). But this is clear, since:

$$\omega(Jv, Jw) \ = < J^2 v, Jw > = < Jv, w > = \ \omega(v, w)$$

thanks to the fact that J is compatible with $<\,,\,>$. $\square$

We now discuss Hermitian forms arising from the data $(V, J, < , >)$ of an almost complex structure compatible with a Euclidean inner product. Define

$$\begin{aligned} h(v,w) &= < v, w > \; - \; \sqrt{-1} \cdot \omega(v,w) \\ &= < v, w > \; - \; \sqrt{-1} \cdot < Jv, w > . \end{aligned} \tag{3.8}$$

This is a Hermitian form on V with respect to the complex structure given by J.[15] For another viewpoint, denote by $< , >_{\mathbf{C}-\mathrm{lin}}$ the $\mathbf{C}$-linear extension of $< , >$ to $V_{\mathbf{C}}$. Then we can extend $< , >$ to a Hermitian inner product $< , >_{\mathrm{herm}}$ on $V_{\mathbf{C}}$ by setting

$$< \mu, \nu >_{\mathrm{herm}} = < \mu, \overline{\nu} >_{\mathbf{C}-\mathrm{lin}} .$$

Then under the isomorphism $(V, J) \cong (V^{1,0}, \cdot\sqrt{-1})$ given by Proposition 3.5, one has

$$\tfrac{1}{2} \cdot h(\; , \;) \; = < , >_{\mathrm{herm}} | V^{1,0}. \tag{3.9}$$

It is convenient to write these definitions explicitly in terms of coordinates. Choose a basis $\{x_i, y_i = J(x_i)\}$ for V as before, and set

$$h_{ij} \; = \; h(x_i, x_j).$$

Then (h_{ij}) is an Hermitian matrix, and

$$h(y_i, y_j) \; = \; h_{ij} \quad , \quad h(x_i, y_j) \; = \; -\sqrt{-1} \cdot h_{ij} \quad , \quad h(y_i, x_j) \; = \; \sqrt{-1} \cdot h_{ij}. \tag{3.10}$$

Moreover, by definition

$$\omega \; = \; -\mathrm{Im}(h) \quad , \quad <,> \; = \; \mathrm{Re}(h),$$

and consequently

$$\begin{aligned} \omega(x_i, x_j) \; = \; \omega(y_i, y_j) \; &= \; -\mathrm{Im}(h_{ij}) \\ \omega(x_i, y_j) \; &= \; \mathrm{Re}(h_{ij}), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} < x_i, x_j > \; = \; < y_i, y_j > \; &= \; \mathrm{Re}(h_{ij}) \\ < x_i, y_j > \; &= \; \mathrm{Im}(h_{ij}). \end{aligned} \tag{3.12}$$

In particular, writing x^i, y^i for the basis dual to x_i, y_i, it follows that

$$\omega \; = \; -\sum_{i<j} \mathrm{Im}(h_{ij}) \cdot \left(x^i \wedge x^j + y^i \wedge y^j \right) \; + \; \sum_{i,j=1}^{n} \mathrm{Re}(h_{ij}) \cdot x^i \wedge y^j. \tag{3.13}$$

Finally, as in (3.2), put $z_i = \frac{1}{2}(x_i - \sqrt{-1}y_i)$ and $\overline{z}_i = \frac{1}{2}(x_i + \sqrt{-1}y_i)$, with dual basis $z^i, \overline{z}^i$. Then (3.11) leads to the basic expression

$$\omega \; = \; \frac{\sqrt{-1}}{2} \cdot \sum_{i,j=1}^{n} h_{ij} \cdot z^i \wedge \overline{z}^j. \tag{3.14}$$

[15] We take Hermitian forms to be $\mathbf{C}$-linear in the first argument, and conjugate linear in the second.

The Hodge $*$-operator. Before proceeding, we pause to recall some facts about the Hodge $*$-operator. Let W be a real vector space of dimension d equipped with a positive definite inner product $g =< , >$. Giving $< , >$ is the same thing as giving a symmetric linear mapping

$$T : W \longrightarrow W^* \quad , \quad T(v)(w) \ =< v, w >$$

with the property that $T(v)(v) > 0$ for all $0 \neq v \in W$. On the other hand, T determines by fuctoriality a symmetric mapping

$$\Lambda^k T : \Lambda^k W \longrightarrow \Lambda^k W^*$$

satisfying the analogous positivity property. In other words, an inner product $< , >$ on W induces one on each exterior product $\Lambda^k W$, which we continue to denote by $< , >$. Concretely, if

$$e_1, \ldots, e_d \ \in \ W \tag{*}$$

is an orthonormal basis for W, then $< , >$ is defined on $\Lambda^k W$ by taking as an orthonormal basis the multi-vectors

$$e_I \ = \ e_{i_1} \wedge \ldots \wedge e_{i_k} \ \text{ for } \ I = \{ i_1 < \ldots < i_k \}.$$

We also fix the volume form

$$\text{vol} \ = \ e_1 \wedge \ldots \wedge e_d \ \in \ \Lambda^d W,$$

arising from the orthonormal basis (*).

There is a natural non-degenerate pairing

$$\Lambda^k W \otimes \Lambda^{d-k} W \longrightarrow \Lambda^d W \quad , \quad \alpha \otimes \beta \mapsto \alpha \wedge \beta,$$

and we have fixed a basis vol for the one-dimensional space on the right. We can then define the Hodge $*$-operator

$$* : \Lambda^k W \longrightarrow \Lambda^{d-k} W \quad , \quad \beta \mapsto *\beta$$

by requiring that

$$\alpha \wedge *\beta \ =< \alpha, \beta > \cdot \, \text{vol} \tag{3.15}$$

for every $\alpha, \beta \in \Lambda^k W$. This extends by linearity to a map

$$* : \Lambda^* W \longrightarrow \Lambda^* W.$$

The basic properties of this operator are summarized in the following Proposition, whose proof we leave to the reader.

Proposition 3.10. *Fix as above an orthonormal basis $e_1, \ldots, e_d \in W$.*

(i). *If $I = \{i_1 < \ldots < i_k\}$ and $J = \{j_1 < \ldots < j_{d-k}\}$ are disjoint multi-indices such that $I \cup J = [1, d]$, then*

$$* \, e_I \ = \ \sigma_{I,J} \cdot e_J,$$

where $e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$, with e_J defined similarly, and $\sigma_{I,J}$ is the sign of the permutation (I, J).

(ii). *Given* $\alpha \in \Lambda^k W, \beta \in \Lambda^{d-k} W$, *one has*

$$< \alpha, *\beta > = \ (-1)^{k(d-k)} < *\alpha, \beta > .$$

(iii). *If* $\alpha \in \Lambda^k W$, *then*

$$* * \alpha \ = \ (-1)^{k(d-k)} \cdot \alpha.$$

(iv). *The operator* $*$ *is an isometry of* $\big(\Lambda^* W, < , > \big)$.

Now let $(V, < , >, J)$ be as above a real Euclidean space of dimension $2n$ with a compatible almost complex structure. We apply the previous discussion to $W = V^*$, which inherits an inner product $< , >$ and orientation from V and J. In particular, one then gets an inner product $< , >$ on $\Lambda^* V^*$, and the Hodge $*$-operator

$$* : \Lambda^* V^* \longrightarrow \Lambda^* V^* \tag{3.16}$$

is defined. Moreover, the inner product $< , >$ on V^* extends to a positive definite Hermitian form $< , >_{\text{herm}}$ on $V^*_{\mathbf{C}}$, which then determines a positive definite Hermitian inner product $< , >_{\text{herm}}$ on $\Lambda^* V^*_{\mathbf{C}}$. Thanks to the fact (Exercise 3.5) that the subspaces

$$V^{*1,0} \ , \ V^{*0,1} \ \subseteq V^*_{\mathbf{C}}$$

are orthogonal with respect to $< , >_{\text{herm}}$, one finds that

$$\Lambda^k V^* \ = \ \oplus_{p+q=k} \Lambda^{p,q} V^*_{\mathbf{C}}$$

is an orthogonal decomposition with respect to $< , >_{\text{herm}}$. Finally, the Hodge $*$-operator (3.16) extends $\mathbf{C}$-linearly to

$$* : \Lambda^k V^*_{\mathbf{C}} \longrightarrow \Lambda^{2n-k} V^*_{\mathbf{C}},$$

and satisfies

$$\alpha \wedge *\beta \ = < \alpha, \overline{\beta} >_{\text{herm}} \cdot \text{vol}. \tag{3.17}$$

Observe that if $\beta \in \Lambda^{p,q} V^*$ and $\alpha \in \Lambda^{p',q'} V^*$, then

$$< \alpha, \overline{\beta} >_{\text{herm}} = \ 0 \ \text{ unless } \ p' = q \ , \ q' = p.$$

It follows that

$$*\big(\Lambda^{p,q} V^*_{\mathbf{C}}\big) \ \subseteq \Lambda^{n-q,n-p} V^*_{\mathbf{C}}.$$

(Note the switch in indices!)

Example 3.11. Let $\{x_i, y_i = Jx_i\}$ be an orthonormal basis of V, and let $\{z^i, \overline{z}^i\}$ be the corresponding bases of $\Lambda^{1,0} V^*_{\mathbf{C}}$ and $\Lambda^{0,1} V^*_{\mathbf{C}}$. Given disjoint subsets $I, J, K \subseteq [1, n]$, consider a form of the type

$$\alpha \ = \ z^I \wedge \overline{z}^I \wedge z^J \wedge \overline{z}^K.$$

Then

$$*\alpha \ = \ (\text{constant}) \cdot z^L \wedge \overline{z}^L \wedge z^J \wedge \overline{z}^K,$$

where L is the complement of $I \cup J \cup K$ in $[1, n]$ (Exercise 3.9).

Example 3.12. One has $*1 = \text{vol}$, $*\text{vol} = 1$, and

$$* \frac{\omega^j}{j!} \;=\; \frac{\omega^{n-j}}{(n-j)!}$$

(Exercise 3.9). In particular, this shows again that $\text{vol} = \frac{1}{n!} \cdot \omega^n$ (Exercise **??**).

Lefschetz operator and decomposition. Consider once again a triple $(V, <\,,\,>, J)$ consisting of a real Euclidean space of dimension $2n$ and a compatible almost complex structure. Denote by

$$\omega \;\in\; \Lambda^2 V^*$$

the corresponding fundamental form.

Definition 3.13. The *Lefschetz operator* L associated to ω is the operator given by wedge product with ω:

$$L : \Lambda^k V^* \longrightarrow \Lambda^{k+2} V^* \quad , \quad \alpha \mapsto \omega \wedge \alpha.$$

The *dual* or *adjoint* Lefschetz operator is the operator

$$\Lambda : \Lambda^{k+2} V^* \longrightarrow \Lambda^k V^*$$

determined by requiring that

$$< \Lambda\alpha, \beta > = < \alpha, L\beta > \quad \text{for all } \beta \in \Lambda^k V^*.$$

We sometimes view L and its adjoint Λ as operators on the full exterior algebra of V:

$$L : \Lambda^* V^* \longrightarrow \Lambda^* V^* \quad , \quad \Lambda : \Lambda^* V^* \longrightarrow \Lambda^* V^*.$$

Note that it follows from Proposition 3.10 that the Hodge $*$-operator is invertible. We can then use it to express Λ in terms of L:

Proposition 3.14. *One has*

$$\Lambda \;=\; *^{-1} \circ L \circ *.$$

Proof. Since

$$< \Lambda\alpha, \beta > = < \alpha, L\beta > = < L\beta, \alpha >,$$

the issue is to show that if $\alpha \in \Lambda^{k+2} V^*$ and $\beta \in \Lambda^k V^*$, then

$$< L\beta, \alpha > \cdot\text{vol} \;= < \beta, *^{-1} L * \alpha > \cdot\text{vol}.$$

But

$$\begin{aligned} < \beta, *^{-1} L * \alpha > \cdot\text{vol} \;&=\; \beta \wedge L(*\alpha) \\ &=\; \beta \wedge \omega \wedge *\alpha \\ &=\; \omega \wedge \beta \wedge *\alpha \\ &= < L\beta, \alpha > \cdot\text{vol}, \end{aligned}$$

as required. □

Passing to $V_{\mathbf{C}}$, one extends L and Λ by $\mathbf{C}$-linearity to

$$L : \Lambda^k V_{\mathbf{C}}^* \longrightarrow \Lambda^{k+2} V_{\mathbf{C}}^* \quad , \quad \Lambda : \Lambda^{k+2} V_{\mathbf{C}}^* \longrightarrow \Lambda^k V_{\mathbf{C}}^*.$$

These are adjoint with respect to $< , >_{\text{herm}}$, ie

$$< \Lambda\alpha, \beta >_{\text{herm}} = < \alpha, L\beta >_{\text{herm}},$$

and the equality in Proposition 3.14 remains valid. Moreover, as $\omega \in \Lambda^{1,1}V^*$, one finds that

$$L\big((V^*)^{pq}\big) \ \subseteq \ (V^*)^{p+1,q+1} \quad , \quad \Lambda\big((V^*)^{p+1,q+1}\big) \ \subseteq \ (V^*)^{pq}.$$

We now come to the Lefschetz decomposition. The first point is to define primitive forms:

Definition 3.15. (Primitive vectors). A k-vector $\alpha \in \Lambda^k V^*$ is *primitive* if

$$\Lambda(\alpha) \ = \ 0.$$

We denote by $P^k \subseteq \Lambda^k V^*$ the subspace of primitive elements.

We will see shortly that $\alpha \in \Lambda^k V^*$ is primitive if and only if $k \leq n$ and $L^{n-k+1}(\alpha) = 0$. One uses the same definition to define the subspace of complex primitive forms $P^k_{\mathbf{C}} \subseteq \Lambda^k V^*_{\mathbf{C}}$.

The basic result is then

Theorem 3.16. *As before let $(V, < , >, J)$ be a Euclidean real vector space of dimension $2n$ together with a compatible almost complex structure.*

(i). *One can express $\Lambda^k V^*$ as a $< , >$-orthoginal direct sum*

$$\Lambda^k V^* \ = \ \bigoplus_{i \geq 0} L^i P^{k-2i}.$$

In other words, any $\alpha \in \Lambda^k V^*$ can be expressed uniquely as a sum

$$\alpha \ = \ \alpha_k \ + \ L\alpha_{k-2} \ + \ L^2\alpha_{k-4} + \ldots + L^{[\frac{k}{2}]}\alpha_{k-2[\frac{k}{2}]},$$

where the $\alpha_j \in P^j$ are primitive forms.

(ii). *The mapping*

$$L^{n-k} : \Lambda^k V^* \longrightarrow \Lambda^{2n-k} V^*$$

is an isomorphism.

(iii). *If $k > n$ then $P^k = 0$, while if $k \leq n$ then*

$$P^k \ = \ \{\alpha \in \Lambda^k V^* \mid L^{n-k+1}\alpha = 0\}.$$

The statement in (i) is the *Lefschetz decomposition*. The assertion in (ii) will later on lead to the Hard Lefschetx theorem. The next few pages of this subsection will be devoted to the proof of the Theorem.

Example 3.17. When $k = 2$ the Lefschetz decomposition is

$$\Lambda^2 V^* \ = \ P^2 \oplus \mathbf{R} \cdot \omega.$$

For $k = 3$, it takes the form $\Lambda^3 V^* = P^3 \oplus \omega \cdot V^*$.

Remark 3.18. In the global setting, when we are dealing with differential forms on a projective manifold, the operator L corresponds to cup product with the cohomology class of a hyperplane. The primitive forms then represent the cohomology classes that don't "come from" a linear space section of the given variety.

It is nowadays traditional to prove Theorem 3.16 by realizing $\Lambda^k V^*$ as a representation of the Lie algebra $\mathfrak{sl}_2$. To this end, we introduce a counting operator H on $\Lambda^* V^*$, and work out the commutation relations among L, Λ and H.

Definition 3.19. Define

$$H : \Lambda^* V \longrightarrow \Lambda^* V$$

to be the operator that acts on the subspace $\Lambda^k V^*$ by multiplication by $(k - n)$, ie

$$H \ = \ \sum_{k=1}^{2n} (k-n) \cdot \pi^k,$$

where $\pi^k : \Lambda^* V^* \longrightarrow \Lambda^k V^*$ is the projection.

We extend this $\mathbf{C}$-linearly to an operator on $\Lambda^* V^*_{\mathbf{C}}$.

We now have three operators

$$H \, , \, L \, , \, \Lambda : \Lambda^* V^* \longrightarrow \Lambda^* V^*.$$

The following very basic result gives the commutation relations among these.

Proposition 3.20. *One has*

$$[H, L] \ = \ 2L \quad , \quad [H, \Lambda] \ = \ -2\Lambda$$
$$[L, \Lambda] \ = \ H.$$

Proof. For the first statement, let $\alpha \in \Lambda^k V^*$. Then $L\alpha \in \Lambda^{k+2} V^*$, and so

$$HL\alpha = (k + 2 - n) \cdot L\alpha \quad , \quad LH\alpha = (k - n) \cdot L\alpha,$$

whence $[H, L]\alpha = 2L\alpha$. Similarly, $[L, \Lambda] = -2\Lambda$. To show that $[L, \Lambda] = H$, one proceeds by induction on $\dim V$. If $\dim_{\mathbf{R}} V = 2$, so that $n = 1$, then $L1 = \omega, \Lambda\omega = 1$, and the assertion is clear. Now suppose that we write $(V, < , >, J)$ as a direct sum

$$(V, < , >, J) \ = \ (V_1, < , >_1, J_1) \oplus (V_2, < , >_2, J_2).$$

of proper subspaces of real dimensions $2n_1$ and $2n_2$ that are preserved by J and are orthogonal for $< , >$. Write ω_i, L_i, Λ_i for the indicated data on $\Lambda^* V_i$, and recall that

$$\Lambda^k V^* \ = \ \bigoplus_{k_1 + k_2 = k} \Lambda^{k_1} V_1^* \otimes \Lambda^{k_2} V_2^*.$$

Then to begin with

$$\omega \;=\; \omega_1 + \omega_2 \;\in\; \Lambda^2 V_1^* \oplus \Lambda_2 V_2^* \;\subseteq\; \Lambda^2\big(V_1 \oplus V_2\big),$$

and so L acts on $\Lambda^{k_1} V_1^* \otimes \Lambda^{k_2} V_2^*$ via

$$L \;=\; L_1 \otimes 1 \,+\, 1 \otimes L_2. \tag{*}$$

We claim that similarly

$$\Lambda \;=\; \Lambda_1 \otimes 1 \,+\, 1 \otimes \Lambda_2. \tag{**}$$

In fact, temporarily denote the right hand side of (**) by Λ'. If $\alpha = \alpha_1 \otimes \alpha_2$ and $\beta = \beta_1 \otimes \beta_2$, then

$$< \alpha, \beta > = < \alpha_1, \beta_1 >_1 \cdot < \alpha_2, \beta_2 >_2,$$

and then a computation using (*) shows that

$$< \alpha, L\beta > = < \Lambda' \alpha, \beta >,$$

which establishes (**). It follows that $[L, \Lambda]$ acts on $\Lambda^{k_1} V_1^* \otimes \Lambda^{k_2} V_2^*$ via

$$[L_1, \Lambda_1] \otimes 1 \;+\; 1 \otimes [L_2, \Lambda_2].$$

But by induction $[L_i, \Lambda_i] = H_i$, and hence

$$\begin{aligned} [L, \Lambda](\alpha_1 \otimes \alpha_2) \;&=\; (k_1 - n_1) \cdot \alpha_1 \otimes \alpha_2 \;+\; (k_2 - n_2) \cdot \alpha_1 \otimes \alpha_2 \\ &=\; \big((k_1 + k_2) - (n_2 + n_2)\big) \cdot \alpha_1 \otimes \alpha_2, \end{aligned}$$

as required. □

We next review the structure of representations of $\mathfrak{sl}_2$: this will lead to the proof of Theorem 3.16. Recall that $\mathfrak{sl}_2$ is the Lie algebra of all 2×2 matrices of trace 0; we will work with real matrices and representations, but nothing changes if one complexifies. Put

$$X \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad B \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad Y \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These satisfy the commutation relations

$$[B, X] \;=\; 2X \quad , \quad [B, Y] \;=\; -2Y \quad , \quad [X, Y] \;=\; B.$$

Lemma 3.21. *Let W be an irreducible finite-dimensional representation of $\mathfrak{sl}_2$. Then there exists a non-negative integer m such that W is the direct sum*

$$W \;=\; W_{-m} \oplus W_{-m+2} \oplus \ldots \oplus W_{m-2} \oplus W_m,$$

where $W_j \subseteq W$ is a one-dimensional eigenspace for B, with eigenvalue j. Moreover, Y maps W_j onto W_{j-2} and X maps W_j onto W_{j+2}. In particular, any vector $u \in W$ can be written uniquely in the form

$$u \;=\; u_0 + Xu_1 + X^2 u_2 + \ldots + X^m u_m$$

where $Yu_j = 0$ for each j.

Proof. Let $v \in W$ be an eigenvector of B with eigenvalue λ.[16] Then

$$\begin{aligned} B(Xv) &= [B,X](v) + XB(v) \\ &= (2+\lambda)X(v), \end{aligned}$$

so Xv is an eigenvector of B with eigenvaule $\lambda+2$. Similarly, Yv is an eigenvector with eigenvalue $\lambda-2$. It follows that we can find a *primitive vector* $w \in W$, ie an eigenvector of B such that $Yw = 0$. There also exists an integer n such that

$$X^m(w) \neq 0 \quad , \quad X^{m+1}(w) = 0.$$

Now the elements $X^j w$ $(j \geq 0)$ span a sub-representation of W, and they are linearly independent since they are eigenvectors of B with distinct eigenvalues. It follows that

$$W = \bigoplus_{j=0}^{m} \mathbf{R} \cdot X^j(w).$$

So it suffices to show that the B-eigenvalue of w is $-m$. To this end, suppose that $Bw = \lambda w$. Then

$$YXw = XYw - Bw = -\lambda w,$$

and

$$\begin{aligned} YX^2 w &= XYXw - BXw \\ &= -\lambda Xw - (\lambda+2)Xw. \end{aligned}$$

In general

$$\begin{aligned} YX^k w &= \Big(-\lambda - (\lambda+2) - \ldots - \big(\lambda + (2(k-1))\big)\Big)(X^{k-1}w) \\ &= (-k\lambda - k^2 + k)X^{k-1}w. \end{aligned}$$

Since $X^{m+1}w = 0$ but $X^m w \neq 0$, this implies that

$$-(m+1)\lambda - (m+1)^2 + (m+1) = 0,$$

ie $\lambda = -m$. □

Remark 3.22. Let $\mathbf{C}^2$ denote the standard two-dimensional representation of $\mathfrak{sl}_2$. Then the representation W considered in the previous lemma arises as $\mathrm{Sym}^m \mathbf{C}^2$.

Corollary 3.23. *Let U be a finite dimensional representation of $\mathfrak{sl}_2$, and let*

$$P = \{u \in U \mid Yu = 0\} \subseteq U$$

be the subspace of primitive vectors. Assume that B acts on P by multiplication by $-m$. Then $m \geq 0$, and

$$U = P \oplus X(P) \oplus X^2(P) \ \ldots \ \oplus X^m(P).$$

Moreover $X^{m+1}(P) = 0$, while

$$X^m(u) \neq 0 \ \ \forall\, 0 \neq u \in P.$$

Proof. In fact, U is a direct sum of irreducible representations, and then the corollary follows from the previous lemma and its proof. □

[16] It will emerge shortly that B actually has real eigenvalues.

We now give the

Proof of Theorem 3.16. Proposition 3.20 asserts that $\Lambda^* V^*$ is a representation of $\mathfrak{sl}_2$, with

$$L \leftrightarrow X \quad , \quad H \leftrightarrow B \quad , \quad \Lambda \leftrightarrow Y.$$

Furthermore, by definition H acts as multiplication by $k-n$ on $P^k V^* \subseteq \Lambda^k V^*$. The existence of the Lefschetz decomposition then follows from the previous corollary as does statement (iii) of the Theorem. Moreover the corollary shows that the mapping

$$L^{n-k} : P^k V^* \longrightarrow \Lambda^{n-k} V^*$$

is injective. It then follows from (i) that

$$L^{n-k} : \Lambda^k V^* \longrightarrow \Lambda^{2n-k} V^*$$

is injective, and since both sides have the same dimension L^{n-k} must be an isomorphism. $\square$

Note that the operators L^{n-k-j} and $*L^j$ both map $\Lambda^k V^*$ to $\Lambda^{2n-k-j} V^*$, so it is natural to ask for a comparison of these. To this end, write

$$\mathbf{J} \ = \ \Lambda^* J^* : \Lambda^* V^* \longrightarrow \Lambda^* V^*$$

for the endomorphism of $\Lambda^* V^*$ determined by the almost complex structure J.

