SYLLABUS

TOPOLOGY / GEOMETRY II

Spring - 2014

Instructor: Blaine Lawson
Office: 5-109.
Office Hours: Tues.-Thurs. 2:30-4:00 or by appointment.

Grader: Seyed Ali Aleyasin
Office: 2-115.
Office Hours: Mon. 3:00-4:00 in MLC and 4:00-5:00 in office. Tues. 5:00-6:00 in MLC.

Schedule: Tues.-Thurs. 11:30 AM – 12:50 PM
Classroom: Physics P123

Grading: Homework 50%, Midterm 25%, Final 25%.

Textbook: *Introduction to Smooth Manifolds* by John Lee. (Second Edition)

Course Description: This is a course in the fundamentals of smooth manifold theory. After introducing the basic definitions we will study the following topics: Vector fields and flows, transversality, Sard's Theorem and Sard's Theorem for Families, The Whitney Embedding Theorem, differential forms, integration and Stokes' Theorem, de Rham cohomology and the de Rham Theorem, Lie groups and Lie algebras, distributions and the Frobenius Theorem.

The course will emphasize examples and applications.

Week 1 (Jan 28 and 30). Smooth manifolds, smooth mappings, and partitions of unity.

Week 2 (Feb. 4 and 6). Tangent vectors and the tangent bundle. Vector bundles. The differential of a smooth mapping.

Week 3 (Feb. 11 and 13). Immersions, submersions and embeddings. Submanifolds and regular values of mappings.

Week 4 (Feb. 18 and 20). Vector fields and flows. Transversality.

Week 5 (Feb. 25 and 27). Sard's Theorem and Sard's Theorem for families. Approximation theorems.

Week 6 (March 4 and 6). The Whitney Embedding Theorem.

Week 7 (March 11 and 13). Review and then:

MIDTERM EXAM – Thursday March 13th

Note the Change.

Spring Break – March 17 to 21

Week 8 (March 25 and 27). Tensor fields on a manifold. Vector bundles, fibre bundles and principal bundles.

Week 9 (April 1 and 3). The cotangent bundle, exterior differential forms, the deRham differential.

Week 10 (April 8 and 10). Integration on manifolds, and Stokes' Theorem.

Week 11 (April 15 and 17). De Rham cohomology and the de Rham Theorem.

Week 12 (April 22 and 24). Vector fields revisited. The Lie derivative, contraction and the infinitesimal homotopy formula.

Week 14 (April 29 and May 1). Lie Groups and Lie algebras.

Week 15 (May 6 and 9). Distributions and the Frobenius Theorem.

FINAL EXAM: Wednesday May 14th, 5:30pm-8:00pm

Disability Support Services: If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

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HOMEWORK

Due Tues, Feb. 4th: Do 8 of the following. I: 1,5,6,7,10, 12, II: 6,7,10,14
Due Tues, Feb. 11th: III: 3,6,8, IV: 5,6,13, V: 6.
Due Tues, Feb. 18th: V:1,7.

Due Tues, Feb. 25th: IX: 3(a), 4, 5, 8,
Due Tues, Mar. 4th: VI: 2, 9,10,11, 13(a) and (c), 14.
Due Tues, Mar. 11th: VI: 3 (use Theorem 2.29), 4, 12, 16(a),(c),(f), and prove following:

Suppose $(M, \partial M)$ is a compact manifold with boundary, and N is any manifold (without boundary). Suppose $F : M \to N$ is a smooth mapping, and $p \in N$ is

(1) a regular value of F, and

(2) a regular value of $F\big|_{\partial M}$.

Then $S \equiv F^{-1}(p)$ is a compact submanifold with boundary $\partial S \subset \partial M$. Furthermore, the embedding of S is transversal to ∂M, in the sense that the tangent space to S at each point $x \in \partial S$ is transversal to ∂M.

Due Tues, April 1st: Chapter 12, nos. 1,2,3,8, and do following:

(1) Let V be a finite-dimensional vector space. Show that the exterior algebra is associative and skew-commutative. That is, for $\varphi \in \Lambda^k V$, $\psi \in \Lambda^\ell V$ and $\eta \in \Lambda^m V$,

(i) $(\varphi \wedge \psi) \wedge \eta \;=\; \varphi \wedge (\psi \wedge \eta)$, and

(ii) $\varphi \wedge \psi \;=\; (-1)^{k\ell}\psi \wedge \varphi$.

(2) Show that there is a natural isomorphism: $U \otimes (V \otimes W) \cong U \otimes (V \otimes W)$.

