MAT 542: Complex Analysis I
Spring 2013

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General Information

Professor: Rubí E. Rodríguez, Math Tower 4-119, Phone 632-8358,

email: rubi@math.sunysb.edu, office hours: M 4:00 - 5:00 PM, W 11:00 - 12:00 AM, and by appointment.

Grader: Anant Atyam, Math Tower 2-105, email: anant@math.sunysb.edu Office hours: W 1:00 – 2:00 PM

Place and time: Physics P-123, MW 2:30 - 3:50 PM

Textbook: Complex Analysis, by R. E. Rodríguez, I. Kra and J. P. Gilman. Springer GTM 245 Second Edition 2013.

Course description: The course will cover at least the topics in the **basic syllabus**, with some variations in the order of presentation, and possible additions on topics of current interest.

Grades policy: Homework problems will be asssigned most weeks, for a total of ten weeks, then collected in class and graded. There will be a midterm exam on Wednesday March 13, 2-3:50 PM, and a final exam on Monday, May 13, 5:30 - 8:00 PM (room P123). Homework, midterm and final will count for 30%, 30% and 40% of your grade respectively.

Syllabus/schedule (subject to change)

M 1/28	Chapter 2.
W 1/30	Chapter 2.
M 2/4	Chapter 3.
W 2/6	Chapter 3. **HW due:** 2.2,2.5,2.7,2.9,2.10,3.20,3.21
W 2/13	Chapter 3.
M 2/18	Chapter 3.
W 2/20	Chapter 3. **HW due:** 3.1,3.6,3.10,3.11,3.13,3.14,3.15
M 2/25	Chapter 3.
W 2/27	Chapter 4.
M 3/4	Chapter 4.
W 3/6	Chapter 4. **HW due:** 2.15,2.16,2.17,3.7,3.12,3.18,3.24
M 3/11	Chapter 4.
W 3/13	Chapter 4.

M 3/25	Chapter 5.
W 3/27	Chapter 5.
M 4/1	Chapter 5.
W 4/3	Chapter 6. **HW due:** 4.4,4.5,4.11,5.3,5.8,5.9
M 4/8	Chapter 6.
W 4/10	Chapter 6.
M 4/15	Chapter 6.
W 4/17	Chapter 6.
M 4/22	Chapter 6.
W 4/24	Chapter 7. **HW due:** 5.10,5.11,5.12,6.2,6.4,6.5,6.15
M 4/29	Chapter 7.
W 5/1	Chapter 8.
M 5/6	Chapter 8.
W 5/8	Review.

Information for students with disabilities

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

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Extra course material

The Fundamental Theorem of complex function theory: the first part of the course will be dedicated to understanding its proof and consequences.

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Exams

Solutions for the **Midterm** and **Final Exam** will appear here.

1.2. The Fundamental Theorem of complex function theory

THEOREM 1.1. *Let $D \subseteq \mathbb{C}$ denote a domain (an open connected set) and let $f = u + \imath v : D \to \mathbb{C}$ be a complex-valued function defined on D. The following conditions are equivalent:*

(1) *The complex derivative*

$$f'(z) \text{ exists for all } z \in D; \qquad \text{(Riemann)}$$

that is, the function f is holomorphic on D.

(2) *The functions u and v are continuously differentiable and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad \text{(Cauchy–Riemann: CR)}$$

Alternatively, the function f is continuously differentiable and satisfies

$$\frac{\partial f}{\partial \overline{z}} = 0. \qquad \text{(CR–complex form)}$$

(3) *For each simply connected subdomain $\widetilde{D}$ of D there exists a holomorphic function $F : \widetilde{D} \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \widetilde{D}$.*

(4) *The function f is continuous on D, and if γ is a (piecewise smooth) closed curve in a simply connected subdomain of D, then*

$$\int_\gamma f(z)\, dz = 0.$$

((1) $\Longrightarrow$ (4): Cauchy's theorem; (4) $\Longrightarrow$ (1): Morera's theorem)

An equivalent formulation of this condition is: The function f is continuous on D and the differential form $f(z)\, dz$ is closed on D.