Proposition 3.24. *Let $\alpha \in P^k$ be a primitive k-form. Then*

$$*L^j(\alpha) \ = \ (-1)^{\frac{k(k+1)}{2}} \cdot \frac{j!}{(n-k-j)!} \cdot L^{n-k-j} \mathbf{J}(\alpha).$$

The Proposition is proven using the Lefschetz decomposition by an induction on $\dim V$, in the spirit of the proof of Proposition 3.20. However the calculations are somewhat involved, and we will not repeat them here. We refer the interested reader to [3, pages 37-38].

Example 3.25. Take $\alpha = 1 \in \Lambda^0 V^*$ (so $k = 0$) and $j = 0$. Then the Proposition asserts that $* \frac{\omega^j}{j!} = \frac{\omega^{n-j}}{(n-j)!}$, as observed in Example 3.12.

The operators L, Λ, H pass by linearity to the complexification $\Lambda^* V^*_{\mathbf{C}}$, and all the results of this subsection remain valid in this setting. Moreover since these operators act bihomogeneously with respect to the (p,q) grading on $\Lambda^* V^*_{\mathbf{C}}$, the Lefschetz decomposition respects this bigrading. Specifically, let

$$P^{p,q} \ = \ P^k_{\mathbf{C}} \cap \Lambda^{p,q} V^*_{\mathbf{C}}.$$

Then $P^k_{\mathbf{C}} = \oplus_{p+q=k} P^{p,q}$, and

$$\Lambda^{p,q} V^*_{\mathbf{C}} \ = \ \bigoplus_{i \geq 0} \ L^j P^{p-i,q-i}.$$

Moreover, since L is real:

$$\overline{P^{p,q}} \ = \ P^{q,p}.$$

Finally, recall that if $\alpha \in P^{p,q}$ then $\mathbf{J}(\alpha) = (\sqrt{-1})^{p-q} \cdot \alpha$, and hence Proposition 3.24 assumes a particularly simple shape in this case.

Example 3.26. Taking $k = 2$, note that $\Lambda^{2,0}V^*$ and $\Lambda^{2,0}V^*$ are primitive for reasons of type. The Lefschetz decomposition in this case takes the form

$$\Lambda^2 V^*_{\mathbf{C}} \ = \ P^{2,0} \oplus \big(P^{1,1} \oplus \mathbf{C}\cdot\omega \big) \oplus P^{0,2}.$$

We conclude by studying the sign of an important quadractic form defined with the help of ω. Define

$$Q : \Lambda^k V^* \times \Lambda^k V^* \longrightarrow \mathbf{R}$$

by requiring that

$$(-1)^{\frac{k(k+1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k} \ = \ Q(\alpha, \beta) \cdot \mathrm{vol}.$$

We extend this by linearity to a symmetric bilinear form

$$Q : \Lambda^k V^*_{\mathbf{C}} \times \Lambda^k V^*_{\mathbf{C}} \longrightarrow \mathbf{C},$$

called the *Hodge-Riemann pairing.* The basic fact is:

Theorem 3.27. (Hodge-Riemann bilinear relations). *If $\alpha \in \Lambda^{p,q}V^*$ and $\beta \in \Lambda^{p',q'}V^*$, then*

$$Q(\alpha, \beta) \ = \ 0 \quad \textit{unless} \quad p = q' \, , \, q = p'.$$

If $p + q \leq n$ and $\alpha \in P^{p,q}$ is a primitive (p, q)-form, then

$$(\sqrt{-1})^{(p-q)} \cdot Q(\alpha, \overline{\alpha}) = \big(n - (p + q)\big)! \cdot < \alpha, \alpha >_{\text{herm}} . \tag{*}$$

In particular $(\sqrt{-1})^{(p-q)} Q(\alpha, \overline{\alpha}) > 0$ if $\alpha \neq 0$.

Proof. Only (*) needs proof, for which we apply Proposition 3.24 with $j = 0$ and $k = p + q$ to the primitive form $\overline{(\alpha)} \in P^{q,p}$. One finds that

$$*\overline{\alpha} \ = \ (-1)^{\frac{k(k+1)}{2}} \cdot \frac{1}{(n-k)!} \cdot (\sqrt{-1})^{q-p} \cdot \overline{\alpha} \wedge \omega^{n-k},$$

so that

$$\overline{\alpha} \wedge \omega^{n-k} \ = \ (-1)^{\frac{k(k+1)}{2}} \cdot (n-k)! \cdot (\sqrt{-1})^{p-q} \cdot *\overline{\alpha}.$$

Recalling that

$$< \alpha, \alpha >_{\text{herm}} \cdot \mathrm{vol} \ = \ \alpha \wedge *\overline{\alpha},$$

(*) follows. □

Example 3.28. Take $k = 2$, and consider the form

$$Q : \Lambda^{1,1}V^* \times \Lambda^{1,1}V^* \longrightarrow \mathbf{R} \quad , \quad (\alpha, \beta) \mapsto \alpha \wedge \beta \wedge \omega^{n-2}$$

(where we identity $\Lambda^n V^*$ with $\mathbf{R}$ via vol). Recall that

$$\Lambda^{1,1}V^* \ = \ P^{1,1} \oplus (\mathbf{R} \cdot \omega).$$

Theorem 3.27 shows that Q is negative definite on $P^{1,1}$, but positive-definite on the one-dimensional subspace spanned by ω. So on $\Lambda^{1,1}V^*$ Q has one positive eigenvalue, and the rest negative. This will lead to the Hodge Index theorem in the global setting.

Example 3.29. Suppose that $\dim_{\mathbf{C}} V = n = 2m$ is even, and let

$$P \;=\; P^{m,m} \;=\; \{\alpha \in \Lambda^{m,m}V^* \mid \alpha \wedge \omega = 0\}.$$

Then the quadratic form

$$P \times P \longrightarrow \mathbf{R} \;\;,\;\; (\alpha, \beta) \mapsto \alpha \wedge \beta$$

is positive definite if m is even, and negative definite if m is odd.

Exercises for Section 3.

Exercise 3.1. Prove the existence of the vectors $x_1, \ldots, x_n \in V$ asserted in Remark 3.2 .

Exercise 3.2. Prove Lemma 3.4.

Exercise 3.3. Let $(V, <\,,\,>)$ be an oriented real Euclidean space with $\dim_{\mathbf{R}} = 4$. Show that the possible compatible complex structures that determine the given orientation are parametrized by a copy of the two-dimensional sphere S^2. (Hint: Think of $\mathbf{R}^4$ as the quaternions.)

Exercise 3.4. Prove that a two-form $\eta \in \Lambda^2 V^*$ is of type $(1,1)$ in $\Lambda^2 V^*_{\mathbf{C}}$ if and only if $\eta(Jv, Jw) = \eta(v, w)$ for all $v, w \in V$. (Hint: see (3.7).)

Exercise 3.5. Prove that with respect to the Hermitian form $<\,,\,>_{\text{herm}}$, $V_{\mathbf{C}}$ is the orthogonal direct sum of $V^{1,0}$ and $V^{0,1}$.

Exercise 3.6. Prove the expression for ω appearing in equation (3.14).

Exercise 3.7. Show that one can always choose an basis $\{x_i, y_i = Jx_i\}$ which is orthonormal with respect to the form $<\,,\,>$. Then show that with respect to this basis,

$$\omega \;=\; \frac{\sqrt{-1}}{2} \cdot \sum_{i=1}^{n} z^i \wedge \overline{z}^i \;=\; \sum_{i=1}^{n} x^i \wedge y^i.$$

Exercise 3.8. With the usual notation, show that

$$\text{vol} \;=\; \frac{1}{n!} \cdot \omega^n.$$

Exercise 3.9. Prove the assertions of Example 3.11, and Example **??**

Exercise 3.10. Show that if $\alpha \in \Lambda^k V^*$, then

$$[L^i, \Lambda](\alpha) \;=\; i(k - n + i - 1) \cdot \alpha.$$

(Hint: Argue by induction on i using Proposition 3.20 together with the identity

$$[L^i, \Lambda] \;=\; L[L^{i-1}, \Lambda] \,+\, [L, \Lambda] L^{i-1}.$$

4. Calculus on Complex Manifolds, I: Local Theory

We now apply the linear algebra of the previous section to the tangent and cotangent spaces of a complex manifold. We start with the local picture, where we can make explicit calculations.

Differential forms on open subsets of $\mathbf{C}^n$. Let $U \subseteq \mathbf{C}^n$ be an open set, and let

$$z_i \ = \ x_i + \sqrt{-1} \cdot y_i$$

be complex coordinates on U. Then for each point $p \in U$, the (real) tangent space T_pU has a natural almost complex structure $J : T_pU \longrightarrow T_pU$. Concretely, we take as a basis of T_pU the tangent vectors

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \ , \ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}, \tag{*}$$

and then J is given by

$$\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i} \quad , \quad \frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}.$$

Thus on the dual basis dx_i , dy_i, the transpose of J – which we again write simply as J – acts by

$$J(dx_i) \ = \ -dy_i \ \ , \ \ J(dy_i) \ = \ dx_i.$$

Write TU for the real tangent bundle of U. We view the vector fields (*) as giving a trivialization of this bundle over U:

$$TU \ = \ U \times \mathbf{R}^{2n},$$

and we can think of J as defining an endomorphism of TU.

We now apply the discussion of the previous section. Denote by

$$T_{\mathbf{C}}U \ = \ TU \otimes_{\mathbf{R}} \mathbf{C}$$

the complexified tangent bundle of U, whose fibres are the complexified tangent spaces $(T_pU)_{\mathbf{C}}$. Then we can decompose $T_{\mathbf{C}}U$ as a sum of two complex sub-bundles

$$T_{\mathbf{C}}U \ = \ T^{1,0}U \oplus T^{0,1}U,$$

on which (the complexification of) J acts as $\sqrt{-1} \cdot \mathrm{Id}$ and $-\sqrt{-1} \cdot \mathrm{Id}$ respectively. These bundles have as a basis the complex tangent vector fields

$$\frac{\partial}{\partial z_i} \ = \ \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_i} \ = \ \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i}\right).$$ [17]

Furthermore, with the almost complex structure on $T^{1,0}U$ determined by multiplication by $\sqrt{-1}$,

$$\left(T^{1,0}U, \sqrt{-1}\right) \ \cong \ (TU, J).$$

Similarly, we get a decomposition

$$T^*_{\mathbf{C}}U \ = \ \Lambda^{1,0}T^*U \oplus \Lambda^{0,1}T^*U$$

of the complexified cotangent bundle of U, with bases

$$dz_i \ = \ dx_i + \sqrt{-1} \cdot dy_i \ \ , \ \ d\overline{z}_i \ = \ dx_i - \sqrt{-1} \cdot dy_i.$$

[17] This explains intrinsically the meaning of these complex vector fields, which were introduced in an ad hoc fashion in the first section.

As in the pointwise situation of the previous section, this leads to a **C**-linear direct sum

$$\Lambda^k T^*_{\mathbf{C}} U \;=\; \bigoplus_{p+q=k} \Lambda^{p,q} T^* U,$$

with a basis for $\Lambda^{p,q}T^*$ being given by forms of the sort

$$dz_I \wedge d\overline{z}_J \;\;,\;\; |I| = p \;,\; |J| = q.$$

This in turn leads to a decomposition of the spaces of smooth differential forms on U. Write $\mathcal{A}^k_{\mathbf{C}}(U)$ for the space of smooth complex-valued differential k-forms on U, i.e. $\mathcal{C}^\infty$ global sections of $\Lambda^k T^*_{\mathbf{C}} U$. Then we have

$$\mathcal{A}^k_{\mathbf{C}}(U) \;=\; \bigoplus_{p+q=k} \mathcal{A}^{p,q}(U), \tag{4.1}$$

where $\mathcal{A}^{p,q}(U)$ is the (infinite-dimensional) complex vector space of smooth sections of $\mathcal{A}^{p,q}(U)$, i.e. complex-valued forms of the type

$$\sum_{|I|=p, |J|=q} f_{I,J} \cdot dz_I \wedge d\overline{z}_J,$$

the $f_{I,J}$ being smooth functions on U. We say that such a form is of *type* (p, q), and we denote by

$$\pi^{p,q} : \mathcal{A}^k_{\mathbf{C}}(U) \longrightarrow \mathcal{A}^{p,q}(U)$$

the projections.

So far all of these constructions are completely algebraic in nature, ie they depend only on the point-wise situation studied in the previous section. Calculus enters the picture with the exterior derivative. Given a smooth **C**-valued function $f \in \mathcal{A}^0_{\mathbf{C}}(U)$ on U, we define

$$\partial f \in \mathcal{A}^{1,0}(U) \;\;,\;\; \overline{\partial} f \in \mathcal{A}^{0,1}(U)$$

to be respectively the $(1, 0)$ and $(0, 1)$ components of $df \in \mathcal{A}^1(U)$, i.e. $\partial f = \pi^{1,0}(df)$ and $\overline{\partial} f = \pi^{0,1}(df)$. Thus $df = \partial f + \overline{\partial} f$, and in terms of local coordinates:

$$\partial f \;=\; \sum_{i=1}^n \frac{\partial f}{\partial z_i} \cdot dz_i$$

$$\overline{\partial} f \;=\; \sum_{i=1}^n \frac{\partial f}{\partial \overline{z}_i} \cdot d\overline{z}_i.$$

So for instance f is holomorphic if and only if $\overline{\partial} f = 0$.

Now consider a "monomial" (p, q)-form

$$\alpha = f \cdot dz_I \wedge d\overline{z}_J,$$

where I, J are multi-indices of sizes p and q respectively. Then $d\alpha = df \cdot dz_I \wedge d\overline{z}_J$, and therefore

$$d\alpha \;=\; (\partial f \wedge dz_I \wedge d\overline{z}_J) \;+\; (\overline{\partial} f \wedge dz_I \wedge d\overline{z}_J). \tag{*}$$

Therefore the derivative of an arbitrary (p,q)-form is the sum of a $(p+1,q)$-form and a $(p,q+1)$-form. It follows that the full exterior derivative

$$d : \mathcal{A}^k_{\mathbf{C}}(U) \longrightarrow \mathcal{A}^{k+1}_{\mathbf{C}}(U)$$

decomposes as a direct sum $d = \partial + \overline{\partial}$ of two operators ∂ and $\overline{\partial}$

$$\begin{aligned} \partial \ &: \ \mathcal{A}^{p,q}(U) \longrightarrow \mathcal{A}^{p+1,q}(U) \\ \overline{\partial} \ &: \ \mathcal{A}^{p,q}(U) \longrightarrow \mathcal{A}^{p,q+1}(U). \end{aligned}$$

determined by the two expressions on the right in (*). Formally,

$$\partial \ = \ \pi^{p+1,q} \circ d \quad , \quad \overline{\partial} \ = \ \pi^{p,q+1} \circ d,$$

and in terms of local coordinates:

$$\begin{aligned} \partial\big(f \cdot dz_I \wedge d\overline{z}_J\big) \ &= \ \left(\sum_{i=1}^n \frac{\partial f}{\partial z_i} \cdot dz_i \right) \wedge dz_I \wedge d\overline{z}_J \\ \overline{\partial}\big(f \cdot dz_I \wedge d\overline{z}_J\big) \ &= \ \left(\sum_{i=1}^n \frac{\partial f}{\partial \overline{z}_i} \cdot d\overline{z}_i \right) \wedge dz_I \wedge d\overline{z}_J. \end{aligned}$$

The following statement summarizes the properties of this operator.

Proposition 4.1. *One has:*

(i). $d = \partial + \overline{\partial}$.

(ii). $\partial^2 = \overline{\partial}^2 = 0$ *and* $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

(iii). *These operators satisfy the Leibniz rule: if* $\alpha \in \mathcal{A}^{p,q}(U)$, *then*

$$\begin{aligned} \partial(\alpha \wedge \beta) \ &= \ \partial(\alpha) \wedge \beta \ + \ (-1)^{p+q} \alpha \wedge \partial(\beta) \\ \overline{\partial}(\alpha \wedge \beta) \ &= \ \overline{\partial}(\alpha) \wedge \beta \ + \ (-1)^{p+q} \alpha \wedge \overline{\partial}(\beta). \end{aligned}$$

$\overline{\partial}$-Poincaré Lemma. In this subsection we will establish the analogue for $\overline{\partial}$ of the classical Poincaré lemma that a closed form is locally exact. Specifically, we aim to prove (most of) the following:

Theorem 4.2. *Let* $\Delta \subseteq \mathbf{C}^n$ *be a polydisk, and let*

$$\alpha \ \in \ \mathcal{A}^{p,q}(\Delta)$$

be a $\overline{\partial}$*-closed smooth* (p,q)*-form on* δ*. If* $q \geq 1$*, then there exists a smooth* $(p, q-1)$*-form* β *on* Δ *such that*

$$\alpha \ = \ \overline{\partial}\beta.$$

It turns out that most of the work occurs in the case $n = 1$. We start by recording a $\mathcal{C}^\infty$ version of the Cauchy integral formula.

Proposition 4.3. *Let $\Delta \subset \mathbf{C}$ be a disk, let $f \in \mathcal{C}^\infty(\overline{\Delta})$ be a smooth function defined on an open neighborhood of the closure of Δ, and fix a point $z \in \Delta$. Then*

$$f(z) \;=\; \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)dw}{w-z} \;+\; \frac{1}{2\pi\sqrt{-1}}\int_{\Delta}\frac{\partial f(w)}{\partial \overline{w}}\frac{dw\wedge d\overline{w}}{w-z}.$$

For the proof, see [2, p.2] or [3, p. 46].

The following proposition essentially proves the $\overline{\partial}$-Poincaré lemma in one variable for the form $f(z)d\overline{z}$ (which is automatically $\overline{\partial}$-closed).

Proposition 4.4. *Let $\Delta \subset \mathbf{C}$ be a disk, and let $f \in \mathcal{C}^\infty(\overline{\Delta})$ be a smooth function on an open neighborhood of $\overline{\Delta}$. Then the function*

$$h(z) \;=\; \frac{1}{2\pi\sqrt{-1}}\int_{\Delta}\frac{f(w)}{w-z}dw\wedge d\overline{w}$$

is defined and $\mathcal{C}^\infty$ in Δ, and satisfies

$$\frac{\partial h}{\partial \overline{z}} \;=\; f.$$

Proof. Fix a point $z_0 \in \Delta$, and choose $\varepsilon > 0$ such that

$$\Delta(z_0,\varepsilon) \;\subset\; \Delta(z_0,2\varepsilon) \;\subset\; \Delta.$$

Starting with a $\mathcal{C}^\infty$ function that is $\equiv 1$ in $\Delta(z_0,\varepsilon)$ and $\equiv 0$ outside $\Delta(z_0,2\varepsilon)$, we may write

$$f \;=\; f_1 + f_2$$

where $f_1 \equiv 0$ outside $\Delta(z_0,2\varepsilon)$ and $f_2 \equiv 0$ in $\Delta(z_0,\varepsilon)$. Then for $z \in \Delta(z_0,\varepsilon)$ the function

$$h_2(z) \;=_{\text{def}}\; \frac{1}{2\pi\sqrt{-1}}\int_{\Delta}\frac{f_2(w)}{w-z}dw\wedge d\overline{w}$$

is $\mathcal{C}^\infty$, and we have $\frac{\partial}{\partial\overline{z}}h_2(z) = 0$ since the integrand is holomorphic in z outside $\Delta(z_0,\varepsilon)$. We next consider the corresponding integral of f_1. Since f_1 has compact support, one can write and write (for $z \in \Delta(z_0,\varepsilon)$):

$$\begin{aligned} h_1(z) \;&=_{\text{def}}\; \frac{1}{2\pi\sqrt{-1}}\int_{\Delta}\frac{f_1(w)}{w-z}dw\wedge d\overline{w} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbf{C}}\frac{f_1(w)}{w}dw\wedge d\overline{w} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbf{C}}\frac{f_1(u+z)}{u}du\wedge d\overline{u}, \end{aligned}$$

where $u = w - z$. Now pass to polar coordinates $u = re^{(\sqrt{-1})\theta}$. One finds that

$$h_1(z) \;=\; -\frac{1}{\pi}\int_{\mathbf{C}} f_1(z + re^{(\sqrt{-1})\theta})e^{-(\sqrt{-1})\theta}dr\wedge d\theta,$$

which is $\mathcal{C}^\infty$ in z. Moreover we have

$$\begin{aligned}\frac{\partial h_1(z)}{\partial \overline{z}} &= -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial f_1}{\partial \overline{z}}(z + re^{(\sqrt{-1})\theta}) \cdot e^{-(\sqrt{-1})\theta} dr \wedge d\theta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\Delta} \frac{\partial f_1(w)}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z} \\ &= f_1(z),\end{aligned}$$

where the last equality arises from Proposition 4.3 and the fact that f_1 vanishes on $\partial\Delta$. Therefore $\frac{\partial h}{\partial \overline{z}} = f$ in $\Delta(z_0, \varepsilon)$, as required. □

We now turn to the proof of (a slightly weakened form of) Theorem 4.2. To begin with, we reduce to proving the theorem in the case $p = 0$. In fact, suppose that

$$\alpha \ = \ \sum_{I,J} f_{I,J}\, dz_I \wedge d\overline{z}_J$$

($\#I = p, \#J = q$) is a form of type (p, q). Then we can write $\alpha = \sum_I dz_I \wedge \alpha_I$, where

$$\alpha_I \ = \ \sum_J f_{I,J} d\overline{z}_J$$

is a $(0, q)$-form. Then $\overline{\partial}\alpha = 0$ if and only if $\overline{\partial}\alpha_I = 0$ for every I, and if each α_I is $\overline{\partial}$-exact then so it α. So it suffices to prove the theorem for $(0, q)$-forms.

Now let $\Delta = \Delta(\varepsilon) \subseteq \mathbf{C}^n$ be a bounded polydisk, and that for $q > 0$ we are given a $\overline{\partial}$-closed $(0, q)$-form $\alpha \in \mathcal{A}^{0,q}(U)$ defined and smooth on an open neighborhood $U \supseteq \overline{\Delta}$. We will show that

$$\alpha \ = \ \overline{\partial}\beta \quad \text{for some } \beta \in \mathcal{A}^{0,q-1}(\Delta).$$ [18]

Write then $\alpha = \sum f_I d\overline{z}_I$, and let $1 \leq k \leq n$ be the least index such that no $d\overline{z}_i$ appears non-trivially in this expression for $i > k$. Then we can write

$$\alpha \ = \ \alpha_1 \wedge d\overline{z}_k \, + \, \alpha_2,$$

where α_2 is a $(0, q)$-form involving only the variables $z_1, \ldots, z_{k-1}$. Setting

$$\partial_i \ = \ \left(\frac{\partial}{\partial \overline{z}_i}\right) \cdot d\overline{z}_i,$$

the assumption $\overline{\partial}\alpha = 0$ implies that

$$\partial_i \alpha_1 \ = \ \partial_i \alpha_2 \ = \ 0$$

for $i > k$, and hence the coefficients f_I of α are holomorphic in the variables $z_{k+1}, \ldots, z_n$.

Now consider the function

$$h_I \ = \ \frac{1}{2\pi\sqrt{-1}} \int_B \frac{f_I(z_1, \ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n)}{w - z_k} dw \wedge d\overline{w},$$

[18]The general case – which allows α to be only defined on Δ is treated by an exhaustion argument. See [2, p. 26] or [3, p. 47] for the details.

the integral being taken over a suitable disk $B \subset \mathbf{C}$. By Proposition 4.4, one has

$$\frac{\partial h_I}{\partial \overline{z}_k} \;=\; f_I$$

on B. Moreover h_I is holomorphic in $z_{k+1}, \ldots, z_n$, and smooth in the other variables. This being said, put

$$\gamma \;=\; \sum_{I \ni k} h_I d\overline{z}_{I-\{k\}}.$$

Then $\overline{\partial}_i \gamma = 0$ for $i > k$, and $\overline{\partial}_k \gamma = \pm \alpha_1$. Thus

$$\alpha \pm \overline{\partial} \gamma$$

is still $\overline{\partial}$-closed, but does not involve any of the $d\overline{z}_k, \ldots, d\overline{z}_n$, and the assertion follows by induction on k.