(3) Show that there is a natural isomorphism: $\mathrm{Hom}(V, W) \cong V^* \otimes W$.

Due Tues, April 8th: Do Exercise 14.22 on page 362, Exercise 14.25 on page 366, Exercises 1, 5, and 7(a) at the end of Chapter 14, and do following:

(1) Given a 1-form $\omega \in \mathcal{E}^1(M)$, and vector fields V, W on M, define

$$d\omega(V, W) \;=\; V\omega(W) - W\omega(V) - \omega([V, W]).$$

Show that $d\omega$ is a 2-form. That is, prove that $d\varphi$ is a skew-symmetric $C^\infty(M)$-bilinear map $\mathcal{V}(M) \times \mathcal{V}(M) \to C^\infty(M)$. (Recall that $[fV, W] = -(Wf)V + f[V, W]$.)

(2) Given a p-form $\varphi \in \mathcal{E}^p(M)$, and vector fields $V_0, ..., V_p$ on M, define

$$d\varphi(V_0, ..., V_p) \;=\; \sum_{k=0}^{p}(-1)^p V_k\varphi(V_0, ..., \widehat{V_k}, ..., V_p) - \sum_{0\le k<\ell\le p}(-1)^{k+\ell}\varphi([V_k, V_\ell], V_0, ..., \widehat{V_k}, ..., \widehat{V_\ell}, ...V_p)$$

Show that when $p = 2$, $d\varphi$ is a 3-form. That is, prove that $d\varphi$ is a skew-symmetric $C^\infty(M)$-trilinear map $\mathcal{V}(M) \times \mathcal{V}(M) \times \mathcal{V}(M) \to C^\infty(M)$.

Due Tues, April 15th: Do Exercises 2,3,5 and 15 from Chapter 15. Do Exercises 4,6 and 9 from Chapter 16, and do the following:

(1) Show that every compact connected 1-dimensional submanifold is diffeomorphic to either S^1 or $[0, 1]$. Hint: First assume $\partial M = \emptyset$. Show M is orientable and consider the flow of a nowhere vanishing vector field. When $\partial M \neq \emptyset$, consider the double.

(2) Let $M = S^2 \times S^2$. Show that

$$\dim H^2_{\mathrm{deR}}(M) \ \geq\ 2.$$

(3) Let M be a compact manifold of dimension $2n$ without boundary.
Suppose there exists a 2-form $\omega \in \mathcal{E}^2(M)$ such that:

(i) $d\omega = 0$, and
(ii) $\omega^n \neq 0$ at all points of M.

Show that $H^{2k}_{\mathrm{deR}}(M) \neq 0$ for all $k = 1, ..., n$

(4) Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form in $\mathbf{R}^n$.
Given a vector field V on $\mathbf{R}^n$ define the contraction $i_V(\Omega) \in \mathcal{E}^{n-1}(\mathbf{R}^n)$ by

$$i_V(\Omega)(V_1, ..., V_{n-1}) \ =\ \Omega(V, V_1, ..., V_{n-1})$$

(i) If

$$V = \sum_{k=1}^n a_k(x) \frac{\partial}{\partial x_k}$$

show that

$$i_V(\Omega) \ =\ \sum_{k=1}^n (-1)^{k-1} a_k(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n.$$

(ii) Let $H \subset \mathbf{R}^n$ be an oriented linear subspace with a preferred unit normal $\nu \perp H$. Define the orientation on H by requiring that $(e_1, ..., e_{n-1})$ has positive orientation if $(\nu, e_1, ..., e_{n-1})$ has positive orientation on $\mathbf{R}^n$. Show that

$$i_V(\Omega)\big|_H \ =\ \langle V, \nu\rangle \mathrm{vol}_H$$

where $\mathrm{vol}_H = e_1^* \wedge \cdots \wedge e_{n-1}^*$ for (any) oriented orthonormal basis of H^*.

(iii) Let $D \subset\subset \mathbf{R}^n$ be a compact domain with smooth boundary ∂D. Let ν be the outward-pointing unit normal vector field to ∂D. Suppose out vector field satisfies the condition

$$\sum_{k=1}^n \frac{\partial a_k}{\partial x_k}(x) \ <\ 0 \quad \text{for all } x \in D.$$

Show that it is impossible to have

$$\langle V, \nu\rangle \ >\ 0 \qquad \text{at all points of } \ \partial D.$$

Due Tues, April 22nd: Do Exercises 1, 5, 7, 10, 12, 13 in Chapter 17. Also do the following:

Suppose $M = U \cup V$ as in the hypothesis of the Mayer-Vietoris Theorem. Assume that for each p the map

$$(i^* - j^*) : \mathcal{E}^p(U) \oplus \mathcal{E}^p(V) \;\to\; \mathcal{E}^p(U \cap V) \qquad \text{is surjective}$$

where $i : U \to M$ and $j : V \to M$ are the inclusions. Define

$$\delta : H^p_{\text{deR}}(U \cap V) \;\to\; H^{p+1}_{\text{deR}}(M)$$

as follows. Fix $\alpha \in H^p_{\text{deR}}(U \cap V)$ and choose $\varphi \in \mathcal{E}^p(U \cap V)$ with $d\varphi = 0$ and deRham class $[\varphi] = \alpha$. By the surjectivity above, choose $(\psi_1, \psi_2) \in \mathcal{E}^p(U) \oplus \mathcal{E}^p(V)$ so that

$$\psi_1 \;-\; \psi_2 \;=\; \varphi \qquad \text{on } \; U \cap V.$$

We define $\delta(\alpha)$ to be the deRham class of the differential form

$$\Psi \;\equiv\; \begin{cases} d\psi_1 & \text{on } U \\ d\psi_2 & \text{on } V \end{cases}$$

Note that $d\psi_1 - d\psi_2 = d\varphi = 0$ on $U \cap V$.

To Show: Prove that δ is well-defined, i.e., that it is independent of the choices involved.

Due Tues, April 29th: Do Exercises 1, 3, 7a, 8, 9 in Chapter 18. When the use of other homework exercises is suggested in the problem, you can use those other exercises without doing them.

Extra Credit: Do problem 7b.

Due Tues, May 6th: Do Exercises 10(a)–(d) in Chapter 12, Exercise 9 in Chapter 14, Also do the following:

(1) Let $\omega \in \mathcal{E}^2(X)$ have the following two properties:

(i) $d\omega \;=\; 0$,

(ii) ω is non-degenerate, i.e., for $V \in T_xM$ (at any point x),

$$\omega(V, W) = 0 \;\text{ for all } W \in T_xM \quad \Rightarrow \quad V = 0.$$

Then we obtain an isomorphism of vector bundles

$$\Phi_\omega : TM \;\longrightarrow\; T^*M \qquad \text{given by } \; \Phi_\omega(V) = i_V\omega.$$

Let

$$\Phi_\omega^{-1} : T^*M \;\longrightarrow\; TM \qquad \text{denote the inverse.}$$

Suppose now that $f \in C^\infty(M)$ and let

$$V \;=\; \Phi_\omega^{-1}(df).$$

Show that: $\varphi_t^*(\omega) = \omega$ where φ_t is the local flow generated by V,

(2) Suppose A is an associative algebra. By a *derivation* of A we mean a linear map $L : A \to A$ such that $L(a_1 a_2) = L(a_1)a_2 + a_1 L(a_2)$ for all $a_1, a_2 \in A$. Show that if L and $\widetilde{L}$ are derivations of A, then so is $[L, \widetilde{L}] \equiv L\widetilde{L} - \widetilde{L}L$. Conclude that for $V, W \in \mathcal{V}(M)$ and $\varphi \in \mathcal{E}^p(M)$, that

$$\{\mathcal{L}_V \mathcal{L}_W - \mathcal{L}_W \mathcal{L}_V\}\,\varphi \;=\; \mathcal{L}_{[V,W]}\varphi$$

(3) Suppose

$$A^* = \bigoplus_{k=1}^{\infty} A^k \qquad \text{with} \quad A^k \cdot A^\ell \;\subset\; A^{k+\ell}$$

is an associative algebra. By a *skew derivation* of A we mean a linear map $L : A^* \to A^*$ such that $L(a_1 a_2) = L(a_1)a_2 + (-1)^k a_1 L(a_2)$ for all $a_1 \in A^k, a_2 \in A$. We say that $L : A^* \to A^*$ has *degree* d if $L(A^k) \subset A^{k+d}$ for all k. Show that if L is a (non-skew) derivation of degree 0 and $\widetilde{L}$ is a skew derivation of degree d, then $[L, \widetilde{L}]$ is a skew derivation of degree d.

Conclude that for $V, W \in \mathcal{V}(M)$ the operators $\mathcal{L}_V$ and i_W on $\mathcal{E}^*(M)$ satisfy

$$[\mathcal{L}_V, i_W] \;=\; i_{[V,W]}.$$

(Hint: Show that both sides are skew derivations of degree -1 on $\mathcal{E}^*(M)$, and they agree on 1-forms.)