(5) *If $\{z \in \mathbb{C} : |z - z_0| \le r\} \subseteq D$ with $r > 0$, then*

$$f(z) = \frac{1}{2\pi\imath} \int_{|\tau - z_0| = r} \frac{f(\tau)}{\tau - z}\, d\tau \qquad \text{(Cauchy's integral formula)}$$

for each z such that $|z - z_0| < r$.

(6) *The n-th complex derivative*

$$f^{(n)}(z) \text{ exists for all } z \in D \text{ and for all integers } n \ge 0.$$

(7) *If* $\{z : |z - z_0| \leq r\} \subseteq D$ *with* $r > 0$, *then there exists a unique sequence of complex numbers* $\{a_n\}_{n=0}^{\infty}$ *such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{(Weierstrass)}$$

for each z *such that* $|z - z_0| < r$. *Furthermore, the series converges uniformly and absolutely on every compact subset of* $\{z : |z - z_0| < r\}$. *The coefficients* a_n *may be computed as follows.*

$$a_n = \frac{1}{2\pi\imath} \int_{|\tau - z_0| = r} \frac{f(\tau)}{(\tau - z_0)^{n+1}}\, d\tau \qquad \text{(Cauchy)}$$

and

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \qquad \text{(Taylor)}$$

(8) *Choose a point* $z_i \in K_i$, *where* $\bigcup_{i \in I} K_i$ *is the connected component decomposition of the complement of* D *in* $\mathbb{C} \cup \{\infty\}$, *and let* $S = \{z_i; i \in I\}$. *Then the function* f *is the limit (uniform on compact subsets of* D*) of a sequence of rational functions with singularities only in* S.

(Runge)

MAT 542 Complex Analysis I
Midterm

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Justify all your answers.

I.- Consider the power series given by

$$f(z) = \sum_{n=1}^{\infty} n(z-5)^n.$$

a) Find its radius of convergence.

b) Compute $f'(9/2)$.

Solution

a) The radius of convergence is given by

$$\limsup_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1.$$

b) Observe that

$$\begin{aligned} f(z) = \sum_{n=1}^{\infty} n(z-5)^n &= (z-5)\sum_{n=1}^{\infty} n(z-5)^{n-1} \\ &= (z-5)\frac{d}{dz}\left(\sum_{n=0}^{\infty}(z-5)^n\right) \\ &= (z-5)\frac{d}{dz}\left(\frac{1}{1-(z-5)}\right) \\ &= \frac{z-5}{(6-z)^2} \end{aligned}$$

for all $|z-5| < 1$.

Since $|9/2 - 5| < 1$, we obtain

$$f'(9/2) = 4/27.$$

II.- Find all zeroes and poles in $\mathbb{C}$, including their corresponding orders, for the function

$$g(z) = \frac{z\ \sin(z)}{(z-2\pi)(z+\pi)^2}.$$

Solution

Candidates for zeroes of g are $z = 0$ and the zeroes of the sin function: $k\pi$, with $k \in \mathbb{Z}$. Similarly, candidates for poles are 2π and $-\pi$.

Since the functions $g_1(z) = z$, $g_2(z) = \sin(z)$, $g_3(z) = z - 2\pi$ and $g_4(z) = (z+\pi)^2$ have power series expansion at each point c of $\mathbb{C}$, the order of g at c is given by

$$\text{order}_c(g) = \text{order}_c(g_1) + \text{order}_c(g_2) - \text{order}_c(g_3) - \text{order}_c(g_4).$$

From the corresponding power series for each g_j at the candidate points we obtain

$$\begin{aligned} \text{order}_0(g) &= 1 + 1 - 0 - 0 = 2; \\ \text{order}_{k\pi}(g) &= 0 + 1 - 0 - 0 = 1\ , \quad \text{for } k \neq 0, 2, -1; \\ \text{order}_{2\pi}(g) &= 0 + 1 - 1 - 0 = 0; \\ \text{order}_{-\pi}(g) &= 0 + 1 - 0 - 2 = -1, \end{aligned}$$

and therefore g has a double zero at $c = 0$, simple zeroes at $c = k\pi$ for $k \neq 0, 2, -1$, a simple pole at $c = -\pi$, and no more zeroes nor poles in $\mathbb{C}$.