Local geometry of Kähler forms. We now bring metrics into the picture.

As before, let $U \subseteq \mathbf{C}^n$ be an open set.

Consider a Riemannian metric $g =< \, , \, >$ on U, i.e a positive definite bilinear form $g_x =< \, , >_x$ on real tangent space $T_x U$ varying smoothly with x. We assume that g is compatible with the (almost) complex structure on U, i.e. we assume that

$$< Jv, Jw > = < v, w >$$

for all local vector fields v, w on U. Then by our discussion of Hermitian linear algebra, g determines a real $(1,1)$-form ω in U given by

$$\omega(v, w) \;=\; < Jv \, , \, w >,$$

and

$$h \;=\; g - \sqrt{-1} \cdot \omega$$

is a positive-definite Hermitian metric on U. As in the pointwise setting, one calls ω the *fundamental form* associated to the metric. If we define

$$h_{ij}(z) \;=\; h\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)_z,$$

then h_{ij} is a positive-definite matrix of smooth functions, and

$$\omega \;=\; \frac{\sqrt{-1}}{2} \cdot \sum_{i,j} h_{ij}(z) \, dz_i \wedge d\overline{z}_j.$$

Since conversely ω determines g and h, one sometimes abusively refers to the fundamental form ω as the compatible metric.

For example, if g is the constant standard metric on U, then $h = \mathrm{Id}$ is the constant standard Hermitian metric, and

$$\omega \;=\; \frac{\sqrt{-1}}{2} \cdot \sum_i dz_i \wedge d\overline{z}_i.$$

We now come to a central definition

Definition 4.5. One says that the compatible metric g (or ω) is *Kähler* if

$$d\omega \;=\; 0.$$

Such metrics will play a central role in what follows. For now we want to prove an important local characterization of Kähler metrics.

We say that the Hermitian metric h on U (or the underlying Riemannian metric g) *osculates to the identity to order* 2 at a point $p \in U$ if in suitable local holomorphic coordinates centered at p one write

$$h \;=\; \mathrm{Id} + O(|z|^2), \tag{4.2}$$

where the term on the right indicates a matrix of functions vanishing to order ≥ 2 at 0. This is a useful condition, because it allows one to check identities among first order differential operators by working with the standard metric.

The basic result is that the Kähler condition is equivalent to the being able to choose coordinates so that g osculates to the identity.

Theorem 4.6. *Let g be a compatible metric on $U \ni p$, with fundamental form ω. Then $d\omega = 0$ if and only if one can choose coordinates centered at p with respect to which g osculates the identity to order* 2.

Sketch of proof. Assuming that $d\omega = 0$ we will show that after a coordinate change we can arrange for (4.2) to hold. To this end, write

$$\omega \;=\; \frac{\sqrt{-1}}{2} \cdot \sum_{i,j} h_{ij}(z) dz_i \wedge d\overline{z}_j,$$

where

$$h_{ij}(z) \;=\; \delta_{ij} + \sum_k a_{ijk} z_k + \sum_k a'_{ijk} \overline{z}_k + O(|z|^2),$$

and $h_{ij} = \overline{h_{ji}}$. Then

$$dh_{ij} \;=\; \sum_k a_{ijk} dz_k + \sum_k a'_{ijk} d\overline{z}_k + O(|z|).$$

Now the condition $d\omega = 0$ implies that the form

$$\sum_{i,j} dh_{ij} \wedge dz_i \wedge d\overline{z_j}$$

vanishes at the origin, which is equivalent to the equalities

$$a_{ijk} \;=\; a_{kji} \;,\; a'_{ijk} \;=\; a'_{ikj}. \tag{*}$$

Moreover, since h is hermitian, $a'_{ijk} = \overline{a}_j ik$. Now set

$$w_j \;=\; z_j + \frac{1}{2} \cdot \sum_{i,k=1}^{n} a_{ijk} z_i z_k.$$

Then a computation using the relations in (*) shows that in these new coordinates,

$$\omega \;=\; \frac{\sqrt{-1}}{2} \cdot \sum dw_j \wedge d\overline{w}_j \;+\; O(|w|^2),$$

as required. (See [3, p. 49] for details.) $\square$

Almost complex structures. Let X be a real manifold of even (real) dimension $2n$. An *almost complex structure* on X is a bundle endomorphism

$$J : TX \longrightarrow TX \quad \text{with } J^2 = \mathrm{Id}.$$

Thus J gives an almost complex structure J_x on each of the (real) tangent spaces T_xX. As in §3, an almost complex structure gives rise to a decomposition into type

$$T^*X \;=\; T^{*1,0} \oplus T^{*0,1}X,$$

and similarly for k-forms. Moreover we can define operators

$$\partial : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p+1,q}(X)\ ,\ \overline{\partial} : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q}(X)$$

by composing the deRham d with projection onto the spaces of $(p+1,q)$ and $(q,p+1)$-forms respectively. Note that on functions d acts as $\partial + \overline{\partial}$, but there is no reason that this has to be true on higher forms, since it could happen eg that there is a $(1,0)$ form α such that $d\alpha$ has a non-trivial $(2,0)$ component.

Evidently every complex manifold has a canonical such almost complex structure, so it is natural to for conditions under which an almost complex structure is *integrable*, i.e arises from a complex structure. This is the content of a famous theorem of Newlander and Nierenberg.

Theorem 4.7. (Newlander–Nierenberg). *Let (X, J) be an almost complex manifold. Then X carries a complex structure giving rise to J if and only if any of the following equivalent conditions are satisfied:*

(i). $d = \partial + \overline{\partial}$ *on* $\mathcal{A}^{1,0}(X)$.
(ii). $d = \partial + \overline{\partial}$ *on* $\mathcal{A}^{*}(X)$.
(iii). $\overline{\partial}^2 = 0$.

The proof that these statements are equivalent is elemantary (see [3, p. 107, 108]). The serious assertion is that these imply the existence of a complex structure on X giving rise to J.

Exercises for Section 4.

Exercise 4.1. Let $f : U \longrightarrow V$ be a holomorphic mapping. Show that the complexified real derivative $D_{\mathbf{C}}(f) : T_{\mathbf{C}}U \longrightarrow T_{\mathbf{C}}V$ maps $T^{1,0}U$ to $T^{1,0}(V)$, and that under the isomorphisms of complex vector spaces

$$TU \;\cong\; T^{1,0} \;\;,\;\; TV \;\cong\; T^{1,0}V,$$

the resulting homomorphism $T^{1,0}U \longrightarrow T^{1,0}V$ corresponds to the $\mathbf{C}$-linear mapping determined by $D_{\mathbf{R}}f$, as in Proposition 1.17.

Exercise 4.2. Use considerations of type to prove statements (ii) and (iii) of Proposition 4.1.

5. Dolbeaut Theorem and Sheaf Cohomology

In this section, we give a brief sketch of the theory of sheaves and their cohomology, and prove Dolbeaut's theorem.

Statement of Dolbeaut's theorem. Let X be a complex manifold. Writing $\mathcal{A}^{p,q}(X)$ for the space of global smooth (p,q)-forms on X, we have the $\overline{\partial}$-operator

$$\overline{\partial} : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q+1}(X)$$

satisfying $\overline{\partial}^2 = 0$. Therefore we can make the

Definition 5.1. The Dolbeaut cohomology groups of X are defined as the vector space of $\overline{\partial}$-closed (p,q)-forms modulo $\overline{\partial}$-exact (p,q)-forms:

$$H^{p,q}_{\overline{\partial}}(X) \;=\; \frac{\ker\left(\overline{\partial} : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q+1}(X)\right)}{\operatorname{im}\left(\overline{\partial} : \mathcal{A}^{p,q-1}(X) \longrightarrow \mathcal{A}^{p,q}(X)\right)}.$$

One can see these groups as holomorphic analogues of the DeRham cohomology groups of a smooth manifold. So it is natural to ask what invariants of X they compute. Dolbeaut's theorem asserts that they are isomorphic to certain basic sheaf-theoretic invariants of X:

Theorem 5.2. (Dolbeaut's theorem, I). *There are canonical isomorphisms:*

$$H^{p,q}_{\overline{\partial}}(X) \;=\; H^q(X, \Omega^p_X).$$

In particular,

$$H^{0,q}_{\overline{\partial}}(X) \;=\; H^q(X, \mathcal{O}_X).$$

The groups on the right are the cohomology of X with coefficients in the sheaf of holomorphic p-forms and the structure sheaf of X, respectively. Our goal in the rest of the section is to explain the meaning of these groups, and to sketch the proof of the theorem. (Along the way we'll give the sheaf-theoretic proof of De Rham's theorem, which follows formally from the d-Poincaré lemma in the same way that the Dolbeaut theorem will follow from the $\overline{\partial}$-Poincaré lemma.) However before turning to sheaf theory, we want to indicate a "twisted" analogue of Dolbeaut cohomolgy.

Specifically, let E be a holomorphic vector bundle of rank e on X, given as above by local data $\{U_i, g_{ij}\}$. Then one can form the complex vector bundle $\Lambda^{p,q}TX^* \otimes_{\mathbf{C}} E$ of E-valued (p,q)-forms, whicn in turn gives rise to the (infinite-dimensional) complex vector space $\mathcal{A}^{p,q}(E) = \mathcal{A}^{p,q}(X,E)$ of smooth E-valued (p,q)-forms: an element

$$\alpha \in \mathcal{A}^{p,q}(X,E)$$

is locally given by a vector of smooth (p,q)-forms which transform via the transition matrices of E. Thus

$$\mathcal{A}^{p,q}(X) \ = \ \mathcal{A}^{p,q}(X,\mathbf{1})$$

is recovered as the space of smooth (p,q)-forms with values in the trivial line bundle.

It is a wonderful fact that given any holomorphic vector bundle E, the $\overline{\partial}$ operator on forms extends canonically to an operator

$$\overline{\partial} \ = \ \overline{\partial}_E : \mathcal{A}^{p,q}(X,E) \longrightarrow \mathcal{A}^{p,q+1}(X,E). \tag{5.1}$$

The critical case to understand is that when $p = 0$:

$$\overline{\partial}_E : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{0,1}(E). \tag{*}$$

In other words, we need to explain how to take the $\overline{\partial}$-derivative of a smooth section of E. On a trivializing open set U_j, we represent $s \in \mathcal{A}^0(E)$ by a vector s_j of smooth sections, and these data patch together via the rule $s_i = g_{ij}s_j$ on U_{ij}. We locally define $\overline{\partial}s$ to be the vector of $(0,1)$-forms obtained by differentiating each of the components of s_j, i.e.

$$\overline{\partial}_E(s) \ =_{\text{locally}} \overline{\partial}(s_j) \ \in \ \mathcal{A}^{0,1}(U_j, E).$$

We need to check that the vectors of $(0,1)$-forms so obtained transform in the required manner. But $\overline{\partial}g_{ij} = 0$ thanks the fact that the g_{ij} are matrices of holomorphic functions, and hence differentiating the relation $s_i = g_{ij}s_j$ one finds that

$$\begin{aligned} \overline{\partial}s_i \ &= \ \overline{\partial}g_{ij} \cdot s_j \ + \ g_{ij} \cdot \overline{\partial}s_j \\ &= \ g_{ij} \cdot \overline{\partial}s_j, \end{aligned}$$

as required. The reader can then check

Lemma 5.3. *The operator $\overline{\partial}_E : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{0,1}(E)$ defined in (*) extends uniquely to an operator*

$$\overline{\partial} \ = \ \overline{\partial}_E : \mathcal{A}^{p,q}(X,E) \longrightarrow \mathcal{A}^{p,q+1}(X,E)$$

satisfying the Leibnitz rule

$$\overline{\partial}_E(f \cdot s) \ = \ (\overline{\partial}f \wedge s) \ + \ f \cdot \overline{\partial}_E(s),$$

and $\overline{\partial}_E^2 = 0$.

We naturally then define the Dolbeaut cohomology of X with coefficients in E to be

$$H^{p,q}(X,E) \;=\; H^{p,q}_{\overline{\partial}}(E) \;=\; \frac{\ker\left(\overline{\partial} : \mathcal{A}^{p,q}(E) \longrightarrow \mathcal{A}^{p,q+1}(E)\right)}{\operatorname{im}\left(\overline{\partial} : \mathcal{A}^{p,q-1}(E) \longrightarrow \mathcal{A}^{p,q}(E)\right)}.$$

The first statement of Dolbeaut's theorem then generalizes to the assertion that these groups involve the sheaf the cohomolgy of X with coefficients in the sheaf of holomorphic sections of E:

Theorem 5.4. (Dolbeaut's theorem, II). *There are canonical isomorphisms:*

$$H^{p,q}_{\overline{\partial}}(X,E) \;=\; H^q(X, \Omega^p_X \otimes \mathcal{O}_X(E)).$$

In particular,

$$H^{0,q}_{\overline{\partial}}(X,E) \;=\; H^q(X, \mathcal{O}_X(E)).$$

Sheaves. One can think of the theory of sheaves as a tool for managing the questions involved in trying to pass from local to global constructions.

Let X be a topological space.

Definition 5.5. A *pre-sheaf* F of additive abelian groups on X is a rule that assigns to each open set $U \subseteq X$ an abelian group $F(U)$, and to each inclusion $V \subseteq U$ a homomorphism

$$\rho_{U,V} : F(U) \longrightarrow F(V),$$

called a restriction map. These data are required to satisfy the following axioms:

(i). $F(\emptyset) = (0)$

(ii). $\rho_{U,U} = \mathrm{Id}$.

(iii). Given $W \subseteq V \subseteq U$, one has

$$\rho_{U,W} \;=\; \rho_{V,W} \circ \rho_{U,V}.$$

More succinctly, one can say that a presheaf is a contravariant functor from the category of open subsets of X to the category of abelian groups. One defines presheaves of rings, or vector spaces, or ..., similarly. One typically calls the elements of $F(U)$ *sections* of F over U.

Example 5.6. (Holomorphic functions and sections). Let X be a complex manifold. The presheaf $\mathcal{O}_X$ of holomorphic functions on X associates to an open set U the holomorphic functions on U:

$$\mathcal{O}_X(U) \;= \{\text{holomorphic functions on } U\},$$

with $\rho_{U,V}$ being given by restriction. Similarly, given a holomorphic vector bundle E on X, we define $\mathcal{O}_X(E)$ by the rule

$$\mathcal{O}_X(E)(U) \;=\; \Gamma(U,E) \;=\; \{\text{holomorphic sections of } E \text{ over } U\},$$

again with the ρ being ordinary restrictions. Thus $\mathcal{O}_X$ is a presheaf of rings (and $\mathcal{O}_X(E)$ is a sheaf of modules over $\mathcal{O}_X$ in a sense that we will define shortly).

Example 5.7. (Ideal of analytic subvariety). Let X be a complex manifold, and let $V \subseteq X$ be an analytic subvariety. Then one defines the ideal presheaf $\mathcal{I}_V$ of V by taking

$$\mathcal{I}_V(U) \;=\; \{f \in \mathcal{O}_X(U) \mid f|V \equiv 0\}.$$

This is a sub-presheaf of $\mathcal{O}_X$ in the sense that $\mathcal{I}_V(U) \subseteq \mathcal{O}_X(U)$ for all $U \subseteq X$.

Example 5.8. (Smooth forms). Let M be a smooth real manifold. Then the presheaf $\mathcal{A}^k_M$ of smooth k-forms on X is given by

$$\mathcal{A}^k_M(U) \;=\; \mathcal{A}^k(U),$$

with the evident restrictions. So for exampe $\mathcal{A}^0_M$ is the presheaf of smooth functions on M. If X is a complex manifold, one defines presheaves $A^{p,q}_X$ in the analogous fashion.

Example 5.9. (Constant presheaf). Let X be a topological space. The constant presheaf $\mathbf{Z}_{\text{const}}$ on X is the presheaf that assigns to $U \subseteq X$ all constant functions $U \longrightarrow \mathbf{Z}$, ie

$$\mathbf{Z}_{\text{const}}(U) \;=\; \mathbf{Z},$$

with all non-trivial restrictions being the identity.

In all the examples so far, the restriction homomorphisms ρ have been actual restrictions of functions, sections or forms. Although this need not be true in general, it is customary to lighten the notation by writing

$$\rho_{U,V}(s) \;=\; s|V$$

for $s \in F(U)$ and $V \subseteq U$.

A sheaf is a pre-sheaf whose sections are determined locally:

Definition 5.10. Let F be a presheaf on a topological space. One says that F is a *sheaf* if the following condition is satisfied:

> Let U be an open set, and let $U = \cup U_i$ be an open covering of U. Suppose given sections $s_i \in F(U_i)$ satisfying
>
> $$s_i|(U_i \cap U_j) \;=\; s_j|(U_i \cap U_j)$$
>
> for all i, j. Then there exists a unique section $s \in F(U)$ such that
>
> $$s_i \;=\; s \mid U_i.$$

In other words, the condition asks that the s_i should patch together to give a section on all of U.

Example 5.11. The presheaves in Examples 5.6, 5.7 and 5.8 are sheaves.

Example 5.12. The presheaf $\mathbf{Z}_{\text{const}}$ from Example 5.9 is *not* in general a sheaf. For example, take $X = \mathbf{R}$, and consider the disconnected subset $U = \mathbf{R} - \{0\}$ with the open cover

$$U \;=\; (-\infty, 0) \cup (0, \infty).$$

Then evidently constant functions

$$s_- \in \mathbf{Z}_{\text{const}}\big((-\infty,0)\big) \quad , \quad s_+ \in \mathbf{Z}_{\text{const}}\big((0,\infty)\big)$$

do not necessarily patch together to give a constant function on U, even though they vacuously agree on intersection of the two subsets of U in question. However for any topological space X, the presheaf $\mathbf{Z}_X$ of *locally* constant functions on X, given by

$$\mathbf{Z}_X(U) \ = \ \{\text{locally constant functions } U \longrightarrow \mathbf{Z}\},$$

is a sheaf. In particular, $\mathbf{Z}_{\text{const}}$ is not the natural object to look at, and one therefore typically refers to $\mathbf{Z}_X$ as the constant sheaf on X.

It will be particularly important to understand the sections of a sheaf F over all of X: this group is written

$$\Gamma(X,F) \ = \ \Gamma(F) \ = \ F(X),$$

and is called the group of *global sections* of F.

There is a natural notion of a homomorphism of sheaves or presheaves, as well as subsheaves.

Definition 5.13. A *morphism* $\phi : F \longrightarrow G$ of sheaves or pre-sheaves consists in giving for each open $U \subseteq X$ a homomorphism

$$\phi_U : F(U) \longrightarrow G(U)$$

such that the ϕ_U are compatible with the restriction morphisms for F and G.

Definition 5.14. Given sheaves or presheaves F, G, one says that F is a *subsheaf* of G if $F(U) \subseteq G(U)$ is a subgroup for every U, compatably with restrictions.

Example 5.15. Let X be a complex manifold. Then (locally) constant sheaf $\mathbf{Z}_X$ sits as a subsheaf

$$\mathbf{Z}_X \subseteq \mathcal{O}_X$$

of the sheaf $\mathcal{O}_X$ of holomorphic functions on X. On the other hand, denote by $\mathcal{O}_X^*$ the sheaf of nowhere-zero holomorphic functions, given by

$$\mathcal{O}_X^*(U) \ = \ \mathcal{O}_X(U)^* :$$

we view this as a sheaf of multplicative abelian groups. Then the exponential defines a sheaf homomorphism

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^*$$

via the maps

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X^*(U) \quad , \quad f \mapsto e^{(2\pi\sqrt{-1}\cdot f)}.$$

In fact, we will see shortly that $\mathbf{Z}_X = \ker \exp$, and that there is a short exact sequence

$$0 \longrightarrow \mathbf{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0 \tag{5.2}$$

which has very beautiful cohomological consequences.

Example 5.16. Let X be a complex manifold, and fix an index $p \geq 0$. Then the $\overline{\partial}$-operator gives rise to a sheaf homomorphism

$$\overline{\partial} : \mathcal{A}_X^{p,q} \longrightarrow \mathcal{A}_X^{p,q+1}$$

of sheaves of vector spaces.[19] In fact, these fit together into a *complex*

$$\ldots \longrightarrow \mathcal{A}^{p,q-1} \longrightarrow \mathcal{A}_X^{p,q} \longrightarrow \mathcal{A}^{p,q+1} \longrightarrow \ldots \tag{5.3}$$

of sheaves of vector spaces. The proof of the Dolbeaut theorem will revolve around the analysis of this complex.

Recall that we defined the germ of a holomorphic function f at a given point x to be the restriction of f to a small neighborhood of x, and identifying two such restrictions if they agree in a possibly smaller neighborhood. Given any presheaf F, we can define in a completely analogous way the germ of a section of F at a point: this leads to the notion of the stalk of F.

Definition 5.17. Let F be presheaf on a topological space X, and let $x \in X$ be a fixed point. The *stalk* F_x of F at x is defined to be

$$F_x \;=\; \Big\{ (U, s) \Big| U \ni x \;,\; s \in F(U) \Big\} \Big/ \sim$$

where $(U_1, s_1) \sim (U_2, s_2)$ if there is an open neighborhood $x \in V \subseteq U_1 \cap U_2$ such that

$$s_1|V \;=\; s_2|V.$$

Given $s \in F(U)$, the *germ* $s_x \in F_x$ of s at x is determined by the equivalence class of (U, s).

Observe that a morphism $\phi : F \longrightarrow G$ of presheaves determines a homomorphism $\phi_x : F_x \longrightarrow G_x$ for every $x \in X$.

Example 5.18. (Sections from stalks). If F is a sheaf, then for any open set $U \subseteq X$, a section $s \in F(U)$ is determined by its germs $s_x \in F_x$ for $x \in U$. However this can fail if F is merely a presheaf (Exercise 5.1).

There are many constructions that lead naturally to a presheaf F, and it is then important to know that there is a canonical way to pass to a sheaf that is "closest" to F. This is the *sheafification* of F, or the *sheaf associated* to F.

Proposition 5.19. *Let F be a presheaf on a topological space X. Then there is a sheaf F^+, together with a morphism*

$$\theta : F \longrightarrow F^+$$

characterized by the property that any homomorphism $\phi : F \longrightarrow G$ from F to a sheaf G factors uniquely through θ. Moreover, the canonical homomorphism

$$\theta_x : F_x \longrightarrow F_x^+$$

is an isomorphism for all $x \in X$.

[19] Note that since $\overline{\partial}$ is not linear over the ring of smooth functions, 5.3 is not a homomorphism of $\mathcal{A}_X^0$-modules.

We refer eg to [] for the proof. However we can quicky explain the construction of F^+. Start by forming the disjoint union

$$U \ = \ \coprod_{x \in X} F_x$$

of all the stalks of F, and fix $U \subseteq X$. Then $F^+(U)$ consists of all functions

$$t : U \longrightarrow T \quad , \quad \text{with } t(x) \in F_x \text{ for all } x,$$

having the property that for any $y \in U$, there is a neighborhood $U \supseteq V \ni y$ and a section $s \in F(V)$ such that

$$t(z) \ = \ s_z \ \text{ for all } z \in V.$$

We can now define the kernel and cokernel of a homomorphism of sheaves.

Definition 5.20. Let

$$\phi : F \longrightarrow G$$

be a homomorphism of sheaves. Then $\ker(\phi)$ is the sheaf:

$$\ker(\phi)(U) \ = \ \ker\big(\phi_U : F(U) \longrightarrow G(U)\big).$$

Similarly, $\mathrm{im}(\phi)$ is the sheaf associated to the presheaf

$$U \mapsto \Big(\mathrm{im}\big(\phi_U : F(U) \longrightarrow G(U)\big)\Big).$$

One says that ϕ is *surjective* if $\mathrm{im}(\phi) = G$.

Example 5.21. Let $\phi : F \longrightarrow G$ be a homomorphism of sheaves. Then ϕ is surjective if and only if given any point $x \in X$,a neighborhood $U \ni x$ and a section $s \in G(U)$, one can find a smaller neighborhood $x \in V \in U$ such that

$$s|V \ = \ \phi_V(t) \ \text{ for some } t \in F(V).$$

Example 5.22. Let X be a complex manifold. Then the exponential mapping

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^*$$

defined above is surjective, and

$$\ker(\exp) \ = \ \mathbf{Z}_X.$$

(Exercise 5.4.) In other words, the exponential sequence (5.2) is exact.

Note in particular that the surjectivity of ϕ does *not* imply the surjectivity of $\phi_U : F(U) \longrightarrow G(U)$ for every open set $U \subseteq X$. For example, let $X = \mathbf{C}$, and consider the exponential map

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^*.$$

As noted in the previous example, this is surjective as a morphism of sheaves. However if $U = \mathbf{C} - \{0\}$, then the function $z \in \mathcal{O}_X^*(U)$ is not in the image of exp (one cannot define a single-valued branch of $\log(z)$ on $\mathbf{C} - \{0\}$).

The quotient of a sheaf by a subsheaf is defined similarly. Specifically, given $F \subseteq G$, one defines F/G to be the sheaf associated to the subsheaf

$$U \mapsto \Big(G(U)/F(U)\Big).$$

Thus a section $s \in (G/F)(U)$ is determined by the condition that one can find an open covering $U = \cup U_i$, together with sections $t_i \in G(U_i)$, such that t_i maps to $s|U_I$.

Having discussed quotients and kernels, one can then define exact sequences of sheaves:

$$0 \longrightarrow G \longrightarrow F \longrightarrow Q \longrightarrow 0,$$

which implies that G is a sub sheaf of F, and that $Q = F/G$. In general, a sequence of sheaves on X is exact if and only if the corresponding sequences of stalks are exact at every point.

Consider again now a complex manifold X.

Definition 5.23. A sheaf F on X is called an $\mathcal{O}_X$-module if $F(U)$ is an $\mathcal{O}_X(U)$-module for each open set $U \subseteq X$, compatibly with the restriction maps.

Example 5.24. The sheaf $\mathcal{O}_X(L)$ of sections of a holomorphic line bundle is an $\mathcal{O}_X$-module: in fact, it is locally isomorphic to $\mathcal{O}_X$. Similarly for vector bundles. In this case $\Gamma(X, \mathcal{O}_X(L))$ is the space of global sections of L as discussed above.

Example 5.25. Fix a point $P \in X$, and consider the sheaf $\mathcal{O}_X(L) \otimes \mathfrak{m}_P$ of sections of L vanishing at P:

$$\big(\mathcal{O}_X(L) \otimes \mathfrak{m}_P\big)(U) \ = \ \{s \in \Gamma(U, L) \mid s(P) = 0\}.$$

This is an $\mathcal{O}_X$-module sitting as a sub-module of $\mathcal{O}_X(L)$, and the quotient

$$G \ = \ \mathcal{O}_X(L)/(\mathcal{O}_X(L) \otimes \mathfrak{m}_P)$$

is a *sky-scraper* sheaf supported at P. The stalks of G are

$$G_x \ = \ \begin{cases} \mathbf{C} & \text{if } x = P \\ 0 & \text{otherwise.} \end{cases}$$

The natural map

$$\mathcal{O}_X(L)(U) \longrightarrow G(U)$$

is given by evaluation at P. We usually denote such a sky-scraper sheaf as $\mathbf{C}_P$.

In our discussion of cohomology, it will be convenient to make a simplifying assumption. Although we will not prove this here, we note the assumption holds in practice for all "reasonable" sheaves of $\mathcal{O}_X$ module on a complex manifold.

Simplifying Assumption 5.26. Let X be a complex manifold, and let

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

be an exact sequence of $\mathcal{O}_X$-modules on X. It will sometimes be convenient to assume that there exist arbitrarily fine open coverings $\mathfrak{A} = \{U_i\}$ of X such that the sequences

$$0 \longrightarrow F(U_I) \longrightarrow E_(U_I) \longrightarrow G(U_I) \longrightarrow 0 \tag{5.4}$$

are exact, where for each multi-index $I = \{i_0 < \ldots < i_p\}$, U_I denotes the intersection of the corresponding elements of the covering.

Assume that for every $x \in X$, the stalks

$$F_x \ , \ E_x \ , \ G_x \ \text{ are finitely generated } \mathcal{O}_{X,x}\text{-modules.}$$

Then in fact this condition is satisfied.

Cohomology of Sheaves. We now introduce (somewhat informally) the cohomology of X with coefficients in a sheaf.

By way of motivation, consider an exact sequence

$$0 \longrightarrow F \longrightarrow E \xrightarrow{\pi} G \longrightarrow 0$$

of sheaves on X, and suppose that we are in the situation of Fact 5.26. Writing as above $\Gamma(X, F) = F(X)$ for the space of global sections of F, we want to investigate when the map

$$\Gamma(X, E) \longrightarrow \Gamma(X, G)$$

induced by π is surjective. For this, use 5.26 to choose an open covering $\mathfrak{A} = \{U_i\}$ of X such that we have a short exact sequence

$$0 \longrightarrow F(U_i) \longrightarrow E(U_i) \xrightarrow{\pi} G(U_i) \longrightarrow 0$$

for each i. Given a section $s \in \Gamma(X, G)$, put

$$s_i \ = \ s \mid U_i \ \in \ G(U_i).$$

Then the s_i lift to $t_i \in E(U_i)$, ie $\pi(t_i) = s_i$. One cannot hope that the t_i patch together to give a global section $t \in \Gamma(X, E)$ – after all, there is some ambiguity in choosing them – but we can study systematically the extent to which this fails. In fact, on $U_{ij} = U_i \cap U_j$ consider the difference

$$r_{ij} \ = \ t_i|U_{ij} - t_j|U_{ij}.$$

Then

$$\pi(r_{ij}) \ = \ \big(\pi(t_i) - \pi(t_j)\big) \mid U_{ij} \ = \ \big(s_i - s_j\big) \mid U_{ij} \ = \ 0.$$

So we may view $r_{ij} \in \ker(\pi)$, i.e we may view as a section of F over U_{ij}:

$$r_{ij} \ \in \ F(U_{ij}).$$

Note that on U_{ijk} the (restrictions of the) r_{ij} satisfy

$$\big(r_{ij} - r_{ik} + r_{kj}\big) \mid U_{ijk} \ = \ 0.$$

The liftings t_i are not unique: if we replace t_i by

$$t_i' \ = \ t_i + r_i \quad , \quad r_i \in F(U_i),$$

then the t_i' are again liftings of s_i. Moreover the t_i' give rise to $r_{ij}' \in F(U_{ij})$ satisfying

$$(r_{ij}' - r_{ij}) \ = \ \big(r_i - r_j\big) \mid U_{ij}.$$

We claim now that s lifts to a section $t \in \Gamma(X, E)$ if and only if we can find $r_i \in F(U_i)$ such that

$$r_{ij} \;=\; \big(r_i - r_j\big) \mid U_{ij}.. \tag{*}$$

In fact, supposing that (*) holds, define

$$t'_i \;=\; t_i - r_i.$$

Then $(t'_i - t'_j) = 0$ on U_{ij}, so the t'_i patch to give a global section $t \in \Gamma(X, E)$ that maps to s.

We can rephrase this discussion more systematically by introducing the group

$$\check{H}^1(\mathfrak{A}, F) \;=\; \frac{\{r_{ij} \in F(U_{ij}) \mid r_{ij} - r_{jk} + r_{ki} = 0\}}{\{r_i - r_j \mid r_i \in F(U_i)\}}.$$

Then in effect we have constructed a homomorphism

$$\delta : \Gamma(X, G) \longrightarrow \check{H}^1(\mathfrak{A}, F)$$

that fits into an exact sequence

$$0 \longrightarrow \Gamma(X, F) \longrightarrow \Gamma(X, E) \longrightarrow \Gamma(X, G) \xrightarrow{\delta} \check{H}^1(\mathfrak{A}, F). \tag{5.5}$$

In other words, we have cooked up a (cohomology) group H^1 which gives the obstruction to the surjectivity of the map $\Gamma(X, E) \longrightarrow \Gamma(X, G)$. As we shall see, in practice it is the H^0 and H^1 that are of most immediate importance, but to get our hands on the H^1 we often need to consider an H^2, etc. So we are led to introduce the whole panaply of cohomology groups.

We now turn to the formal definitions. Let X be a topological space, let $\mathfrak{A} = \{U_i\}$ be an open covering of X, and let F be a sheaf on X.

Definition 5.27. The group of Cech p-cochains is

$$C^p(\mathfrak{A}, F) \;=\; \prod_{i_0, i_1, \ldots, i_p} F(U_{i_0 i_1 \ldots i_p}).$$

Thus an element $\gamma \in C^p(\mathfrak{A}, F)$ consists in giving for each $p+1$-fold intersection of the U_i a section

$$\gamma(i_0, \ldots, i_p) \;\in\; F(U_{i_0 i_1 \ldots i_p}$$

of F over the corresponding open set.

Note that we do not require that the U_i appearing in the definition be distinct.

We next define the Cech boundary operator

$$\delta : C^p(\mathfrak{A}, F) \longrightarrow C^{p+1}(\mathfrak{A}, F)$$

by the rule

$$\delta(\gamma)(i_0, \ldots, i_{p+1}) \;=\; \sum_{j=0}^{p+1} (-1)^j \mathrm{res}\big(\gamma(i_0, \ldots, \widehat{i_j}, \ldots i_{p+1})\big),$$

where each of the sections on the right is restricted to the open set $U_{i_0 \ldots i_{p+1}}$.

In the usual way, one checks that $\delta^2 = 0$, and then we can take cohomology:

Definition 5.28. The Cech cohomology of F with respect to the covering $\mathfrak{A}$ the cohomology of the Cech complex just constructed:

$$\check{H}^p(\mathfrak{A}, F) \ = \ H^p\big(C^*(\mathfrak{A}, F)\big).$$

Example 5.29. For any sheaf F and any open covering $\mathfrak{A}$,

$$\check{H}^0(\mathfrak{A}, F) \ = \ \Gamma(X, F).$$

(Exercise 5.6.)

Example 5.30. The reader should check that the case $p = 1$ of Definition 5.28 agrees with the group constructed by hand above.

Of course one would like to remove the dependence on the covering. Although we will gloss over this point, for this one passes to refinements. In brief, suppose that $\mathfrak{A}' \prec \mathfrak{A}$ is a refinement of $\mathfrak{A}$. One can then construct chain maps

$$C^p(\mathfrak{A}, F) \longrightarrow C^p(\mathfrak{A}', F)$$

which are uniquely defined up to homotopy. This gives rise to canonically defined homomorphisms

$$\check{H}^p(\mathfrak{A}, F) \longrightarrow \check{H}^p(\mathfrak{A}', F),$$

and one then passes to a direct limit:

Definition 5.31. The Cech cohomology of X with coefficients in a sheaf F is detined to the the direct limit

$$\check{H}^p(X, F) = \varinjlim \check{H}^p(\mathfrak{A}, F),$$

where the limit is taken over all open covers.

It turns out that for most of the sheaves F that arise in day to day complex or algebraic geometry – specifically "coherent analytic" or "coherent algebraic" sheaves – there is a single cover $\mathfrak{A}$ with the property that all the cohomology groups $\check{H}^p(X, F)$ can be computed as $H^p(\mathfrak{A}, F)$. So in practice one usually doesn't need to pass to the limit.

Example 5.32. Given a complex manifold X, denote by $\mathcal{O}_X^*$ the sheaf of nowhere-vanishing holomorphic functions on X: this is a sheaf of abelian groups under multiplication. Then

$$\check{H}^1(X, \mathcal{O}_X^*) \ = \ \Big\{\text{isomorphism classes of holomorphic line bundles on } X\Big\}.$$

In fact, $\check{H}^1(\mathfrak{A}, \mathcal{O}_X^*)$ classifies isomorphism classes of line bundles that trivialize on the open cover $\mathfrak{A}$: the corresponding transition functions $g_{ij} \in \mathcal{O}_X^*(U_{ij})$ give the cocycle determined by a line bundle.

The most important property of the theory is that a short exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0 \tag{*}$$

of sheaves on X should give rise to a long exact sequence on cohomology extending (5.5). This is not universally true for Cech cohomology, but it does work on paracompact Hausdorff spaces.

Theorem 5.33. *Let X be a paracompact Hausdorff space, and suppose given a short exact sequence (*) of sheaves on X. Then (*) induces a long exact sequence*

$$0 \longrightarrow \check{H}^0(X,F) \longrightarrow \check{H}^0(X,E) \longrightarrow \check{H}^0(X,G) \longrightarrow \check{H}^1(X,F) \longrightarrow \check{H}^1(X,E) \longrightarrow \check{H}^1(X,E)\check{H}^2(X,F) \longrightarrow \dots$$

of cohomology groups.

We will not try to prove this in general, but the result is easy to derive assuming that we are in the situation of the simplfying ssumption 5.26. In fact, it follows from (5.4) and the definitions that the short exact exact sequence (*) gives rise to a short exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{A},F) \longrightarrow C^\bullet(\mathfrak{A},E) \longrightarrow C^\bullet(\mathfrak{A},G) \longrightarrow 0.$$

This gives rise to a long exact sequence of the cohomology groups of $\mathfrak{A}$, and since we assume that 5.26 holds for arbitrarily fine covers, we get the conclusion of 5.33 by passing to a limit.

Example 5.34. Consider the exponential sequence

$$0 \longrightarrow \mathbf{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

on a compact connected complex manifold X. Since

$$H^0(X,\mathcal{O}_X) = \mathbf{C}\ ,\ H^0(X,\mathcal{O}_X^*) = \mathbf{C}^*$$

(the only holomorphic functions on X are constant), the corresponding long exact sequence on cohomology effectivelly starts with H^1 and takes the form:

$$0 \longrightarrow H^1(X,\mathbf{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow H^2(X,\mathbf{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow \dots.$$

Recalling that $H^1(X,\mathcal{O}_X^*)$ classifies isomorphism classes of holomorphic line bundles on X, this will later give us a rather precise description of all such. In particular this sequence will lead to the proof of the Lefschetz $(1,1)$-theorem.

Proof of the Dolbeaut theorem. We're now ready to indicate the proof of the Dolbeaut theorem.

Let X be a complex manifold of (complex) dimension n.

Definition 5.35. A holomorphic p-form on X is a $(p,0)$-form η which can locally be expressed as

$$\eta \ = \ \sum f_I dz_I,$$

where the f_I are holomorphic functions. Equivalently, η is a holomorphic section of the holomorphic vector bundle $\Lambda^{p,0}T_X^*$. We denote by Ω_X^p the holomorphic locally free sheaf of holomorphic p-forms. Given a holomorphic vector bundle E on X, we speak similarly of E-valued holomorphic p-forms, corresponding to the sheaf $\Omega_X^p \otimes E$.

We denote by $\mathcal{A}^{p,q} = \mathcal{A}^{p,q}_X$ the sheaf of smooth (p,q) forms on X. Thus the space $\mathcal{A}^{p,q}(X)$ of global (p,q)-forms on X is just the group of global sections of this sheaf. Fixing an integer $p \geq 0$, we get as in Example 5.16 a complex of sheaves $\mathcal{A}^{p,\bullet}$:

$$0 \longrightarrow \mathcal{A}^{p,0} \longrightarrow \mathcal{A}^{p,1} \longrightarrow \ldots \longrightarrow \mathcal{A}^{p,q} \longrightarrow \ldots \tag{5.6}$$

Remainder of section not yet written...

Exercises for Section 5.

Exercise 5.1. Let F be a sheaf, and let $U \subseteq X$ be an open set. Suppose that $s, t \in F(U)$ are two sections such that

$$s_x = t_x \in F_x$$

for all $x \in U$. Show that then $s = t$. However give an example to show that this can fail if F is merely a presheaf.

Exercise 5.2. Let $F = \mathbf{Z}_{\text{const}}$ be the constant pre-sheaf on a space X. Show that the sheafification F^+ of F is the sheaf $\mathbf{Z}_X$ of locally constant integer valued functions on X.

Exercise 5.3. Let $\phi : F \longrightarrow G$ be a morphism of sheaves. Show that

$$\ker(\phi)_x = \ker(\phi_x : F_x \longrightarrow G_x) \quad , \quad \operatorname{im}(\phi)_x = \operatorname{im}(\phi_x : F_x \longrightarrow G_x).$$

In particular, ϕ is injective or surjective if and only if all the maps ϕ_x on stalks are so.

Exercise 5.4. Prove the assertions of Example 5.22

Exercise 5.5. (Locally free sheaves) Prove that a sheaf F on X is of the form $\mathcal{O}_X(L)$ for a holomorphic line bundle on X if and only if there is an open covering $\{U_i\}$ of X such that $F|U_i \cong \mathcal{O}_{U_i}$ for each i. State and prove the analogous assertion for vector bundles of higher rank.

Exercise 5.6. Prove the assertion of Example 5.29

Exercise 5.7. Work out in detail the assertion of Example 5.32. Show similarly that isomorphism classes of smooth complex line bundles on a complex manifold X are classified by

$$\check{H}^1(X, \mathcal{A}^*),$$

where $\mathcal{A}^*$ denotes the sheaf of nowhere vanishing smooth $\mathbf{C}$-valued functions on X.

6. Kähler Manifolds

In this section, much of the preceeding material comes together in the Kähler package, which shows that compact complex manifolds that carry a Kähler metric have a number of truly miraculous prorperties.

The Kähler condition. Let X be a complex manifold of (complex) dimension n, and let J be the corresponding almost complex structure. We suppose that X carries a Riemannian metric $g = < \, , \, >$ that is compatible with J, so that the fundamental two-form

$$\omega \in \mathcal{A}^{1,1}(X) \ , \ \omega(v, w) \ = < Jv \, , \, w >$$

is defined. As in Definition 4.5, we say that (X, J, ω) is a Kähler manifold – and that g (or ω) is a Kähler metric if $d\omega = 0$.[20]

Example 6.1. Any Riemann surface (complex manifold of dimension = 1) is Kähler.

Example 6.2. The standard constant metric g on $\mathbf{C}^n$, with fundamental form

$$\omega \ = \ \frac{\sqrt{-1}}{2} \sum dz_i \wedge d\overline{z}_i,$$

is Kähler. Since the metric is invariant under translations, it follows that any complex torus $X = \mathbf{C}^n / \Lambda$ is Kähler.

Example 6.3. (Submanifolds) If (X, g) is a Kähler manifold, then any complex submanifold $Y \subseteq X$ inherits a natural Kähler structure.

Example 6.4. Let (X, g, ω) be a compact Kahler manifold of dimension n. Since ω is closed, it defines a class

$$[\omega] \in H^2(X, \mathbf{C}).$$

Moreover since ω^n is (a multiple of) the volume form determined by g, one has

$$\int_X \omega^n \ > \ 0,$$

and hence $[\omega] \neq 0$. In particular, if X is a compact Kähler manifold, then

$$H^2(X, \mathbf{C}) \ \neq \ 0,$$

and in fact all the even Betti numbers of X are non-zero. (In due time we will see that the Hodge decomposition gives some much more subtle topological restrictions to a compact complex manifold being Kähler, eg that the odd Betti numbers must be even.)

Example 6.5. (Hopf manifolds). We constructed in Example 2.16 a compact complex manifold X of dimension n that is diffeomorphic to $S^{2n-1} \times S^1$. It follows from the previous example that if $n > 1$ then X cannot carry a Kähler metric.

Fubini-Study metric. What gives the Kähler condition its real importance is that projective space carries a natural $\mathrm{SU}(n+1)$-invariant Kähler metric, the so-called Fubini–Study metric. Therefore any complex submanifold of projective space also carries a Kahler metric. We construct the Fubini-Study metric by building an $\mathrm{SU}(n+1)$-invariant Hermitian metric H_{FS} on $\mathbf{P}^n$. The Fubini–Study form ω_{FS} will then arise as the negative imaginary part $\omega_{\mathrm{FS}} = -\mathrm{Im}\, H_{\mathrm{FS}}$ of this Fubini–Study metric.[21]

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[20] One sometimes says that a manifold in Kähler if it admits a Kähler metric, but for now we'll generally suppose given a particular Kähler metric.

[21] The following paragraphs are lifted from Section 1.2.C of my positivity book.

Consider the standard Hermitian inner product $h(v,w) = {}^t v \cdot \overline{w}$ on $V = \mathbf{C}^{n+1}$. Set $V^0 = V - \{0\}$ and denote by

$$\rho : V^0 \longrightarrow \mathbf{P}^n = \mathbf{P}(V)$$

the canonical map. Define to begin with a Hermitian metric H' on V^0 by associating to $x \in V^0$ the Hermitian inner product

$$H'_x(v,w) = h\left(\frac{v}{|x|}, \frac{w}{|x|}\right) \qquad \text{for } v\,,\, w \in T_x V^0 = V$$

and $|x| = \sqrt{h(x,x)}$. The metric H' is constructed so as to be invariant under the natural $\mathbf{C}^*$-action on V^0. Now $\rho^* T\mathbf{P}^n$ is canonically a quotient of TV^0, and so H' induces in the usual manner a $\mathbf{C}^*$-invariant metric on $\rho^* T\mathbf{P}^n$, which then descends to a Hermitian metric H_{FS} on $T\mathbf{P}^n$.

More explicitly, write $W_x \subseteq V$ for the H'_x-orthogonal complement to $\mathbf{C} \cdot x \subseteq V$, and let $\pi_x : V \longrightarrow W_x$ be orthogonal projection:

$$\pi_x(v) = v - \frac{h(v,x)}{h(x,x)} \cdot x.$$

Then W_x is identified with $T_{\rho(x)}\mathbf{P}^n$ and π_x with $d\rho_x$, and

$$\begin{aligned} H_{\rho(x)}\big(d\rho_x v, d\rho_x w\big)_{\mathrm{FS}} &= H'_x(\pi_x v, \pi_x w) \\ &= \frac{h(v,w)h(x,x) - h(v,x)h(x,w)}{h(x,x)^2}. \end{aligned}$$

If we take the usual affine local coordinates $z_1, \ldots, z_n$ on $\mathbf{P}^n$ — corresponding to $x = (1, z_1, \ldots, z_n) \in V^0$ — then one finds[22] that

$$\begin{aligned} \omega_{\mathrm{FS}} &=_{\mathrm{def}} -\mathrm{Im}\, H_{\mathrm{FS}} \\ &=_{\mathrm{locally}} \frac{\sqrt{-1}}{2} \cdot \left(\frac{\sum dz_\alpha \wedge d\bar{z}_\alpha}{1 + \sum |z_\alpha|^2} - \frac{\big(\sum \bar{z}_\alpha dz_\alpha\big) \wedge \big(\sum z_\alpha d\bar{z}_\alpha\big)}{\big(1 + \sum |z_\alpha|^2\big)^2} \right). \end{aligned}$$

By construction H_{FS} is invariant under the natural $\mathrm{SU}(n+1)$-action on $\mathbf{P}^n$, and hence so too is ω_{FS}.

We next verify that ω_{FS} is indeed a Kähler form, i.e. that ω_{FS} is closed. Following Mumford, one can use the $\mathrm{SU}(n+1)$ invariance to give a quick proof of this. In fact, given $p \in \mathbf{P}^n$ choose an element $\gamma \in \mathrm{SU}(n+1)$ such that $\gamma(p) = p$ while $d\gamma_p = -\mathrm{Id}$. Then for any three tangent vectors $u, v, w \in T_p\mathbf{P}^n$ one has

$$d\omega_{\mathrm{FS}}(u,v,w) = \gamma^*(d\omega_{\mathrm{FS}})(u,v,w) = d\omega_{\mathrm{FS}}(-u,-v-w),$$

and hence $d\omega_{\mathrm{FS}} = 0$. □

[22] It may be useful to consider here an n-dimensional vector space $W = \mathbf{C}^n$, with its standard Hermitian product $h(u,v) = {}^t u \cdot \overline{v}$. Then as w varies over W the expressions

$$\eta_w(u,v) = -\mathrm{Im}\big(h(u,v)\big) \quad , \quad \eta'_w(u,v) = -\mathrm{Im}\big(h(u,w)h(w,v)\big)$$

define $(1,1)$-forms η and η' on W, which in terms of standard linear coordinates $w_1, \ldots, w_n$ on W are given by $\eta = \frac{\sqrt{-1}}{2} \cdot \sum dw_\alpha \wedge d\overline{w}_\alpha$ and $\eta' = \frac{\sqrt{-1}}{2} \cdot \big(\sum \overline{w}_\alpha dw_\alpha\big) \wedge \big(\sum w_\alpha d\overline{w}_\alpha\big)$.