III.- Let

$$D = \{z \in \mathbb{C} : |z - 1| < 5\}$$

and let $h : D \to \mathbb{C}$ be a function having power series expansion at each point of D.

Assume that

$$h\left(\frac{n-1}{n}\right) = \frac{4(n-1)^3}{n^3}$$

for all natural numbers n.

Evaluate

$$h'''(1).$$

Solution

Observe that the sequence $z_n = \frac{n-1}{n}$ is contained in D, and converges to 1, also a point in D.

The function $H(z) = h(z) - 4z^3$ has a power series expansion at each point of D, since h and the polynomial $-4z^3$ both do, and H vanishes on the sequence z_n convergent in D.

The Identity Principle then implies that $H \equiv 0$ in D, and therefore

$$h(z) = 4z^3$$

for all z in D, from where

$$h'''(1) = 24.$$

MAT 542 Complex Analysis I
Exam solutions sketch

Name: ________________________________

Justify all your answers.

I.- Compute the value of the following expressions.

a)

$$\sup\{|\sin(z)| : z = x + iy, 0 \le x, y \le 2\pi\}$$

b)

$$\int_\gamma \left(\frac{\exp(\pi z)}{1+z^2}) + \cos(\frac{1}{z}) + \frac{1}{\exp(z)} \right) dz$$

where $\gamma(t) = 1 + i + 2\exp(-2\pi\, i\, t)$, $0 \le t \le 1$.

Solution

a) The function $\sin(z)$ is analytic in $\mathbb{C}$, and nonconstant. By the MMP, the maximum of its modulus on a bounded set will be achieved at the boundary.

Now for $z = x + iy$

$$|\sin(z)| = \frac{1}{2}((\exp(y) + \exp(-y))^2 - 4(\cos(x))^2)^{\frac{1}{2}}.$$

Since $\exp(y) + \exp(-y)$ is increasing for $0 \le y \le 2\pi$, its max is given by $\exp(2\pi) + \exp(-2\pi)$. Also, $\min\{(\cos(x))^2 : 0 \le x \le 2\pi\} = 0$, when $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$. Therefore

$$\sup\{|\sin(z)| : z = x + iy, 0 \le x, y \le 2\pi\} = \frac{1}{2}(\exp(2\pi) + \exp(-2\pi)).$$

b) The curve $\gamma(t) = 1 + i + 2\exp(-2\pi\, i\, t)$, $0 \le t \le 1$ is a Jordan curve, negatively oriented, and homotopic to a point in $\mathbb{C}$.

The function $f(z) = \frac{\exp(\pi z)}{1+z^2}) + \cos(\frac{1}{z}) + \frac{1}{\exp(z)}$ is holomorphic in the plane, except at $\{0, \pm i\}$. The points 0 and i are in the interior of γ, and $-i$ is not.

Then, by the Residue Theorem,

$$\int_\gamma \left(\frac{\exp(\pi z)}{1+z^2}) + \cos(\frac{1}{z}) + \frac{1}{\exp(z)} \right) dz = -2\pi i(\mathrm{Res}(f, 0) + \mathrm{Res}(f, i))$$

.