Another approach to the Fubini–Study metric involves the Hopf map. Keeping the notation of the previous example, consider the unit sphere

$$\mathbf{C}^{n+1} \;\supseteq\; S^{2n+1} \;=\; S$$

with respect to the standard inner product $\langle\ ,\ \rangle$, with

$$p \,:\, S \longrightarrow \mathbf{P}^n$$

the Hopf mapping. Denote by ω_{std} the standard Kähler form on $\mathbf{C}^{n+1}$, i.e.

$$\omega_{\text{std}} \;=\; \sum dx_\alpha \wedge dy_\alpha,$$

where $z_\alpha = x_\alpha + iy_\alpha$ are the usual complex coordinates on $\mathbf{C}^{n+1}$. Then ω_{FS} is characterized as the unique symplectic form on $\mathbf{P}^n$ having the property that

$$p^*\omega_{\text{FS}} \;=\; \omega_{\text{std}} \mid S.$$

(This follows from the construction in the previous example.)

Remark 6.6. (Normalization) There are various different normalizing conventions in the literature. As shown in [3, p. 119] (where Huybrechts makes a different choice), one has

$$\int_{\mathbf{P}^1} \omega_{FS} \;=\; \pi.$$

Noting that the Fubini-Study metric on $\mathbf{P}^n$ restricts to the Fubini–Study metric on any linear subspace, it follows that $\frac{1}{\pi} \cdot \omega_{\text{FS}}$ represents a generator of

$$H^2(\mathbf{P}^n, \mathbf{Z}) \;=\; \mathbf{Z}.$$

The Kähler Identitites. We next study various pointwise and differential operators on a complex manifold, and the relations that hold among them in the Kähler setting.

As in the beginnng of this section, consider a complex manifold X of (complex) dimension n with a compatible metric $g =<,>$ and fundamental form ω. Then all the constructions and results from Section 3 extend to give pointwise operators on the tangent and cotangent bundles of X and their exterior powers, as well as on the spaces of forms on X. Thus one has the Hodge $*$-operator

$$* : \Lambda^k T^*X \longrightarrow \Lambda^{2n-k}T^*X \quad , \quad * : \mathcal{A}^k(X) \longrightarrow \mathcal{A}^{2n-k}(X),$$

characterized by the property that

$$\alpha \wedge *\beta \;=\; <\alpha, \beta> \cdot \,\text{vol}.$$

Similarly the Lefschetz operator $\alpha \mapsto \alpha \wedge \omega$ gives

$$L : \Lambda^k T^*X \longrightarrow \Lambda^{k+2}T^*X \quad , \quad L : \mathcal{A}^k(X) \longrightarrow \mathcal{A}^{k+2}(X),$$

and its dual operator $\Lambda =^{-1} \circ L \circ *$ gives

$$\Lambda : \Lambda^k T^*X \longrightarrow \Lambda^{k-2}T^*X \quad , \quad \Lambda : \mathcal{A}^k(X) \longrightarrow \mathcal{A}^{k-2}(X).$$

Note that while one can view L as a morphism of homolorphic vector bundles, this is not the case for Λ. As before, we can extend these – under the same names – to the complexified cotangent spaces.

We now bring differentiation into the picture. Specifically, we introduce three operators:

$$\begin{aligned} d^* &: \mathcal{A}^k(X) \longrightarrow A^{k-1}(X) \\ \partial^* &: \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p-1,q}(X) \\ \overline{\partial}^* &: \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q-1}(X) \end{aligned} \tag{6.1}$$

defined as the compositions

$$\begin{aligned} d^* &= -*\,d\,* \\ \partial^* &= -*\,\overline{\partial}\,* \\ \overline{\partial}^* &= -*\,\partial\,* \end{aligned} \tag{6.2}$$

Thus $d^* = \partial^* + \overline{\partial}^*$.

One can interpret these operators as (formal) adjoints to $d, \partial, \overline{\partial}$ with respect to natural inner products on the spaces of forms in question. Specifically, given a compact complex manifold (X, g) as above define a Hermitian inner product on the spaces $\mathcal{A}^k(X)$, $\mathcal{A}^{p,q}(X)$ by the formula

$$(\alpha, \beta)_X \;=\; \int_X \alpha \wedge *\overline{\beta}. \tag{6.3}$$

If as in §3 we denote by $<\,,\,>_{\text{herm}}$ the Hermitian extension of g, then it follows from equation (3.17) that

$$(\alpha, \beta)_X \;=\; \int_X <\alpha, \beta>_{\text{herm}}\; d\text{vol}. \tag{6.4}$$

The adjoint property of $\overline{\partial}^*, \partial^*, d^*$ are given by

Proposition 6.7. *For every $\alpha \in \mathcal{A}^{p,q-1}(X)$ and $\beta \in \mathcal{A}^{p,q}(X)$ one has*

$$(\overline{\partial}\alpha, \beta)_X \;=\; (\alpha, \overline{\partial}^*\beta)_X.$$

and analogously for ∂^* and d^*.

Proof. Note to begin with that

$$\overline{\partial}\left(\alpha \wedge *\overline{\beta}\right) \;=\; \overline{\partial}\alpha \wedge *\overline{\beta} \;+\; (-1)^{p+q-1}\,\alpha \wedge \overline{\partial} * \overline{\beta}.$$

Now $d = \overline{\partial}$ on forms of type $(n, n-1)$ and hence by Stoke's theorem

$$(\overline{\partial}\alpha, \beta)_X \;=\; (-1)^{p+q} \int_X \alpha \wedge \overline{\partial} * \overline{\beta}.$$

On the other hand, since $\overline{\partial\eta} = \overline{\partial}\overline{\eta}$, and since $*^2 = (-1)^k$ on k-forms, one has

$$\overline{- * * \partial * \beta} \;=\; (-1)^{p+q}\overline{\partial} * \overline{\beta}.$$

Therefore

$$(\alpha, \overline{\partial}^*\beta)_X \;=\; \int_X \alpha \wedge (\overline{- * * \partial * \beta}) \;=\; (-1)^{p+q} \int_X \alpha \wedge \overline{\partial} * \overline{\beta},$$

as required. □

We next study the commutation relations among these operators when X is Kähler. The first remark is

Proposition 6.8. *Assume that X is Kähler. Then*

$$[\overline{\partial}, L] \; = \; [\partial, L] \; = \; 0 \tag{6.5}$$

$$[\overline{\partial}^*, \Lambda] \; = \; [\partial^*, \Lambda] \; = 0. \tag{6.6}$$

Proof. Note that $\partial\omega$ and $\overline{\partial}\omega$ are respectively the $(2,1)$ and $(1,2)$ components of $d\omega$, and hence $d\omega = 0$ if and only if $\partial\omega = \overline{\partial}\omega = 0$. Therefore

$$\partial(\alpha \wedge \omega) \; = \; \partial\alpha \wedge \omega \quad , \quad \overline{\partial}(\alpha \wedge \omega) \; = \; \overline{\partial}\alpha \wedge \omega,$$

which gives (6.5). For (6.6) one first uses that $\Lambda = *^{-1}L*$ (Proposition 3.14) and the fact that $*^2 = (-1)^k$ on k-forms to prove that

$$[\overline{\partial}^*, \Lambda] \; = \; - * [\partial, L]*,$$

and the assertion follows. □

The deeper identity involves the commutation relations between Λ and ∂ and $\overline{\partial}$.

Theorem 6.9. *Assume that (X, g) is Kähler. Then*

$$[\Lambda, \overline{\partial}] \; = \; -(\sqrt{-1})\, \partial^* \quad , \quad [\Lambda, \partial] \; = \; (\sqrt{-1})\, \overline{\partial}^* \tag{6.7}$$

$$[L, \overline{\partial}^*] \; = \; = -(\sqrt{-1})\, \partial \quad , \quad [L, \partial^*] \; = \; (\sqrt{-1})\, \overline{\partial} \tag{6.8}$$

Turning to a sketch of the proof of the theorem, note that the assertion is an identity between first-order differential operators. Therefore thanks to Theorem 4.6 it suffices to prove it in a neighborhood of the origin in $\mathbf{C}^n$ with the standard (constant) metric. While one could work on $\mathbf{C}^n$ with compactly supported forms, it is convient to keep dealing with compact manifolds. Therefore, as do Griffiths–Harris, we will work in the following paragraphs with a complex torus $T = \mathbf{C}^n/\Lambda$ with the flat Kähler metric coming from $\mathbf{C}^n$. I will closely follow Schnell's presentation in [4].

In the sequel, we write $dz_I, d\overline{z}_J$ etc for the forms on T induced by the indicated translation invariant forms on $\mathbf{C}^n$. We assume that the flat metric g is normalized so that $\mathrm{vol}_g(T) = 1$, and we denote by h the corresponding Hermetian metric on T. Note that

$$h(dz_I \wedge d\overline{z}_J, dz_I \wedge d\overline{z}_J) \; = \; 2^{|I|+|J|},$$

and hence

$$(dz_I \wedge d\overline{z}_J, dz_I \wedge d\overline{z}_J)_T \; = \; 2^{|I|+|J|}.$$

Given an index $1 \leq i \leq n$, we define operators:

$$\begin{aligned} e_i : \mathcal{A}^{p,q}(X) \longrightarrow A^{p+1,q}(T) \quad &, \quad \alpha \mapsto dz_i \wedge \alpha \\ \overline{e}_i : \mathcal{A}^{p,q}(X) \longrightarrow A^{p,q+1}(T) \quad &, \quad \alpha \mapsto d\overline{z}_i \wedge \alpha. \end{aligned} \tag{6.9}$$

Thus

$$L(\alpha) \;=\; \frac{\sqrt{-1}}{2}\Big(\sum dz_i \wedge d\overline{z}_i\Big) \wedge \alpha \;=\; \frac{\sqrt{-1}}{2}\Big(\sum e_i \overline{e}_i\Big)(\alpha).$$

One next defines the adjoint

$$e_i^* : \mathcal{A}^{p,q}(T) \longrightarrow \mathcal{A}^{p-1,q}(T)$$

by the condition that e_i^* be adjoint with respect to $<,>_{\text{herm}}$ of the point-wise operator given by the same formula as e_i, so that

$$(e_i\alpha, \beta)_T \;=\; (\alpha, e_i^*\beta)_T.$$

A combinatorial argument now proves the:

Lemma 6.10. *Fix an index* $1 \le i \le n$.

(i). *If* $i \notin J$ *then*

$$e_i^*(dz_J \wedge d\overline{z}_K) \;=\; 0 \quad , \quad e_i^*\big(dz_i \wedge dz_J \wedge d\overline{z}_K\big) \;=\; 2dz_J \wedge d\overline{z}_K.$$

(ii).

$$e_j e_i^* + e_i^* e_j \;=\; \begin{cases} 2 \cdot \mathrm{Id} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

See [4, Lemma 22.1] for the proof.

Next, define differential operators

$$\begin{aligned} \partial_i : \mathcal{A}^{p,q}(T) \longrightarrow \mathcal{A}^{p,q}(T) \quad , \quad \sum \phi_{J,K}\, dz_I \wedge d\overline{z}_K \mapsto \sum \frac{\partial \phi_{J,K}}{\partial z_i}\, dz_J \wedge d\overline{z}_K \\ \overline{\partial}_i : \mathcal{A}^{p,q}(T) \longrightarrow \mathcal{A}^{p,q}(T) \quad , \quad \sum \phi_{J,K}\, dz_J \wedge d\overline{z}_K \mapsto \sum \frac{\partial \phi_{J,K}}{\partial \overline{z}_i}\, dz_J \wedge d\overline{z}_K \end{aligned} \tag{6.10}$$

Thus

$$\partial \;=\; \sum \partial_i\, e_i \quad , \quad \overline{\partial} \;=\; \sum \overline{\partial}_i\, \overline{e}_i. \tag{6.11}$$

Evidently these operators commute with each other, and with e_i and e_i^*. Furthermore, integrating by parts shows:

Lemma 6.11. *The adjoints of* ∂_i *and* $\overline{\partial}_i$ *are given by*

$$\partial_i^* \;=\; -\overline{\partial}_i \quad , \quad \overline{\partial}_i^* \;=\; -\partial_i$$

in the sense that

$$(\partial_i\alpha, \beta)_T \;=\; -(\alpha, \overline{\partial}_i\beta)_T \quad , \quad (\overline{\partial}_i\alpha, \beta)_T \;=\; -(\alpha, \partial_i\beta)_T. \quad \square$$

We now prove the identity

$$[\Lambda, \partial] \;=\; \sqrt{-1}\,\overline{\partial}^*,$$

the other assertion in (6.7) being similar. Since $L = \frac{\sqrt{-1}}{2}\sum e_i\overline{e}_i$, taking adjoints yields $\Lambda = -\frac{\sqrt{-1}}{2}\sum \overline{e}_i^* e_i^*$. Hence:

$$\begin{aligned}\Lambda\partial - \partial\Lambda &= -\tfrac{\sqrt{-1}}{2}\sum_{i,j}\big(\overline{e}_i^* e_i^*\partial_j e_j \ - \partial_j e_j \overline{e}_i^* e_i^*\big)\\ &= -\tfrac{\sqrt{-1}}{2}\sum_{i,j}\partial_j\big(\overline{e_i}^* e_i^* e_j - e_j\overline{e}_i^* e_i^*\big)\\ &= -\tfrac{\sqrt{-1}}{2}\sum_{i,j}\partial_j\overline{e}_i^*\big(e_i^* e_j + e_j e_i^*\big).\end{aligned}$$

Using Lemma 6.10 (ii), it follows that

$$[\Lambda, \partial] \ = \ -(\sqrt{-1})\sum \partial_j \overline{e}_j^*. \tag{*}$$

On the other hand, taking adjoints in the relation $\overline{\partial} = \sum \overline{\partial}_j\overline{e_j}$ and using Lemma 6.11 we see that

$$-\sum \partial_j \overline{e}_j^* \ = \ \sum \overline{\partial}_j^* \overline{e}_j^* \ = \ \overline{\partial}^*.$$

Thus (*) shows that $[\Lambda, \partial] = (\sqrt{-1})\,\overline{\partial}^*$, as required. Finally (6.8) follows from (6.7) by taking adjoints.

Laplacians and harmonic forms. Let (X, g) be a compact complex manifold with a compatible metric. Associated to each of the operators d, ∂, ∂^* there is a Laplace operator whose kernel consists (by definition) of harmonic forms. The Hodge theorem will assert that any cohomolgy class has a unique harmonic representative. In general there need be no particular connection between d-, ∂ and ∂^*-harmonic forms, but in the case of Kähler manifolds the three Laplacians essentially coincide. As we shall see, fact has many remarkable consequences.

We start by defining the three Laplace operators in question. As in the previous paragraph, let (X, g) be a compact complex manifold with a compatable metric.

Definition 6.12. We define the Laplace operators associated to $d, \partial, \overline{\partial}$ to be the operators

$$\begin{aligned}\Delta_d &: \mathcal{A}^k(X) \longrightarrow \mathcal{A}^k(X)\\ \Delta_\partial &: \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q}(X)\\ \Delta_{\overline{\partial}} &: \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q}(X)\end{aligned} \tag{6.12}$$

given by:

$$\begin{aligned}\Delta_d &= dd^* + d^*d\\ \Delta_\partial &= \partial\,\partial^* + \partial^*\partial\\ \Delta_{\overline{\partial}} &= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}.\end{aligned} \tag{6.13}$$

We will be particularly interested in those forms killed by the Laplacian:

Definition 6.13. A form $\alpha \in \mathcal{A}^k(X)$ is said to be *d-harmonic* if

$$\Delta_d \, \alpha \;=\; 0,$$

and similarly $\alpha \in \mathcal{A}^{p,q}(X)$ is ∂- or $\overline{\partial}$-harmonic if

$$\Delta_\partial \, \alpha \;=\; 0 \quad \text{or} \quad \Delta_{\overline{\partial}} \, \alpha \;=\; 0.$$

We denote by

$$\begin{aligned} \mathcal{H}^k_d(X) \;&=\; \mathcal{H}^k_d(X, g) \;\subseteq\; \mathcal{A}^k(X) \\ \mathcal{H}^{p,q}_\partial(X) \;&=\; \mathcal{H}^k_\partial(X, g) \;\subseteq\; \mathcal{A}^{p,q}(X) \\ \mathcal{H}^{p,q}_{\overline{\partial}}(X) \;&=\; \mathcal{H}^{p,q}_{\overline{\partial}}(X, g) \;\subseteq\; \mathcal{A}^{p,q}(X) \end{aligned} \tag{6.14}$$

the spaces of harmonic forms for the indicated Laplacians.

We emphasize that these spaces depend on the chosen metric on X. It will turn out that they are finite dimensional. Note that in the case of d, one only needs (X, g) to be a compact oriented Riemannian manifold.

Example 6.14. To explain the terminolgy, consider the Laplace operator Δ_d acting on functions $f \in \mathcal{A}^0_c(\mathbf{R}^n)$ where $\mathbf{R}^n$ is given the usual flat Euclidean metric.[23] Then $d^* f = 0$ since f is a zero-form, hence with Euclidean coordinates $x_1, \ldots, x_n$ one has

$$\begin{aligned} \Delta_d(f) \;=\; d^* d\, f \;&=\; - * d * \left(\sum \frac{\partial f}{\partial x_i} dx_i \right) \\ &=\; - * d \left(\pm \sum \frac{\partial f}{\partial x_i} \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \right) \\ &=\; - * \left(\sum \frac{\partial^2 f}{\partial x_i^2} \, dx_1 \wedge \ldots \wedge dx_n \right) \\ &= - \left(\sum \frac{\partial^2 f}{\partial x_i^2} \right). \end{aligned}$$

So up to a sign, $\Delta_d(f)$ is just the classical Laplacian of f. One can compute $\Delta_\partial(f)$ and $\Delta_{\overline{\partial}}(f)$ similarly (see [2, p. 83]).

Proposition 6.15. *Let X be a compact complex manifold. Then a form $\alpha \in \mathcal{A}^k(X)$ is d-harmonic if and only if*

$$d\alpha \;=\; d^*\alpha \;=\; 0,$$

with the analogous condition for a form $\alpha \in \mathcal{A}^{p,q}(X)$ to be ∂- or $\overline{\partial}$-harmonic.

Proof. We use Proposition 6.7 (or more precisely its analogue for d^*). In fact, that result shows that

$$\begin{aligned} (\Delta_d \alpha \,, \alpha)_X \;&=\; (d\, d^* \alpha, \alpha)_X \;+\; (d^* d\alpha, \alpha)_X \\ &=\; (d^* \alpha, d^* \alpha)_X \;+\; (d\alpha, d\alpha)_X. \end{aligned}$$

Since both terms on the right are in any event non-negative, the assertion follows. ☐

[23] Assume for simplicity that n is even to avoid sign worries involving the definition of d^* given above.

The crucial point for us is that on a Kähler manifold, the three different Laplacians that we have defined essentially coincide.

Theorem 6.16. *Let (X, g) be a compact Kähler manifold. Then*

$$\Delta_\partial \ = \ \Delta_{\overline{\partial}} \ = \ \tfrac{1}{2}\Delta_d,$$

and Δ_d commutes with the operators $*, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*, L$ and Λ, and it preserves the decomposition into type.

Proof. We first show that $\Delta_\partial = \Delta_{\overline{\partial}}$. Using equation (6.7) in Theorem 6.9 and recalling that ∂ and $\overline{\partial}$ anticommute we find:

$$\begin{aligned}
\Delta_\partial \ &= \ \partial^*\,\partial + \partial\,\partial^* \\
&= \ (\sqrt{-1})\left([\Lambda, \overline{\partial}]\,\partial + \partial\,[\Lambda, \overline{\partial}]\right) \\
&= \ (\sqrt{-1})\left(\Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial + \partial\Lambda\overline{\partial} - \partial\,\overline{\partial}\Lambda\right) \\
&= (\sqrt{-1})\left(\left(\Lambda\overline{\partial}\,\partial - \left(\overline{\partial}[\Lambda, \partial] + \overline{\partial}\,\partial\Lambda\right) + \left([\partial, \Lambda]\overline{\partial} + \Lambda\partial\,\overline{\partial}\right) - \partial\,\overline{\partial}\Lambda\right)\right) \\
&= (\sqrt{-1})\left(\left(\Lambda\overline{\partial}\,\partial - (\sqrt{-1})\overline{\partial}\,\overline{\partial}^* - \overline{\partial}\,\partial\Lambda\right) + \left(-(\sqrt{-1})\overline{\partial}^*\overline{\partial} + \Lambda\partial\,\overline{\partial} - \partial\,\overline{\partial}\Lambda\right)\right) \\
&= (\sqrt{-1})\left(-(\sqrt{-1})\overline{\partial}\,\overline{\partial}^* - (\sqrt{-1})\overline{\partial}^*\overline{\partial}\right) \\
&= \ \Delta_{\overline{\partial}}.
\end{aligned}$$

Note next that

$$\partial\,\overline{\partial}^* \ + \ \overline{\partial}^*\,\partial \ = \ 0.$$

In fact, by Theorem 6.9

$$\begin{aligned}
(\sqrt{-1})\left(\partial\,\overline{\partial}^* \ + \ \overline{\partial}^*\,\partial\right) &= \partial[\Lambda, \partial] \ + [\Lambda, \partial]\partial \\
&= \ \partial\Lambda\partial - \partial\Lambda\partial \\
&= \ 0.
\end{aligned}$$

Conjugating, it follows that likewise

$$\overline{\partial}\,\partial^* \ + \ \partial^*\,\overline{\partial} \ = \ 0.$$

We now compute:

$$\begin{aligned}
\Delta_d \ &= \ (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) \ + \ (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\
&= \ \Delta_\partial \ + \ \Delta_{\overline{\partial}} + \left(\partial\,\overline{\partial}^* + \overline{\partial}^*\,\partial\right) \ + \ \left(\overline{\partial}\,\partial^* + \partial^*\,\overline{\partial}\right) \\
&= \ \Delta_\partial \ + \Delta_{\overline{\partial}}.
\end{aligned}$$

The proofs that Δ_d commutes with all the holomorphic and anti-holomorphic operators are similar, and are left to the reader. That Δ_d commutes with the projection operators $\pi^{p,q} : \mathcal{A}^k(X) \longrightarrow \mathcal{A}^{p,q}(X)$ follows from the facts that $\Delta_d = 2\Delta_\partial$ and that Δ_∂ preserves the type decomposition since it is the compostion of operators that do. $\square$

These identities already have an interesting corollary. Recall that a *holomorphic* p-form on a complex manifold X is a $(p,0)$-form $\eta \in \mathcal{A}^{p,0}(X)$ such that $\overline{\partial}\eta = 0$. Equivalently, η has a local expression of the form

$$\eta \ =_{\text{locally}} \ \sum f_I \, dz_I$$

where the f_I are holomorphic. In complex dimension 1 any holomorphic $(1,0)$-form is closed simply because $d = \overline{\partial}$ in $(1,0)$-forms. However in higher dimensions there is no local reason that a holomorphic form should be closed: eg $\eta = z\,dw$ is not closed on $\mathbf{C}^2$. However in the global Kähler setting a holomorphic form is automatically closed:

Corollary 6.17. *If X is a compact Kähler manifold, then any holomorphic form η on X is closed.*

We will see later (Exercise 6.4) that if $\eta \neq 0$, then moreover η cannot be exact.

Proof. We have $\partial^*\eta = 0$ for reasons of type, hence η is $\overline{\partial}$-harmonic thanks to Proposition 6.15. Therefore η is d-harmonic, and then the same Proposition shows that it is d- and d^*-closed. □

We close this subsection with a proposition that provides yet another characterization of harmonic forms. It will be used in the next subsection to give an intuitive heuristic for the Hodge decomposition theorem.

Proposition 6.18. *With X as above, consider the Hermitian inner product* (6.3)*, and let*

$$\alpha \ \in \ \mathcal{A}^k(X)$$

be a d-closed form on X. Then α is d-harmonic if and only if α has minimial length in its deRham cohomology class, i.e.

$$\alpha \in \mathcal{H}^k_d(X) \iff ||\alpha + d\eta||^2 \ > \ ||\alpha||^2$$

for every $\eta \in \mathcal{A}^{k-1}(X)$ with $d\eta \neq 0$. The analogous statements hold for ∂ and $\overline{\partial}$ cohology and Δ_∂ and $\Delta_{\overline{\partial}}$-harmonic forms.

Proof of Proposition 6.18. First assume that α is harmonic, so that $d\alpha = d^*\alpha = 0$. Then for any $\eta \in \mathcal{A}^{k-1}(X)$ one has

$$\begin{aligned} ||\, \alpha + d\eta \,||^2 \ &= \ ||\alpha||^2 \ + \ (\alpha, d\eta)_X \ + +(d\eta, \alpha) + X\,||d\eta||^2 \\ &= \ ||\alpha||^2 \ + \ (d^*\alpha, \eta)_X \ + \ (\eta, d^*\alpha)_X \ + \ ||d\eta||^2 \\ &= \ ||\alpha||^2 \ + \ ||d\eta||^2, \end{aligned}$$

so α has minimal norm. Conversely, if α has minimal norm, then taking $\eta = t\,d^*\alpha$ in the above we find that

$$||\alpha + t\,dd^*\alpha|| \ = \ ||\alpha||^2 + 2t||d^*\alpha||^2 \ + \ t^2\,||dd^*\alpha||^2.$$

This has a minimum at $t = 0$ if and only if $||d^*\alpha|| = 0$, ie iff α is harmonic. □

The Hodge Theorem. We now turn to the Hodge theorem and its consequences.

Let (X, g) be a compact connected complex manifold with a compatible complex structure. The Hodge theorem – which we will momentarily state precisely – asserts that there is a unique harmonic representative in each $d-, \partial-$ or $\overline{\partial}-$ cohoomology class. The proof of the theorem is analytic in nature – it's ultimately a consequence of regularity theorems for elliptic PDE – and we do not attempt to give anything like a real explanation. However there is an enjoyable heuristic argument (which most of us learned from Griffiths-Harris) suggesting why the statement is not unreasonable.

The starting point is Proposition 6.18, which asserts that a form is harmonic (with respect to any of the three Laplacians) if and only it has minimal length in its (d-, ∂- or $\overline{\partial}$-) cohomology class. Focusing for concreteness on the d-Laplacian Δ_d and DeRham cohomolgy, imagine – which unfortunately is not the case – that $\mathcal{A}^k(X)$ were a Hilbert space with respect to the inner product $(,)_X$ and that

$$\operatorname{Im}\left(d : \mathcal{A}^{k-1}(X) \longrightarrow \mathcal{A}^k(X)\right) \ \subseteq\ \mathcal{A}^k(X)$$

were a closed subspace. Then we could find a unique element of minimal lengh in the affine subspace

$$[\alpha] \ =\ \alpha + \operatorname{Im}(d),$$

which according to the Proposition would be the unique harmonic form in this cohomology class. This provides at least some intuitive reason to imagine that something like the Hodge theorem could be true.

Here is the formal statement of the Hodge decomposition theorem. To clarify where this is and isn't used, for the moment we do not impose the Kähler condition.

Theorem 6.19. (Hodge Theorem). *Let (X, g) be a compact complex manifold with a compatible metric. Then the three spaces*

$$\mathcal{H}^k_d(X, g) \subseteq \mathcal{A}^k(X) \quad \textit{and} \quad \mathcal{H}^{p,q}_{\partial}(X, g)\,,\, \mathcal{H}^{p,q}_{\overline{\partial}}(X, g) \ \subseteq \mathcal{A}^{p,q}(X)$$

of harmonic forms are finite dimensional. Morover:

(i). *There is an orthogonal direct sum decomposition*

$$\mathcal{A}^k(X) \ =\ d\,\mathcal{A}^{k-1}(X) \oplus \mathcal{H}^k_d(X, g) \oplus d^*\mathcal{A}^{k+1}(X).$$

(ii). *Similarly, there are orthogonal direct sum decompositions*

$$\mathcal{A}^{p,q}(X) \ =\ \partial\,\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\partial}(X, g) \oplus \partial^*\mathcal{A}^{p+1,q}(X)$$
$$\mathcal{A}^{p,q}(X) \ =\ \overline{\partial}\,\mathcal{A}^{p,q-1}(X) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}(X, g) \oplus \overline{\partial}^*\mathcal{A}^{p,q+1}(X).$$

The orthogonality in the theorem is of course with respect to the Hermitian inner produce (6.3). We will say a few words in Remark **??** about the general PDE facts underlying the theorem.

Corollary 6.20. *In the situation of the Theorem, there are canonical isomorphisms*

$$\mathcal{H}^k_d(X,g) \xrightarrow{\cong} H^k_{dR}(X)$$
$$\mathcal{H}^{p,q}_{\overline{\partial}}(X,g) \xrightarrow{\cong} H^{p,q}_{\overline{\partial}}(X)$$
$$\mathcal{H}^{p,q}_{\partial}(X,g) \xrightarrow{\cong} H^{p,q}_{\partial}(X).$$

Lemma 6.21. *One has*

$$\ker\big(d : \mathcal{A}^k(X) \longrightarrow \mathcal{A}^{k+1}(X)\big) \ = \ \mathcal{H}^k_d(X,g) \oplus d\,A^{k-1}(X), \tag{6.15}$$

with analogous statements for $\ker \partial$ *and* $\ker \overline{\partial}$.

Proof. Evidently the RHS of (6.15) contains the LHS. For the reverse inclusion, suppose that $d\alpha = 0$. By the Hodge decomposition one can write $\alpha = d\alpha_1 + h + d^*\alpha_2$, where h is harmonic. It suffices to show that $\alpha_2 = 0$. Now since $d\alpha = 0$, we have $dd^*\alpha_2 = 0$, but then

$$||\alpha_2^2|| \ = \ (d^*\alpha_2, d^*\alpha_2) \ = \ (dd^*\alpha_2, \alpha_2) \ = \ 0,$$

as required. □

Proof of Corollary 6.20. We prove the first statement. Since a harmonic form is d-closed, there is a natural map

$$\mathcal{H}^k_d(X,g) \longrightarrow H^k_{\mathrm{dR}}(X) \ \ , \ \ \alpha \mapsto [\alpha]. \tag{*}$$

It follows from the orthogonality of the Hodge decomposition that a non-zero harmonic form cannot be exact, hence (*) is injective. Moreover, equation (6.15) implies that any closed form is cohomologous to a (unique) harmonic form, hence (*) is surjective. □

Assume now that (X,g) is compact Kähler. Then the three Laplacians in play all coincide, so we expect the Hodge decomposition to have particularly striking consequences. This is very much the case. In fact, by the deRham theorem we have

$$H^k(X,\mathbf{C}) \ = \ H^k_d(X)$$

and we write $H^{p,q}(X) = H^{p,q}_{\overline{\partial}}(X)$ for the $\overline{\partial}$- cohomogy of X, so that

$$H^{p,q}(X) \ = \ H^q(X, \Omega^p_X)$$

thanks to the Dolbeaut theorem.

Then we have the famous

Theorem 6.22. (Hodge Decomposition) *If X is a compact Kähler manifold, then there is a natural decomposition*

$$H^k(X,\mathbf{C}) \ \cong \ \bigoplus_{p+q} H^{p,q}(X), \tag{6.16}$$

which is independent of the Kähler metric on X. Moreover there are isomorphisms

$$\begin{aligned} H^{q,p}(X) \ &= \ \overline{H^{p,q}(X)} \\ H^{p,q}(X) \ &= \ H^{n-q,n-p}(X). \end{aligned} \tag{6.17}$$

Proof. Fix a Kähler metric g on X, and consider the resulting spaces of harmonic forms. In the first place, by Corollary 6.20 we have isomorphisms

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X,g) \xrightarrow{\cong} H^{p,q}(X),$$
$$\mathcal{H}^{k}_{d}(X,g) \xrightarrow{\cong} H^{k}(X,\mathbf{C}).$$

Furthermore, since Δ_d commutes with taking (p,q)-components (Theorem 6.16), we have a decomposition

$$\mathcal{H}^{k}_{d}(X,g) \;=\; \oplus_{p+q=k}\mathcal{H}^{p,q}_{d}(X,g).$$

On the other hand, thanks to Theorem 6.16 it is the same to be harmonic for d or for $\overline{\partial}$, and therefore $\mathcal{H}^{p,q}_{d}(X,g) = \mathcal{H}^{p,q}_{\overline{\partial}}(X,g)$. Putting this together, we get isomorphisms

$$H^k(X,\mathbf{C}) \;\cong\; \mathcal{H}^{k}_{d}(X,g) \;=\; \oplus\,\mathcal{H}^{p,q}_{\overline{\partial}}(X,g) \;\cong\; \oplus\; H^{p,q}(X).$$

It remains to show that the resulting isomorphism between the outer terms is independent of the choice of Kahler metric. Fixing a second Kähler metric g_1, this amount to showing that if we have harmonic forms

$$\alpha \,\in\, \mathcal{H}^{p,q}(X,g) \quad , \quad \alpha_1 \,\in\, \mathcal{H}^{p,q}(X,g_1)$$

representing the same cohomology class in $H^{p,q}(X)$ then α and α_1 represent the same class in $H^{p+q}(X,\mathbf{C})$. The hypothesis means that

$$\alpha - \alpha_1 \;=\; \overline{\partial}\beta,$$

and by the directness of the decomposition in Theorem 6.19 (ii) for $\overline{\partial}$, it follows that $\alpha - \alpha_1$ is orthogonal to $\mathcal{H}^{p,q}_{\overline{\partial}}(X,g) = \mathcal{H}^{p,q}_{d}(X,g)$. Since $d(\alpha-\alpha_1) = 0$, it follows from Proposition 6.21 that $\alpha - \alpha_1 = d\gamma$, as required. Finally if $\alpha \in \mathcal{H}^{p,q}(X)$, then

$$\overline{\alpha} \,\in\, \mathcal{H}^{q,p}(X) \quad , \quad *\alpha \,\in\, \mathcal{H}^{n-q,n-p}(X)$$

(Exercise 6.5), which gives the isomorphisms (6.17). □

We conclude this section with a useful result, called the $\partial\overline{\partial}$- Lemma.

Proposition 6.23. *Let X be a compact Kähler manifold, and let*

$$\alpha \,\in\, \mathcal{A}^{p,q}(X)$$

be a d-closed (p,q)-form. Then the following are equivalent:

(i). *The form α is d-exact, i.e. $\alpha = d\beta$ for some $\beta \in \mathcal{A}^{p+q-1}(X)$;*
(ii). *The form α is ∂-exact, i.e. $\alpha = \partial\beta$ for some $\beta \in \mathcal{A}^{p-1,q}(X)$;*
(iii). *The form α is $\overline{\partial}$-exact, i.e. $\alpha = \overline{\partial}\beta$ for some $\beta \in \mathcal{A}^{p,q-1}(X)$;*
(iv). *The form α is $\partial\overline{\partial}$-exact, i.e. $\alpha = \partial\overline{\partial}\beta$ for some $\beta \in \mathcal{A}^{p-1,q-1}(X)$;*
(v). *The form α is perpendicular to the space $\mathcal{H}^{p,q}(X,g)$ of harmonic forms for an arbitrary Käher metric g on X.*[24]

[24]Since we are in the Kähler setting, the same (p,q)-forms are harmonic for $\Delta_d, \Delta_\partial$ and $\Delta_{\overline{\partial}}$, so we don't need to specify which we refer to here.

Proof. Since α is a form of pure type (p,q), it is d closed if and only if it is both ∂- and $\overline{\partial}$-closed. Therefore it follows from Theorem 6.19 that each of the other conditions implies (v), and clearly (iv) implies (i) – (iii). So it suffices to show that (v) $\Rightarrow$ (iv). Since $\partial\alpha = 0$ and α is perpendicular to $\mathcal{H}^{p,q}_{\partial}(X)$, Lemma 6.21 for ∂ implies that $\alpha = \partial\beta$ for some β. Now use the Hodge decomposition for $\overline{\partial}$ to write

$$\beta \ = \ \overline{\partial}\gamma_1 \,+\, \gamma_2 \,+\, \overline{\partial}^*\gamma_3,$$

where γ_2 is harmonic. Therefore

$$\begin{aligned}\alpha \ = \ \partial\beta \ &= \ \partial\overline{\partial}\gamma_1 \,+\, \partial\overline{\partial}^*\gamma_3 \\ &= -\overline{\partial}\partial\gamma_1 \,-\, \overline{\partial}^*\partial\gamma_3.\end{aligned}$$

On the other hand, $\overline{\partial}\alpha = 0$, so $\overline{\partial}\overline{\partial}^*\partial\gamma_3 = 0$, which implies that

$$(\overline{\partial}\,\overline{\partial}^*\partial\gamma_3, \partial\gamma_3)_X \ = \ ||\overline{\partial}^*\partial\gamma_3||^2 \ = \ 0,$$

and hence

$$-\partial\overline{\partial}^*\gamma_3 \ = \ \overline{\partial}^*\partial\gamma_3 \ = \ 0.$$

Thus $\alpha = \partial\overline{\partial}\gamma_1$, as required. $\square$

The Lefschetz theorems. We start by discussing some topological consequences of the Hodge theorem.

Let (X,g) be a compact Kähler manifold, with Kähler form ω. Denote as usual by L and Λ the Lefschetz operator and it's dual. Since these commute with the Laplacian $\Delta = \Delta_d$, they give rise to homomorphisms

$$L : \mathcal{H}^{p,q}(X) \longrightarrow \mathcal{H}^{p+1,q+1}(X) \ \ , \ \ \mathcal{H}^{p,q}(X) \longrightarrow \mathcal{H}^{p-1,q-1}(X),$$

and hence to

$$L : H^{p,q}(X) \longrightarrow H^{p+1,q+1}(X) \ \ , \ \ \Lambda : H^{p,q}(X) \longrightarrow H^{p-1,q-1}(X).$$

Note that L is given by cup product with $[omega] \in H^{1,1}(X)$, and hence depends only on this class. One can show that the same is true of Λ.

The first main result is the

Theorem 6.24. (Hard Lefschetz Theorem) *Let X be a compact Kähler manifold of complex dimension n. Then the $(n-k)$-fold iteration of the Lefschetz operator gives an isomorhism*

$$L^{n-k} : H^k(X, \mathbf{C}) \xrightarrow{\cong} H^{2n-k}(X, \mathbf{C}).$$

Note that the two groups appearing here are Poincaré dual, so the deep fact is not that there exists an isomorphism between the groups, but rather that $(n-k)$-fold iteration of L does the job.

Proof. Thanks to the Hodge decomposition it suffices to show that

$$L^{n-k} : \mathcal{H}^k(X) \longrightarrow \mathcal{H}^{2n-k}(X) \tag{*}$$

is an isomorphism, and since both source and target have the same dimension it is enough to prove that it is injective. But this follows from our pointwise analysis in §3. In fact, Theorem 3.16 implies that

$$L^{n-k} : \Lambda^k T^*_{\mathbf{C}} X \longrightarrow \Lambda^{2n-k} T^*_{\mathbf{C}} X$$

is an isomorphism of smooth vector bundles, and hence

$$L^{n-k} : \mathcal{A}^k(X) \longrightarrow \mathcal{A}^{2n-k}(X)$$

is bijective. Therefore (*) is indeed injective. □

We turn next to the Lefschetz decomposition. The *primitive cohomology* of X is defined to be

$$\begin{aligned} P^k(X, \mathbf{C}) \ &= \ \ker\left(\Lambda : H^k(X, \mathbf{C}) \longrightarrow H^{k-2}(X, \mathbf{C})\right) \\ P^{p,q}(X, \mathbf{C}) \ &= \ \ker\left(\Lambda : H^{p,q}(X, \mathbf{C}) \longrightarrow H^{p-1,q-1}(X, \mathbf{C})\right) \end{aligned}$$

Theorem 6.25. (Lefschetz decomposition). *Let X be a compact Kähler manifold of dimension n. Then*

$$\begin{aligned} H^k(X, \mathbf{C}) \ &= \ \bigoplus_{i \geq 0} L^i P^{k-2i}(X, \mathbf{C}) \\ H^{p,q}(X) \ &= \ \bigoplus_{i \geq 0} L^i P^{p-i,q-i}(X). \end{aligned}$$

Moreover

$$P^k(X) \ = \ \ker\left(L^{n-k+1} : H^k(X, \mathbf{C}) \longrightarrow H^{2n-k+2}(X, \mathbf{C})\right),$$

and similarly for $P^{p,q}$.

Proof. By the Hodge decomposition theorem, primitive classes are (uniquely) represented by primitive harmonic forms:

$$\mathcal{P}^k(X) \ =_{\text{def}} \ \ker\left(\Lambda : \mathcal{H}^k(X) \longrightarrow \mathcal{H}^{k-2}(X)\right), \tag{*}$$

and similarly for $P^{p,q}$. So it is equivalent to prove the Lefschetz decomposition for harmonic forms. For this one can either reduce to Theorem 3.16, or one can note that L and Λ a generate an SL_2 action on primitive forms, and argue as in §3. We leave details to the reader. □

Remark 6.26. Note that the condition (*) defining primitive harmonic forms operates pointwise. Therefore if $\alpha \in \mathcal{P}^k$ is a primitive harmonic form, then $\alpha(x) \in \Lambda^k T^* X$ is a primitive vector for every $x \in X$ (although possibly $\alpha(x) = 0$ at some $x \in X$.

Finally, we have the Hodge-Riemann bilinear relations.

Theorem 6.27. *Let X be a compact Kähler manifold of dimension n, with Kähler class ω. Let $0 \neq \alpha \in P^{p,q}(X)$. Then*

$$(\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-(p+1)} \ > \ 0.$$

Proof. It is enough to prove the statement when α is a primitive harmonic form. Fix a point $x \in X$ at which $\alpha(x) \neq 0$. Then thanks to Remark 6.26 the assertion follows from the pointwise statement Theorem 3.27 established in §3. □

We now turn to the remarkable Lefschetz $(1,1)$-theorem, which describes the holomorphic line bundles on a compact Kähler manifold. Recall first that isomorphism classes of smooth complex line bundles on X are classified by $H^2(X, \mathbf{Z})$: this arises from the isomorphism

$$H^1(X, \mathcal{A}^*) \xrightarrow{\cong} H^2(X, \mathbf{Z}) \quad , \quad L \mapsto c_1(L) \tag{*}$$

deduced from the exponential sequence $0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^* \longrightarrow 0$ where $\mathcal{A}$ is the sheaf of smooth $\mathbf{C}$-valued functions on X. Here $c_1(L) \in H^2(X, \mathbf{Z})$ is the first Chern class of a line bundle L: for now one can take (*) as the definition of $c_1(L)$, but we will discuss other manifestations shortly.

It is then natural to ask the important

Question 6.28. Which elements in $H^2(X, \mathbf{Z})$ are the first Chern classes of *holomorphic* line bundles on X?

For the answer, one considers the subgroup

$$H^{1,1}(X, \mathbf{Z}) \ \subseteq \ H^2(X, \mathbf{Z})$$

defined as the set of all integral classes which, when mapped to $H^2(X, \mathbf{C})$, have type $(1,1)$ under the Hodge decomposition of $H^2(X, \mathbf{C})$. In other words,

$$H^{1,1}(X, \mathbf{Z}) \ = \ \big\{ \gamma \in H^2(X, \mathbf{Z}) \mid \iota(\gamma) \in H^{1,1}(X) \big\}$$

where $\iota : H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathbf{C})$ is the natural map.[25] In other words, if we ignore torsion and pretend that $H^2(X, \mathbf{Z})$ embeds in $H^2(X, \mathbf{C})$, then one has the more picuresque relation

$$H^{1,1}(X, \mathbf{Z}) \ = \ H^2(X, \mathbf{Z}) \cap H^{1,1}(X).$$

Theorem 6.29. (Lefschetz (1,1) theorem) *A class $\gamma \in H^2(X, \mathbf{Z})$ first Chern class of a holomorphic line bundle if and only $\gamma \in H^{1,1}(X, \mathbf{Z})$.*

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[25]Thus ι embeds $H^2(X, \mathbf{Z})$ modulo its torsion subgroup into

$$H^2(X, \mathbf{C}) = H^2(X, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}.$$

We note that some authors define $H^{1,1}(X, \mathbf{Z})$ to be the intersection

$$\mathrm{im}\Big(H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathbf{C}) \Big) \ \cap \ H^{1,1}(X).$$

Proof. We consider the exponential seqence $0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$, which gives rise to an exact sequence

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathcal{O}). \tag{6.18}$$

Our question is to understand the image of the map

$$\mathrm{Pic}(X) \ = \ H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbf{Z}),$$

so the Theorem is equivalent to the assertion that

$$H^{1,1}(X, \mathbf{Z}) \ = \ \ker \Big(H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathcal{O}) \Big). \tag{*}$$

We assert:

Claim 6.30. Under the Dolbeaut isomorphism

$$H^2(X, \mathcal{O}_X) \ = \ H^{0,2}(X),$$

the map $H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathcal{O}_X)$ coming from the exponential sequence associates to a class $\gamma \in H^2(X, \mathbf{Z})$ the $(0, 2)$-component of its image in $H^2(X, \mathbf{C})$. (Exercise 6.9.)

Granting this, one immediately gets the inclusion $\subseteq$ in (*), and the other inclusion follows from the observation that since the image of $H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathbf{C})$ consists of real classes,

$$\iota(\gamma)^{0,2} = 0 \iff \overline{\iota(\gamma)^{0,2}} = \Big(\overline{\iota(\gamma)}\Big)^{2,0} = \iota(\gamma)^{2,0} = 0.$$

$\square$

To complete the picture, we say a word about the Picard torus of a compact Kähler manifold. Define

$$\mathrm{Pic}^0(X) \ = \ \ker \Big(\mathrm{Pic}(X) \longrightarrow H^2(X, \mathbf{Z}) \Big)$$

to be the group of topologically trivial holomorphic line bundles on X. This fits into the exact sequence

$$0 \longrightarrow H^1(X, \mathbf{Z}) \longrightarrow H^1(X, \mathcal{O}) \longrightarrow \mathrm{Pic}^0(X) \longrightarrow 0.$$

Now it follows from the Dolbeaut isomorphism that $H^1(X, \mathcal{O}_X) = H^{0,1}(X)$, and as above the map $H^1(X, \mathbf{Z}) \longrightarrow H^{0,1}(X)$ occuring here is the composition

$$H^1(X, \mathbf{Z}) \longrightarrow H^1(X, \mathbf{C}) \longrightarrow H^{0,1}(X)$$

coming from the Hodge decomposition of H^1. Now note that

$$\text{rank } H^1(X, \mathbf{Z}) = b_1(X) \ = \ 2 \cdot h^{01}(X) \ = 2 \dim_{\mathbf{C}} H^1(X, \mathcal{O}_X).$$

This suggests that one might hope that $H^1(X, \mathbf{Z})$ sits as a *lattice* inside $H^1(X, \mathcal{O}_X)$. In fact, this is the case (Exercise 6.10). Therefore

$$\mathrm{Pic}^0(X) \ = \ H^1(X, \mathcal{O}_X)/H^1(X, \mathbf{Z})$$

has in the natural way the structure of a complex torus, called the Picard torus of X. Thus we have the beautiful picture that the set of isomorphism classes of holomorphic line bundles having a given topological type is parametrized by a torus of complex dimension $h^{0,1}(X) = \frac{1}{2} b_1(X)$.

Exercises for Section 6.

Exercise 6.1. Prove the assertion of Example 6.3.

Exercise 6.2. Let ω be a real closed $(1,1)$-form on a complex manifold X. Assume that ω is *positive* in the sense that it can locally be written in the form

$$\omega \ =_{\text{locally}} \frac{\sqrt{-1}}{2} \sum h_{ij}(z) \cdot dz_i \wedge d\overline{z}_j,$$

where $h_{ij}(z)$ is a smooth family of positive definite Hermitian matrices on X. Prove that then X carries a Kähler metric h of which ω is the corresponding fundamental 2-form.

Exercise 6.3. Prove that on a compac Kähler manifold, the Kähler form ω is harmonic.

Exercise 6.4. (a). Prove that a non-zero holomorphic form on a compact ähler manifold is never exact.

(b). Give an example of a non-zero exact holomorphic form on $\mathbf{C}^2$.

Exercise 6.5. Let X be a compact Kähler manifold, and let $\alpha \in \mathcal{H}^{p,q}(X)$. Prove that then

$$\overline{\alpha} \in \mathcal{H}^{q,p}(X) \ \ , \ \ *\alpha \in \mathcal{H}^{n-q,n-p}(X).$$

Exercise 6.6. (Functoriality of Hodge decomposition). . Prove that the Hodge decomposition (6.16) is functorial in the sense that if $f : Y \longrightarrow X$ is a holomorphic map of compact Kähler manifolds, then the pull-back mapping $f^* : H^k(X, \mathbf{C}) \longrightarrow H^k(Y, \mathbf{C})$ coincides under the Hodge decomposition with the direct sum of the pullbacks

$$f^* : H^{p,q}(X) \longrightarrow H^{p,q}(Y).$$

(Hint: First use harmonic representatives to prove that this is true for an inclusion $Y \subseteq X$ of Y as a submanifold of X, and then deduce the general case from this by considering the graph of f.)