Now observe that $\cos(\frac{1}{z}) + \frac{1}{\exp(z)}$ is holomorphic at i and $\mathrm{Res}(f, i) = \mathrm{Res}(\frac{\exp(\pi z)}{1+z^2}, i) = \frac{\exp(\pi i)}{2i} = \frac{-1}{2i}$. Similarly, $\frac{\exp(\pi z)}{1+z^2} + \frac{1}{\exp(z)}$ is analytic at $z = 0$, and $\cos(\frac{1}{z}) = 1 - \frac{1}{2}\frac{1}{z^2} + \ldots$, for all $z \ne 0$, from where $\mathrm{Res}(f, 0) = 0$.

Therefore

$$\int_\gamma \left(\frac{\exp(\pi z)}{1+z^2} + \cos(\frac{1}{z}) + \frac{1}{\exp(z)} \right) dz = \pi.$$

II.- Use complex analysis to give two different proofs of the Fundamental Theorem of Algebra.

Variety of possible proofs, we covered several in the course: for instance using Rouché's Thm, or Liouville's Thm.

III.- Decide whether the following statements are true or false. Justify carefully.

a) Let

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let $h, g : \overline{D} \to \mathbb{C}$ be two continuous functions, analytic in D.

Suppose the real part of h and the real part of g coincide at every z with $|z| = 1$. Then h and g are identical.

b) There exists a sequence $\{f_n\}$ of holomorphic functions converging uniformly to a function f in D, and such that the sequence $\{f_n'\}$ of its derivatives does not converge uniformly in D to f'.

Solution

a) Since for any such function h we can obtain many different functions g satisfying the hypothesis by adding any imaginary constant to h, the statement is false.

Challenge: is this the only way to satisfy the hypothesis?

b) Let $f_n(z) = \dfrac{z^n}{n}$, for n in $\mathbb{N}$ and z in D. Then f_n is holomorphic in D and $f_n'(z) = z^{n-1}$ for each n.

The sequence $\{f_n\}$ converges uniformly to $f \equiv 0$ in D, since for every positive ϵ there exists N in $\mathbb{N}$ such that $1/N < \epsilon$. If $n \geq N$, then

$$|f_n(z) - 0| = \frac{|z|^n}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

for all z in D.

We know that then $\{f_n'\}$ converges uniformly to 0 on compact subsets of D. We now show that $\{f_n'\}$ does not converge uniformly to 0 in D. Indeed, take $\epsilon_0 = 1/10$. Then for every N in $\mathbb{N}$ we can find $n = N + 1 > N$ and $z_0 = (1/2)^{\frac{1}{N}}$ (positive real value) in D such that

$$|f_n'(z_0) - 0| = z_0^{n-1} = z_0^N = 1/2 > 1/10 = \epsilon_0$$

IV.- Show that if $f(z)$ is holomorphic for $|z| < 1$, if $f(0) = f'(0) = \ldots = f^{(N)}(0) = 0$ for some $N \geq 0$, and if $|f(z)| \leq 1$ para $|z| < 1$, then

$$|f(i/3)| \leq 3^{-N-1}.$$

Solution

Since $f(0) = f'(0) = \ldots = f^{(N)}(0) = 0$, the power series expansion of f at 0 is of the form

$$f(z) = z^{N+1}(a_{N+1} + a_{N+2}z + \ldots),$$

with radius of convergence $\rho \geq 1$.

Therefore the function

$$g(z) = a_{N+1} + a_{N+2}z + \ldots$$

is analytic for $|z| < 1$.

Note that $g(z) = \dfrac{f(z)}{z^{N+1}}$ for $z \neq 0$. Then for $0 < r < 1$ and all $|z| = r$ we have

$$|g(z)| = \frac{|f(z)|}{|z|^{N+1}} = \frac{|f(z|)}{r^{N+1}} \leq \frac{1}{r^{N+1}};$$

by the MMP, the same inequality holds for all $|z| \leq r$. Letting r approach 1, we obtain

$$|g(z)| \leq 1$$

for all $|z| < 1$; equivalently,

$$|f(z)| \leq |z|^{N+1}$$

for all $|z| < 1$, and the result follows.