Exercise 6.7. (The Kähler cone). Let X be a compact Kähler manifold. The *Kähler cone* of X is the open convex cone

$$\mathcal{K} \ \subseteq \ H^{1,1}(X, \mathbf{R})$$

consisting of the classes of Kähler metrics on X.

(a). Prove that this is indeed an open convex cone. (Cf. [3, p. 130].)

(b). Describe the Kähler cone explitly for $X = \mathbf{P}^1 \times \mathbf{P}^1$.

We remark that in all but the simplest cases it can be difficult to compute this cone explicitly, especially if one wants to know the subspace spanned by the integral classes. For example, if X is a compact Riemann surface of genus $g > 2$, then $\mathcal{K} \cap H^2(X, \mathbf{Z})$ can depend on the intrinsic geometry of X, and in fact it is not known what $\mathcal{K} \cap H^2(X, \mathbf{Z})$ is for "general" X. See [PAG, §1.5.B] for a survey of the algebro-geometric analogue of this question. □

Exercise 6.8. Let X be a compact Kähler maifold.

(a). Prove that the odd betti numbers of X are even.

(b). Prove that $\pi_1(X)$ cannot be isomorphic to the free group on three generators.

(c). Prove that $\pi_1(X)$ cannot be isomorphic to the free group on two generators. [Hint: Otherwise there would be a two-sheeted covering space of X whose π_1 is free on three generators, and such a covering is again a compact Kähler manifold.]

Exercise 6.9. Prove Claim 6.30. (See [3, Lemma 3.3.1].)

Exercise 6.10. Using the facts that

$$H^1(X,\mathbf{C}) \ = \ H^{1,0}(X) \oplus H^{0,1}(X) \ \ , \ \ H^{0,1} = \overline{H^{1,0}},$$

show that the image of $H^1(X,\mathbf{Z}) \longrightarrow H^{0,1}(X)$ is a lattice. (See [3, Cor. 3.3.6].)

Exercise 6.11. (The case of complex tori) Let $\Lambda \subseteq V$ be a lattice in an n-dimensional complex vector space V. We consider the complex torus $X = V/\Lambda$ with the flat (constant) metric on V. As usual, we may consider forms on V that are invariant under translation by Λ to be forms on X.

(a). Prove that the harmonic forms on X are just the constant ones of the sort

$$\sum a_{I,J} dz_I \wedge d\overline{z}_J, \tag{*}$$

where $a_{I,J} \in \mathbf{C}$. Deduce that

$$\mathcal{H}^{p,q}(X) \ = \ \Lambda^{p,q}V^*,$$

and deduce that the Hodge decompostion takes the form

$$\Lambda^k V_{\mathbf{C}} \ = \ \oplus \Lambda^{p,q}V^*.$$

(b). Choose a basis

$$\lambda_1, \dots, \lambda_{2n} \ \in \ V$$

for Λ, and denote by $\lambda_i \wedge \lambda_j$ the two (real) dimensional "subtorus" of X spanned by λ_i, λ_j. Given a $(1,1)$-form

$$\eta \ = \ \sum a_{ij}\, dz_i \wedge d\overline{z}_j \ \in \ \Lambda^{1,1}V^*,$$

show that

$$\eta \in H^{1,1}(X,\mathbf{Z}) \iff \int_{\gamma_i \wedge \gamma_j} \eta \in \mathbf{Z} \quad \forall i,j. \tag{*}$$

(c). Show by explicit computation that there exist $\Lambda \subset V$ that determine tori X for which $H^{1,1}(X,\mathbf{Z}) = 0$. [Hint: By a suitable choice of coordinates one can suppose that

$$\lambda_{n+1} \ = \ e_1 \ \ , \dots , \ \ \lambda_{2n} = e_n,$$

where e_i are the standard basis vectors. Then compute the integrals appearing in (*) in terms of the coordinates of $\lambda_1, \dots, \lambda_n$, and check that for sufficiently general choices of these λ_i there are no solutions to (*).]

7. Positivity for Line Bundles and the Kodaira Theorems

In this final section we study the notion of a positive line bundle on a Kähler manifold, and we prove the Kodiara vanishing and embedding theorems.

Positive Line Bundles. We start with the notion of positivity for a $(1,1)$-form on a complex vector space.

Definition 7.1. Let (V, J) be a real vector space with an almost complex structure. A $(1,1)$-form $\eta \in \Lambda^{1,1}V^*$ is *positive* if

$$\eta(v, Jv) \ > \ 0 \quad \text{for all } \ 0 \neq v \in V.$$

Equivalently, with notation as in §3, η is positive if one can write

$$\eta \ = \ \frac{\sqrt{-1}}{2} \sum h_{ij}\, z^i \wedge \overline{z}^j$$

where h_{ij} is a positive definite Hermitian metric. □

Yet another way to phrase the definition is to require that if $W \subseteq V$ is any J-stable subspace of real dimension 2 – so that $(W, J|W)$ is a two-dimensional almost complex vector space – then $\eta|W = c\,\mathrm{vol}_W$ for some $c > 0$.

Similarly:

Definition 7.2. Let X be a complex manifold, with J the corresponding almost complex structure, and let η be a $(1,1)$-form on X. We say that η is *positive* if

$$\eta_x \ \in \ \Lambda^{1,1}T^*_x X$$

is a positive form for every $x \in X$.

Equivalently, one asks that

$$\eta \ =_{\text{locally}} \ \frac{\sqrt{-1}}{2} \sum h_{ij}(z)\, dz_i \wedge d\overline{z}_j$$

where $h_{ij}(z)$ is a Hermitian matrix of smooth functions that is positive definite at every point.

Example 7.3. Let (X, g) be a complex manifold with a compatible Riemannian metric, and let $\omega \in \mathcal{A}^{1,1}(X)$ be the associated $(1,1)$-form. Then ω is positive, and conversely any positive $(1,1)$-form arises in this manner (cf Exercise 6.2).

The plan now is to introduce now a positivity notion for a holomorphic line bundle L by asking in effect that $c_1(L)$ be representable by a positive $(1,1)$-form.

Let L be a holomorphic line bundle on a X. A *hermitian metric* h on L is a smoothly varying (positive definite) hermitian metric on all the fibres of L, and a *hermitian holomorphic line bundle* (L, h) is a holomorphic line bundle together with the choice of a hermitian metric on L. (Hermitian metrics on vector bundles of higher rank are defined similarly.)

Suppose that (L, h) is a hermitian line bundle, and consider a holomorphic section $s \in \Gamma(U, L)$ of L on an open subset $U \subset X$. Then we take the fibre-wise norm square of s to get a non-negative function

$$|s|^2 \;=\; |s|_h^2 \;=\; h(s, s)$$

on U. If $L|U = U \times \mathbf{C}$ is trivial, then conversely a hermitian metric on $L|U$ is specified by giving a positive smooth function

$$h : U \longrightarrow \mathbf{R}_{>0} \tag{7.1}$$

which we declare to be the (square) length of the section $x \longrightarrow 1 \in \mathbf{C}$. In general, if L is described by transition data (U_i, g_{ij}) then a metric is given by smooth functions

$$h_i : U_i \longrightarrow \mathbf{R}_{>0}$$

such that

$$h_i \;=\; |g_{ij}|^2\, h_j \quad \text{on} \quad U_{ij}. \tag{7.2}$$

A simple argument with partitions of unity shows that any holomorphic (or, for that matter, smooth) line bundle admits (many different) hermitian metrics. (Exercise 7.1.)

Let (L, h) be a hermitian line bundle described by local data (U_i, h_i) as in (7.2). We claim that the $(1, 1)$-forms

$$t_i \;=_{\text{def}}\; -\partial\overline{\partial} \log h_i$$

patch together to give a globally defined form

$$\Theta(L, h) \;\in\; \mathcal{A}^{1,1}(X),$$

which is called the *curvature form* of (L, h). In fact, on $U_i \cap U_j$ one has

$$\log h_i \;=\; \log |g_{ij}|^2 \;+\; \log h_j,$$

and

$$\begin{aligned} \partial\overline{\partial} \log |g_{ij}|^2 \;&= \partial\overline{\partial}\big(g_{ij}\overline{g}_{ij}\big) \\ &=\; \partial\overline{\partial} \log g_{ij} \;-\; \overline{\partial}\partial \log \overline{g}_{ij} \\ &=\; 0 \end{aligned}$$

since g_{ij} is holomorphic. Evidently $\Theta(L, h)$ is closed.

Remark 7.4. Let $s_U \in \Gamma(U, L)$ be a local section which is nowhere-vanishing on U. Then one can alternatively define

$$\Theta(L, h)_U \;=\; -\partial\overline{\partial}\, |s_U|^2,$$

it following from a computation as above that these local expressions are independent of the choice of s_U.

Remark 7.5. We are avoiding here a systematic discussion of curvature of vector bundles. For this, see eg [3, Chapter 4] or [2].

A basic fact is that $\Theta(L,h)$ essentially represents the first Chern class of L:

Proposition 7.6. *One has*

$$c_1(L) \;=\; \tfrac{\sqrt{-1}}{2\pi}\,\Theta(L,h) \;\in\; H^2(X,\mathbf{C}).$$

For a proof, see [].

Now we come to the basic definition

Definition 7.7. A holomorphic line bundle on a compact complex manifold is *positive* (in the sense of Kodaira) if it carries a hermitian metric h such that $\frac{\sqrt{-1}}{2\pi}\Theta(L,h)$ is a positive $(1,1)$-form.

Remark 7.8. In algebraic geometry, the analogous notion is for an algebraic line bundle to be *ample*. In fact, at the end of the day (thanks to the Kodaira embedding theorem, Serre's GAGA theorem, and the Nakai-Moisheson criterion for amplitude) on a compact Kähler manifold these turn out to be essentially equivalent notions.

The prototypical example is the hyperplane line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ on projective space:

Proposition 7.9. *The hyperplane line bundle on $\mathbf{P}^n$ carries a positive hermitian metric.*

Proof. We will show that $\mathcal{O}_{\mathbf{P}^n}(1)$ carries a Hermitian metric h whose curvature form is a multiple of the Fubini–Study form ω_{FS}. In fact, the standard Hermitian product $\langle v,w\rangle = {}^t v\cdot\overline{w}$ on $V=\mathbf{C}^{n+1}$ gives rise to a Hermitian metric on the trivial bundle $V_{\mathbf{P}^n}$ on $\mathbf{P}^n=\mathbf{P}(V)$. Then $\mathcal{O}_{\mathbf{P}^n}(-1)$ inherits a metric as a sub-bundle of the trivial bundle $\mathbf{P}^n\times V$, which in turn determines a metric h on $\mathcal{O}_{\mathbf{P}^n}(1)$. Very explicitly, write $[x]\in\mathbf{P}^n$ for the point corresponding to a vector $x\in V-\{0\}$ and consider a section

$$s \;\in\; V^* = H^0\big(\mathbf{P}^n,\mathcal{O}_{\mathbf{P}^n}(1)\big).$$

Then h is determined by the rule

$$\big|s\big(\,[x]\,\big)\big|_h^2 \;=\; \frac{|s(x)|^2}{\langle x,x\rangle},$$

where the numerator on the right is the squared modulus of the result of evaluating the linear functional s on the vector x.

If we work with the usual affine coordinates $z=(z_1,\ldots,z_n)$ on $\mathbf{P}^n$ corresponding to the point $x=(1,z_1,\ldots,z_n)\in V$ and take $s\in V^*$ to be the functional given by projection onto the zeroth coordinate, then

$$\big|s\big(\,[x]\,\big)\big|_h^2 \;=\; \frac{1}{1+\sum|z_\alpha|^2}.$$

An explicit calculation [2, p. 30] shows that

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\Theta(\mathcal{O}_{\mathbf{P}^n}(1),h) \ &=_{\text{locally}} \ -\frac{\sqrt{-1}}{2\pi}\cdot\partial\bar\partial\log\left(\frac{1}{1+\sum|z_\alpha|^2}\right)\\ &= \ \frac{\sqrt{-1}}{2\pi}\cdot\left(\frac{\sum dz_\alpha\wedge d\bar z_\alpha}{1+\sum|z_\alpha|^2} \ - \ \frac{\left(\sum \bar z_\alpha dz_\alpha\right)\wedge\left(\sum z_\alpha d\bar z_\alpha\right)}{\left(1+\sum|z_\alpha|^2\right)^2}\right)\\ &= \ \frac{1}{\pi}\cdot\omega_{\mathrm{FS}}.\end{aligned}$$

In particular, $\mathcal{O}_{\mathbf{P}^n}(1)$ is positive in the sense of Kodaira. □

Remark 7.10. If L is a positive line bundle on a complex manifold X, and if $Y\subseteq X$ is a submanifold, then the restriction $L|Y$ of L to Y is positive. (Exercise!) It follows that any complex submanifold of $\mathbf{P}^n$ carries a postive line bundle.

Remark 7.11. Be aware that it typically happens that neither L nor L^* is positive. For example, the line bundle $\mathcal{O}(a,b)$ on $\mathbf{P}^1\times\mathbf{P}^1$ is positive if and only if $a,b>0$.

An important fact is that on a Kähler manifold, the positivity of a line bundle L depends only on the cohomology class of $c_1(L)$.

Proposition 7.12. *Assume that X is a compact Kähler manifold. Consider a hermitian line bundle such that*

$$[\tfrac{\sqrt{-1}}{2\pi}\,\Theta(L,h_0)] \ = \ [\omega] \ \in \ H^2(X,\mathbf{C}).$$

where ω is a positive $(1,1)$ form. Then L carries a hermitian metric h such that

$$\tfrac{\sqrt{-1}}{2\pi}\,\Theta(L,h) \ = \ \omega.$$

In particular, L is positive.

Proof. Given h_0, we look for a real-valued function $\phi : X\longrightarrow\mathbf{R}$ such that

$$h \ =_{\text{def}} \ e^{\phi}\,h_0$$

has curvature form ω. Now

$$\Theta(L,e^{\phi}\,h_0) \ = \ -\partial\overline{\partial}\phi \ + \ \Theta(L,h_0),$$

so we need to find ϕ satisfying

$$\tfrac{\sqrt{-1}}{2\pi}\Theta(L,h_0) \ - \ \omega \ = \ \tfrac{\sqrt{-1}}{2\pi}\,\partial\overline{\partial}\phi. \qquad (*)$$

But by assumption the left hand side of (*) is d-exact, hence by the $\partial\overline{\partial}$ lemma (Proposition 6.23) it is $\partial\overline{\partial}$-exact, as required. □

The very basic theorem for which we are aiming is this:

Theorem 7.13. (Kodaira-Nakano Vanishing Theorem) *Let L be a positive holomorphic line bundle on a compact Kähler manifold X of dimension n. Then*

$$H^{n,q}(X, L) \ = \ 0 \ \ \textit{for} \ \ q > 0,$$

i.e. $H^q(X, \Omega_X^n \otimes L) = 0$ *when* $q > 0$. *More generally, one has*

$$H^{p,q}(X, L) \ = \ 0$$

for $p + q > n$.

The first statement is the Kodaira vanishing theorem, while the second is usually called Nakano vanishing.

Harmonic theory for hermitian line bundles. The first ingredient going into Theorem 7.13 is an extension of the Hodge decomposition theorem for hermitian line bundles. The analysis underlying this result (which in any event we ignore) is nothing beyond that which goes into the classical statement, once things have been set up properly. This set-up is what we now discusss. Most of this material generalizes with no change to the case of hermitian vector bundles of higher rank, but for simplicity we stick with line bundles.

Consider then a hermitian holomorphic line bundle (L, h) on a compact complex manifold (X, g) of dimension n with a compatible Riemannian metric. The metric h allows us first of all to define a Hodge $*$-operator on L-valued forms. Namely, one can view h as giving a $\mathbf{C}$-anti-linear isomorphism $h : L \longrightarrow L^*$, and then one can define a $\mathbf{C}$-anti-linear isomorphism

$$\overline{*}_L \ : \ \Lambda^{p,q}T^*X \otimes L \longrightarrow \Lambda^{n-p,n-q}T^*X \otimes L^*$$

by the rule

$$\overline{*}_L\big(\alpha \otimes s\big) \ = \ \overline{*\alpha} \otimes h(s) \ \big(= *\overline{\alpha} \otimes h(s) \big).$$

The metrics h and g also give rise to a global hermitian product on the spaces $\mathcal{A}^{p,q}(L)$ of L valued forms via

$$\big(\alpha \otimes s, \beta \otimes t\big)_L \ = \ \int_X h(s,t) \cdot (\alpha \wedge *\overline{\beta}), \tag{7.3}$$

extending the definition in (6.3). In particular, if $\phi \in \mathcal{A}^{p,q}(X, L)$, then

$$\int_X \phi \wedge \overline{*}_L \, \phi \ = \ (\phi, \phi)_L.$$

We now bring $\overline{\partial}$ into the picture. Recall from (5.1) that because L is holomorphic there is a canonically defined operator

$$\overline{\partial}_L : \mathcal{A}^{p,q}(L) \longrightarrow \mathcal{A}^{p,q+1}(L).$$

We then define the adjoint

$$\overline{\partial}_L^* \ : \ \mathcal{A}^{p,q}(L) \longrightarrow \mathcal{A}^{p,q-1}(L)$$

by the formula

$$\overline{\partial}_L^* \ = \ -\overline{*}_{L^*} \circ \overline{\partial}_{L^*} \circ \overline{*}_L.$$

(See Exercise 7.4). As in Proposition 6.7, this is adjoint to $\overline{\partial}_L$ with respect to the inner product (7.3):

Lemma 7.14. *For any $\alpha \in \mathcal{A}^{p,q}(L)$ and $\beta \in \mathcal{A}^{p,q+1}(L)$ one has*

$$\big(\alpha\,,\,\overline{\partial}^*_L\beta\big)_L \;=\; \big(\overline{\partial}_L\alpha\,,\,\beta\big)_L.$$

(See [3, Lemma 4.1.12].)

Next one defines the Laplace operator for (L,h):

Definition 7.15. The Laplacian associated to (L,h) is the self-adjoint elliptic operator

$$\Delta_L \,:\, \mathcal{A}^{p,q}(L) \longrightarrow \mathcal{A}^{p,q}(L) \quad , \quad \Delta_L \;=\; \overline{\partial}^*_L\overline{\partial}_L \,+\, \overline{\partial}_L\overline{\partial}^*_L.$$

An L-valued form $\alpha \in A^{p,q}(L)$ is *harmonic* if

$$\Delta_L(\alpha) \;=\; 0.$$

One denotes by

$$\mathcal{H}^{p,q}(L) \;\subseteq\; \mathcal{A}^{p,q}(L)$$

the space of all such. As before an L-valued α is harmonic if and only if

$$\overline{\partial}_L(\alpha) \;=\; \overline{\partial}^*_L(\alpha) \;=\; 0.$$

Then one has:

Theorem 7.16. *Let (L,h) be a Hermitian line bundle on a compact complex manifold X. Then $\mathcal{H}^{p,q}(L)$ is finite dimensional, and one has an orthogonal direct sum decomposition*

$$\mathcal{A}^{p,q}(X,L) \;=\; \overline{\partial}_L\mathcal{A}^{p,q-1}(X,L) \,\oplus\, \mathcal{H}^{p,q}(L) \,\oplus\, \overline{\partial}^*_L\mathcal{A}^{p,q+1}(L).$$

And just as in §6, one concludes that every class in $H^{p,q}_{\overline{\partial}}(X,L) = H^q(X,\Omega^p_X \otimes L)$ has a unique harmonic representative:

Corollary 7.17. *The natural map*

$$\mathcal{H}^{p,q}(X,L) \longrightarrow H^{p,q}_{\overline{\partial}}(X,L)$$

is an isomorphism.

Before going on, we pause to note an important consequence of this picture, namely the famous Serre duality theorem. Let L be a holomorphic line bundle on a compact complex manifold X of dimension n. Specifically, there is a natural pairing

$$H^{p,q}(X,L) \times H^{n-p,n-q}(X,L^*) \longrightarrow H^{n,n}(X) \;=\; \mathbf{C} \quad , \quad (\alpha,\beta) \mapsto \int_X \alpha \wedge \beta. \tag{7.4}$$

Here we view $\alpha \wedge \beta$ as an (n,n)-form with values in $L \otimes L^* = \mathbf{1}_X$.

Theorem 7.18. **(Serre duality).** *The map (7.4) is a perfect pairing. In particular,*

$$H^{p,q}(X,L) \;\cong\; H^{n-p,n-q}(X,L^*)^*.$$

Proof. Fixing metrics on X and L, we can view (7.4) as the map

$$\mathcal{H}^{p,q}(X,L) \times \mathcal{H}^{n-p,n-q}(X,L^*) \longrightarrow \mathbf{C} \quad (\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

on harmonic forms. Given $0 \neq \alpha \in \mathcal{H}^{p,q}(X,L)$, we need to show that α pairs non-trivially with some $\beta \in \mathcal{H}^{n-p,n-q}(X,L^*)$. But $\beta =_{\text{def}} \overline{*}_L\alpha$ is harmonic (Exercise 7.5), and

$$\int_X \alpha \wedge \beta \ = \ \int_X \alpha \wedge \overline{*}_L\alpha \ = \ (\alpha,\alpha)_L \ > \ 0,$$

as required. $\square$

Generalized Kähler identities and proof of Kodaira-Nakano vanishing. Let X be a compact Kähler manifold of dimension n, and let (L,h) be a holomorphic line bundle on X with a hermitian metric. The Kodaira vanishing theorem will follow from some analogues of the Kähler identities, involving $\overline{\partial}_L$ in place of $\overline{\partial}$. For this we need an analogue for L of ∂, and this involves the notion of a connection. This presentation closely follows [4].

Recall then that a *connection* on L is a $\mathbf{C}$-linear map

$$\nabla = \nabla_L : \mathcal{A}^0(L) \longrightarrow \mathcal{A}^1(L)$$

that satisfies the Leibniz rule:

$$\nabla(f \cdot s) \ = \ df \otimes s \ + \ f \cdot \nabla(s).$$

One says that ∇ is *hermitian* if it satisfies the identity

$$dh(s_1,s_2) \ = \ h(\nabla s_1, s_2) \ + \ h(s_1, \nabla s_2); \tag{7.5}$$

here the the terms on the right are the 1-forms determined by the rules

$$h(\nabla s_1, s_2)(\xi) \ = \ h(\nabla_\xi s_1, s_2) \quad , \quad h(s_1, \nabla s_2)(\xi) \ = \ h(s_1, \nabla_{\overline{\xi}} s_2)$$

where $\nabla_\xi : \mathcal{A}^0(L) \longrightarrow \mathcal{A}^0(L)$ is the "derivation" associated to a tangent vector ξ via ∇. Using the decomposition $\mathcal{A}^1 = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$ such a connection can be written as a sum $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$, where

$$\nabla^{1,0} : \mathcal{A}^0(L) \longrightarrow \mathcal{A}^{1,0}(L) \quad , \quad \nabla^{0,1} : \mathcal{A}^0(L) \longrightarrow \mathcal{A}^{0,1}(L).$$

One says that ∇_L is *compatible with the complex structure* if $\nabla^{0,1}_L = \overline{\partial}_L$.

Proposition 7.19. *Given a holomorphic line bundle (L,h) with a hermitian metric on a complex manifold X, there is a unique hermitian connection ∇_L on L that is compatible with the complex structure.*

Sketch of Proof. We will give the local construction of ∇_L. Note first that the difference of any two connections on L is a 1-form on X. That said, choose a local trivialization $L_U \cong U \times \mathbf{C}$ of L over some open set $U \subseteq X$. Then on U sections of L are identified with smooth functions, and $\nabla^{0,1}_L = \overline{\partial}_L$ acts simply as $\overline{\partial}$. On the other hand, $\nabla^{1,0}_L$ differs from ∂ by wedge product with a $(1,0)$-form. In other words, one can write locally on U:

$$\nabla_L \ = \ \big(\partial \ + \ \theta\big) \ + \ \overline{\partial}$$

where $\theta \in \mathcal{A}^{1,0}(U)$. It remains to see what is the condition on θ imposed by the requirement that ∇ be hermitian. For this, let $e \in \Gamma(U, L)$ be the section determined by the constant vector $1 \in \mathbf{C}$ under the local trivialization of L. Then

$$h(e,e) \;=\; h$$

is the smooth function on U determining the metric as in equation (7.1), and $\partial e = \overline{\partial} e = 0$. Hence (7.5) yields

$$dh \;=\; h \cdot \theta \;+\; h \cdot \overline{\theta}.$$

Since $dh = \partial h + \overline{\partial} h$ and since θ has type $(1,0)$, we find:

$$\theta \;=\; \frac{1}{h} \cdot \partial h \;=\; \partial \log(h). \tag{7.6}$$

This proves the existence and uniqueness of ∇_L. □

One views $\nabla_L^{1,0}$ as the L-valued analogue of $\overline{\partial}_L$, and to emphasize this we shall subsequently write $\partial_L = \nabla_L^{1,0}$. One extends these to operators

$$\partial_L : \mathcal{A}^{p,q}(L) \longrightarrow \mathcal{A}^{p+1,q}(L) \;\;,\;\; \overline{\partial}_L : \mathcal{A}^{p,q}(L) \longrightarrow \mathcal{A}^{p,q+1}(L)$$

via the Leibnitz rule. As before

$$\partial_L^2 \;=\; \overline{\partial}_L^2 \;=\; 0, \tag{7.7}$$

but now these operators no longer anti-commute. In fact:

Lemma 7.20. *One has*

$$\partial_L \overline{\partial}_L \;+\; \overline{\partial}_L \partial_L \;=\; \Theta_L, \tag{7.8}$$

in the sense that the composition of operators on the left acts by wedge product with the curvature form Θ_L.

Remark 7.21. In view of (7.7), one has

$$\partial_L \overline{\partial}_L \;+\; \overline{\partial}_L \partial_L \;=\; \nabla_L^2,$$

and this is by definition the curvature of a connection. Here we are taking a low-tech approach that avoids a general discussion of curvature.

Proof of Lemma 7.20. For simplicity, we check the stated identity when the two sides act on smooth sections of L. Working locally, we may identify such a section with a smooth function s. Then, using the local description of ∂_L derived in the proof of Proposition 7.19, we have

$$\partial_L \overline{\partial}_L(s) \;=\; \partial(\overline{\partial} s) + \theta \wedge \overline{\partial}(s)$$

where $\theta = \partial(\log h)$ is the 1-form appearing in (7.6). On the other hand,

$$\begin{aligned} \overline{\partial}_L \partial_L(s) \;&=\; \overline{\partial}_L\big(\partial s \;+\; \theta \wedge s\big) \\ &=\; \overline{\partial}\partial(s) + \overline{\partial}\theta \wedge s \;-\; \theta \wedge \overline{\partial} s. \end{aligned}$$

Therefore

$$\big(\partial_L \overline{\partial}_L \;+\; \overline{\partial}_L \partial_L\big)(s) \;=\; \big(\overline{\partial}\partial \log(h)\big) \cdot s,$$

as required. □

There are three more operators that come into the picture. First, wedging with the $(1,1)$ form ω determined by the metric g gives in the natural way an extenstion of the Lefschetz operator

$$L : \mathcal{A}^{p,q}(X,L) \longrightarrow \mathcal{A}^{p+1,q+1}(X,L) \quad , \quad \alpha \otimes s \mapsto \omega \wedge \alpha \otimes s.$$ [26]

As before, we have the adjoint operator Λ:

$$\Lambda : \mathcal{A}^{p,q}(X,L) \longrightarrow \mathcal{A}^{p-1,q-1}(X,L) \quad , \quad \alpha \otimes s \mapsto \Lambda\alpha \otimes s.$$

Note that these are point-wise operators, which coincide with the previously studied versions when we take a local trivialization of L. One can check moreover that Λ is the global adjoint of L with respect to the hermitian inner product $(\ ,\ )_L$ introduced above. Finally we denote by $\partial_L^* : \mathcal{A}^{p,q}(L) \longrightarrow \mathcal{A}^{p-1,q}(L)$ the adjoint of ∂_L with respect to $(\ ,\ )_L$. Then the Kähler identity (6.7) generalizes to

Proposition 7.22. *Let X be a compact Kähler manifold. Then*

$$[\Lambda, \overline{\partial}_L] \ = \ -\sqrt{-1}\,\partial_L^*. \tag{7.9}$$

This is sometimes called the Nakano identity. It fairly easily reduces to (6.7): see for instance [3, Lemma 5.2.3].

We can now give the proof of Kodaira-Nakano vanishing.

Proof of Theorem 7.13. Since L is positive, it carries a hermitian metric h such that

$$\omega \ =_{\text{def}} \frac{\sqrt{-1}}{2\pi}\,\Theta(L,h)$$

is a closed positive $(1,1)$-form. We may then choose a Kähler metric on X with ω as the corresponding Kähler form. With these choices, (7.8) becomes

$$\partial_L\overline{\partial}_L \ + \ \overline{\partial}_L\partial_L \ = \ -2\pi\sqrt{-1}\,L,$$

where L is the Lefschetz operator. Taking adjoints, we find that

$$\partial_L^*\overline{\partial}_L^* \ + \ \overline{\partial}_L^*\partial_L^* \ = \ 2\pi\sqrt{-1}\,\Lambda. \tag{7.10}$$

Now consider a harmonic form

$$\alpha \ \in \ \mathcal{H}^{p,q}(X,L).$$

[26] We apologize for the two conflicting meanings of the symbol L, but there doesn't seem to be any easy way to avoid this, and hopefully no confusion will result.

Thanks to Corollary 7.17, the issue is to show that $\alpha = 0$ if $p+q > n$. For this, we compute:

$$\begin{aligned}
\big(\Lambda\alpha\,,\,\Lambda\alpha\big)_L &= \tfrac{\sqrt{-1}}{2\pi}\big(\Lambda\alpha\,,\,(\partial_L^*\overline{\partial}_L^* + \overline{\partial}_L^*\partial_L^*)\alpha\big)_L && \text{(by (7.10))}\\
&= \tfrac{\sqrt{-1}}{2\pi}\big(\Lambda\alpha\,,\overline{\partial}_L^*\partial_L^*\alpha\big)_L && \text{(since } \overline{\partial}_L^*\alpha = 0)\\
&= \tfrac{\sqrt{-1}}{2\pi}\big(\overline{\partial}_L\Lambda\alpha\,,\,\partial_L^*\alpha)_L\\
&= \tfrac{\sqrt{-1}}{2\pi}\big(-[\Lambda,\overline{\partial}_L]\alpha,\partial_L^*\alpha\big)_L && \text{(since } \overline{\partial}_L\alpha = 0)\\
&= \tfrac{\sqrt{-1}}{2\pi}\big(\sqrt{-1}\partial_L^*\ \alpha,\partial_L^*\alpha\big)_L && \text{(by Proposition 7.22)}\\
&= -\tfrac{1}{2\pi}\big(\partial_L^*\alpha,\partial_L^*\alpha\big)_L.
\end{aligned}$$

But by comparing the signs of the first and last expressions we find that $(\Lambda\alpha, \Lambda\alpha)_L = 0$, and hence $\Lambda\alpha = 0$. But this means that α is locally represented by a primitive form. On the other hand, by Theorem 3.16 (iii), there are no non-vanishing primitive forms of degree $> n$. Hence $\alpha = 0$, and we are done. □

Corollary 7.23. *Let L be a positive line bundle on a compact Kähler manifold X. Then given any line bundle P on X, there exists a constant $m_0 = m_0(L, P)$ such that if $m \geq m_0$ then*

$$H^q(X, L^{\otimes m} \otimes P) \ = \ 0$$

for $q > 0$. □

This is a special case of the Serre vanishing theorem. The reader is asked to prove the statement in Exercise 7.6.

The Kodaira embedding theorem. We now come to our final theorem, which ties together much of what we've studied up to now.

Theorem 7.24. (Kodaira Embedding Theorem). *Let X be a compact Kähler manifold, and let L be a positive holomorphic line bundle on X. Then for $m \gg 0$ there exists an embedding*

$$\phi = \phi_m : X \longrightarrow \mathbf{P}^{r_m}$$

(for some large $r = r_m$) such that

$$L^{\otimes m} \ = \ \phi^*\mathcal{O}_{\mathbf{P}^r}(1).$$

In other words, a compact Kähler manifold admits a projective embedding if and only if it carries a positive line bundle.

The plan is to show first of all that $L^{\otimes m}$ is globally generated for $m \gg 0$, so that $L^{\otimes m}$ defines a holomorphic mapping $\phi : X \longrightarrow \mathbf{P}^r$. Then we will show that – possibly after increasing m – the resulting mapping is actually an embedding. We will concentrate on proving the first point, the second being similar.

Fix a point $x \in X$. We want to show that if $m \gg 0$ then $L^{\otimes m}$ has a holomorphic section that doesn't vanish at x. To this end, denote by $\mathfrak{m}_x \subseteq \mathcal{O}_X$ the ideal sheaf of holomorphic functions vanishing at x, and consider the exact sequence

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\{x\}} \longrightarrow 0,$$

so that $\mathcal{O}_{\{x\}}$ is a one-dimensional sky-scraper sheaf supported at x. Tensoring by $L^{\otimes m}$ we arrive at

$$0 \longrightarrow L^{\otimes m} \otimes \mathfrak{m}_x \longrightarrow L^{\otimes m} \longrightarrow L^{\otimes m} \otimes \mathcal{O}_{\{x\}} \longrightarrow 0,$$

and the long exact cohomology sequence gives

$$H^0(X, L^{\otimes m}) \xrightarrow{\mathrm{ev}_x} \mathbf{C} \longrightarrow H^1(X, L^{\otimes m} \otimes \mathfrak{m}_x). \tag{7.11}$$

Here $\mathbf{C} = H^0(L^{\otimes m} \otimes \mathcal{O}_{\{x\}})$ and the map ev_x is evaluation at x. Now suppose one knew that

$$H^1(X, L^{\otimes m} \otimes \mathfrak{m}_x) \ = \ 0 \quad \text{for } m \gg 0. \tag{7.12}$$

Then evaluation the evaluation map ev_x would be surjective, and we could conclude that $L^{\otimes m}$ has a global section that is non-vanishing at x.[27] The question being reduced to a vanishing statement involving a positive line bundle, one can hope to apply Theorem 7.13. Unfortunately it doesn't apply directly because – when $n = \dim X \geq 2$ – the ideal sheaf $\mathfrak{m}_x \subseteq \mathcal{O}_X$ is not a line bundle. However we can circumvent this problem by blowing up the point $x \in X$, which reduces one to a question about locally free sheaves.

Blowing up a point. Recall that in Example 2.35 we discussed the construction of the blow-up of a point in $\mathbf{C}^n$. Since this construction is local about a neighborhood of $0 \in \mathbf{C}^n$, we will be able carry it over to any complex manifold X to produce a new manifold X' in which the ideal sheaf of a point is in effect replaced by a line bundle.

We start by reviewing and analyzing the construction in $\mathbf{C}^n$. Viewing $\mathbf{P}^{p-1}$ as the space of lines through the origin in $\mathbf{C}^n$, consider

$$Z \ = \ \big\{ ([a], v) \mid v \in \mathbf{C} \cdot a \big\} \ \subseteq \mathbf{P}^{n-1} \times \mathbf{C}^n.$$

Write

$$b : Z \longrightarrow \mathbf{C}^n \quad , \quad f : Z \longrightarrow \mathbf{P}^{n-1}$$

for the two projections. Thus f realizes Z as the total space of the line bundle $\mathcal{O}_{\mathbf{P}^{n-1}}(-1)$ over $\mathbf{P}^{n-1}$. In particular, the fibre of b over $0 \in \mathbf{C}^n$ is a copy of $\mathbf{P}^{n-1}$ which we is called the *exceptional divisor* of Z:

$$E \ = \ \mathbf{P}^{n-1} \ = \ b^{-1}(0) \ \subseteq \ Z.$$

Note that E is naturally identified as the projective space $\mathbf{P}T_0\mathbf{C}^n$ of lines in the (holomorphic) tangent space of $\mathbf{C}^n$ at 0. Furthermore, note that if Δ is any neighborhood of $0 \in \mathbf{C}^n$, and if $\Delta^* = \Delta - \{0\}$, then b restricts to an isomorphism

$$b^{-1}(\Delta^*) \xrightarrow{\cong} (\Delta^*). \tag{7.13}$$

We set

$$\Delta' = \mathrm{Bl}_0(\Delta) \ =_{\mathrm{def}} \ b^{-1}(\Delta) :$$

[27] A similar argument with $\mathfrak{m}_x$ replaced by the ideal $\mathfrak{m}_{x,y}$ of two points $x, y \in X$ would show that the morphism ϕ is one-to-one, and replacing $\mathfrak{m}_x$ by $\mathfrak{m}_x^2$ yields that $d\phi$ is injective.

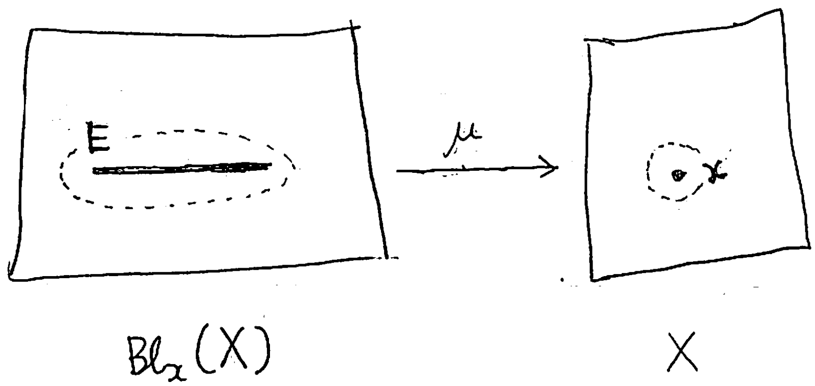

FIGURE 7. Blowing up a point $x \in X$.

this is the blowing-up of Δ at 0. Thus $\Delta' - E = b^{-1}(\Delta^*)$.

It is useful for computations to observe that if $U_i \subseteq \mathbf{P}^{n-1}$ are the standard open subsets of $\mathbf{P}^{n-1}$, then there are natural isomorphisms $f^{-1}(U_i) \cong \mathbf{C}^n$ under which b is locally identified with the maps $b : \mathbf{C}^n \longrightarrow \mathbf{C}^n$ given by

$$(w_0, w_1, \ldots, w_{n-1}) \ \mapsto \ \big(w_i w_0\,,\, w_i w_1\,,\, \ldots\,,\, w_i\,,\, \ldots\,,\, w_i w_{n-1}\big) \tag{7.14}$$

(Exercise 7.8). In these local coordinates on Z, the exceptional divisor E is (locally) defined in Z by the equation $w_i = 0$, and f is given by $(w_0, \ldots, w_i, \ldots w_{n-1}) \mapsto [w_0, \ldots, 1, \ldots, w_{n-1}]$.

Denote by $\mathcal{O}_Z(-E) \subseteq \mathcal{O}_Z$ the ideal sheaf of E, i.e. the sheaf of holomorphic functions vanishing on E. Then, using the local description just given, one can check (Exercise 7.8):

Lemma 7.25. *The ideal sheaf $\mathcal{O}_Z(-E)$ is a line bundle on Z, and*

$$\mathcal{O}_Z(-E) \ \cong \ f^*\mathcal{O}_{\mathbf{P}^{n-1}}(1). \tag{7.15}$$

(See Exercise 7.8.) This will allow us shortly to put a metric on $\mathcal{O}_Z(-E)$. The reader should also check by taking the Jacobian determinant of (7.14) that

$$b^*\Omega^n_{\mathbf{C}^n} \ = \ \Omega^n_Z \otimes \mathcal{O}_Z(-(n-1)E);$$

very concretely, the pull-back via b of the form $dz_1 \wedge \ldots \wedge dz_n$ vanishes to order $(n-1)$ along E.

We now globalize this discussion. Given a complex manifold X, fix a point $x \in X$. Choose a neighborhood U of $x \in X$ isomorphic to a neighborhood $\Delta \ni 0$ of the origin in $\mathbf{C}^n$, and write $U^* = U - \{x\}$. Using the local discussion above, and the identification of U with Δ, we can construct the blowing-up $U' = \mathrm{Bl}_x(U)$ of U at x, and then we construct a new manifold $X' = \mathrm{Bl}_x(X)$ by replacing U by U' via the isomorphism (7.13). The blowing-up $X' = \mathrm{Bl}_x(X)$ comes with a mapping

$$\mu \,:\, \mathrm{Bl}_x(X) \longrightarrow X$$

that is an isomorphism over $X - \{x\}$. Moreover X' contains an *exceptional divisor*

$$E \;=\; \mu^{-1}(x)$$

which is canonically idenfitied with the projective space $\mathbf{P}T_xX$ of one-dimensional subspaces in the holomorphic tangent space T_xX. This process is illustrated schematically in Figure 7. As in the local picture one has

$$\mu^*\Omega^n_X \;=\; \Omega^n_{X'} \otimes \mathcal{O}_{X'}(-(n-1)E). \tag{7.16}$$

The fastidious reader can check that up to isomorphism $\mathrm{Bl}_x(X)$ does not depend on the particular neighborhood $U \ni x$ we took in the construction.

Remark 7.26. In an essentially identical manner, one can form the blowing up of X at a finite number of points. With more effort, one can also construct the blow-up of X along any closed submanifold $Y \subseteq X$: in this case the exceptional divisor is isomorphic to the projectivization $\mathbf{P}N_{Y/X}$ of the complex normal bundle to Y in X.

The next point is to study the positivity of the relevant line bundles.

Lemma 7.27. *Let L be a positive line bundle on X, and fix an integer $a > 0$ and an arbitrary line bundle P on X. Then for $m \gg 0$ the line bundle*

$$\mu^*(L^{\otimes m} \otimes P) \otimes \mathcal{O}_{X'}(-aE))$$

is positive on X'.

Proof. Fix a positive metric h_L on L and an arbitrary metric h_P on P. Then $h_L^{\otimes m} \otimes h_P$ pulls back to a metric h_m on $\mu^*(L^{\otimes m} \otimes P)$. By taking $m \gg 0$ we can arrange that it is positve away from E, but it won't be positive along E itself since $\Theta(h_m)$ vanishes on vectors tangent to E. On the other hand, via the local description (7.15) of $\mathcal{O}_{X'}(-E)$, we can find (using a partition of unity) a metric h_a on $\mathcal{O}_{X'}(-aE)$ that in a neighborhood of E agrees with a power of the pullback $f^*h_{\mathrm{FS}}^{\otimes a}$ of the Fubini-study metric on $\mathcal{O}_{\mathbf{P}^{n-1}(1)}$ (but we don't know anything about h_a away from E). We claim that if $m \gg 0$ then $h_m \otimes h_a$ is everywhere positive. This will follow from Exercise 7.3 (a) away from E, so it remains to check the positivity along E. For this, note that by construction $\Theta(h_a)$ is positive on tangent directions to E. On the other hand, $\Theta(h_m)$ is positive on directions normal to E since (as we may assume) it's the pull-back of a positive $(1,1)$-form on X. Therefore $h_m \otimes h_a$ is positive everywhere on X' for $m \gg 0$. □

Corollary 7.28. *Let L be a positive line bundle on X. If $m \gg 0$, then*

$$H^1\big(X', \mu^*(L^{\otimes m}) \otimes \mathcal{O}_{X'}(-E)\big) \;=\; 0.$$

Proof. We apply the previous Lemma with $P = (\Omega^n_X)^*$ and $a = n = \dim X$. Then by (7.16), one has:

$$\mu^*L^{\otimes m} \otimes \mathcal{O}_{X'}(-E) \;=\; \Omega^n_{X'} \otimes \Big(\mu^*\big(L^{\otimes m} \otimes (\Omega^n_X)^*\big) \otimes \mathcal{O}_{X'}(-nE)\Big),$$

so the assertion follows from Kodaira vanishing. □

We now indicate the proof of the Kodaira embedding theorem. Given a point $x \in X$, we focus on showing that $L^{\otimes m}$ has a section that doesn't vanish at x for $m \gg 0$: this implies that the mapping ϕ_m appearing in Theorem 7.24 is well-defined, and as indicated above the argument that it is one-to-one with injective derivative is similar. For simplicity we assume $n \geq 2$ (the case of Riemann surfaces being elementary given what we already know).

To this end, consider the blowing-up

$$\mu : X' = \mathrm{Bl}_x(X) \longrightarrow X.$$

We claim first that pull-back of sections yields an isomorphism

$$H^0(X, L^{\otimes m}) \xrightarrow{\cong} H^0\big(X', \mu^*(L^{\otimes m})\big).$$

In fact, the map is obviously injective, so consider a section

$$s' \in H^0\big(X', \mu^*(L^{\otimes m})\big).$$

Then the restriction of s' to $X' - E$ can be viewed as a section $s_0 \in \Gamma(X - \{x\}, L^{\otimes m})$ of $L^{\otimes m}$ on the complement of x. But by Hartog's theorem, s_0 extends to a section $s \in \Gamma(X, L^{\otimes m})$ which pulls back to s'.

Now consider the commutative diagram

$$\begin{tikzcd}
H^0\big(X', \mu^*(L^{\otimes m})\big) \arrow[r] \arrow[d] & H^0\big(X', \mu^* L^{\otimes m} \otimes \mathcal{O}_E\big) \arrow[d] \\
H^0\big(X, L^{\otimes m}\big) \arrow[r, "\mathrm{ev}_x"] & H^0\big(X, L^{\otimes m} \otimes \mathcal{O}_{\{x\}}\big)
\end{tikzcd}$$

The vertical map on the left is an isomorphism, and since $\mu^* L$ is trivial along E, so is the vertical map on the right. Thus we can identify the top map with evaluation of sections at $x \in X$. On the other hand, it follows from Corollary 7.28 – and this is the serious point – that the top vertical map is surjective if $m \gg 0$. Therefore ev_x is surjective when $m \gg 0$, and so we have produced a section

$$s_x \in \Gamma\big(X, L^{\otimes m}\big) \quad \text{with} \quad s_x(x) \neq 0.$$

The starting value of m here might depend on x. However s_x is non-vanishing in a neighborhood of x, and then by compactness we can find one value of m that works at every point of x. QED.

Exercises for Section 7.

Exercise 7.1. Prove that any smooth complex line bundle on a manifold admits a smooth hermitian metric. [Hint: If h_1, h_2 are hermitian metrics on X, and $\rho : X \longrightarrow \mathbf{R}_0$ is a smooth function, then $h_1 + h_2$ and $\rho\, h_1$ are also Hermitian metrics. This allows one to glue local metrics together via a partition of unity.)]

Exercise 7.2. If L_1, L_2 are positive line bundles on a compact complex manifold X, then so is $L_1 \otimes L_2$.

Exercise 7.3. Let X be a compact complex manifold.

(a). Let ω be a positive $(1,1)$-form on X, and let η be an arbitrary $(1,1)$-form. Show that if $m \gg 0$, then $m\omega + \eta$ is positive.
(b). Let L be a positive holomorphic line bundle on X, and let P be an arbitrary holomorphic line bundle on X. Show that if $m \gg 0$, then $L^{\otimes m} \otimes P$ is again a positive line bundle.

Exercise 7.4. Show that if $L = X \times \mathbf{C}$ is the trivial line bundle with the constant metric, then $\overline{\partial}_L^* = \overline{\partial}^*$.

Exercise 7.5. Prove that if $\alpha \in \mathcal{H}^{p,q}(X, L)$, then

$$\overline{*}_L \alpha \ \in \ \mathcal{H}^{n-p,n-q}(X, L^*).$$

Exercise 7.6. Prove Corollary 7.23. (Hint: Use Exercise 7.3 to write $L^{\otimes m_0} \otimes P = \Omega_X^n \otimes N$ where N is a positive line bundle.)

Exercise 7.7. Prove that a compact Kähler manifold X admits a projective embedding if and only if it carries a Kähler form ω with rational periods, ie with the property that

$$\int_\gamma \omega \ \in \ \mathbf{Q}$$

for all $\gamma \in H_2(X, \mathbf{Z})$. (Hint: The condition is equivalent to the condition that $\omega \in H^2(X, \mathbf{Q})$, and then after replacing ω by multiple one can assume that $\omega \in H^2(X, \mathbf{Z})$. Now apply the Lefschetz $(1,1)$-theorem.) Such a manifold is sometimes called a Hodge manifold.

Exercise 7.8. Prove the local description of blowing up given in equation (7.14). Use this to verify Lemma 7.25.

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