Math 531: Topology/Geometry II

2:20-3:40p, Spring 2008
Department of Mathematics
SUNY at Stony Brook

This course will offer an introduction to the elementary theory of smooth manifolds. Topics will include: vector fields, differential forms, vector bundles, Lie derivatives, Stokes' theorem, and de Rham cohomology.

Instructor: Dr. Corbett Redden
Math Tower 3-114. Phone: 632-8261. email: *redden at math dot sunysb dot edu*
Office Hours: M 4-5p, Tu 10a-12p, or drop-in, or by appointment.

Grader: Yakov Savelyev
e-mail: *yasha at math dot sunysb dot edu*

Homework: Working homework problems is the only way to really learn the material. While you are encouraged to work with others, you must write up all solutions on your own. Homework sets will usually be collected in class on Thursdays.
Due 2/7: 2. 6, 8, 18
Due 2/14: 2. 26, 28, 32, 33
Due 2/21: §3: 5, 7, 10, 23
Due 2/28: §3: 20, 24, 26, 30
Due 3/6: §4: 1, 2, 6, 8
Due 3/13: §5: 2, 6, 12, 14ab
Recommended by 3/22: Review/Practice Exam (and solutions)
Due 3/28: Midterm Exam (and solutions)
Due 4/3: §6: 5. §7: 2, 18, 20, 27
Due 4/10: §8: 2, 3, 13, 14
Due 4/17: §8: 17, 23, 32
Due 4/24: Short HW
Due 5/1: §11: 2, 3
Due 5/13: Ch 11 #11, 13

Midterm: Take Home midterm due Friday, March 28. Review/Practice HW, Review Solutions, MIDTERM EXAM. Midterm Solutions

Final Exam: Thursday, May 15. 2:00p.m. Physics P129. Final Exam.

Textbook:: *A Comprehensive Introduction to Differential Geometry, Volume 1* (3rd edition), by Michael Spivak. Publish or Perish Inc., 1999.

Course Grade: Homework: 50%, Midterm: 20%, Final: 30%

Disabilities: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.www.ehs.stonybrook.edu/fire/disabilities.shtml

Math 531 Midterm Review

The midterm will be a timed, take-home exam (taken on the honor system). You will have 3 hours to complete the exam. You may use your notes and Spivak, but you may not obtain outside help from anyone. The problems will be a mix of abstract theorems and concrete calculations. There will be multiple problems, and you will be able to choose which ones you do (e.g. work 2 out of 3 problems). Here are a few good problems to practice with:

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1.$$

For which points $p = (0, 0), p = (\frac{1}{3}, \frac{1}{3}), p = (\frac{-1}{3}, \frac{-1}{3})$, is $f^{-1}(f(p))$ an embedded submanifold in $\mathbb{R}^2$?

2. Let M be a compact manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be a smooth map. Does f have to have a critical point? In other words, must there exist a point $p \in M$ such that f_{*p} is not injective?

3. Let S^2 be the 2-sphere. Let $U_N = S^2 - \{N\}, U_S = S^2 - \{S\}$ be the open sets obtained by removing the "North Pole" and the "South Pole," respectively. On both U_N and U_S there exist standard stereographic projections to $\mathbb{R}^2$ (dealt with in a previous homework). These coordinate charts give a trivialization of the tangent bundle over each open set. QUESTION: Compute the transition functions for the tangent bundle on the overlap. i.e. if ϕ_N, ϕ_S are the local trivializations of TS^2 induced by the two stereographic projections ($\phi_N : TU_N \to U_N \times \mathbb{R}^2$, $\phi_S : TU_S \to U_S \times \mathbb{R}^2$), calculate $\phi_S \circ \phi_N^{-1}$.

4. Is the Klein bottle orientable? Justify your answer.

5. On $\mathbb{R}^3$ with standard coordinates, consider the vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$
$$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Calculate the Lie bracket on the vector fields X, Y, Z. What sort of structure does this look like?

Math 531 Midterm Review Solutions

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1.$$

For which points $p = (0, 0), p = (\frac{1}{3}, \frac{1}{3}), p = (\frac{-1}{3}, \frac{-1}{3})$, is $f^{-1}(f(p))$ an embedded submanifold in $\mathbb{R}^2$?

The answer here is quite subtle. The points $p = (1/3, 1/3)$ and $p = (-1/3, -1/3)$ give submanifolds, while $p = (0, 0)$ does not. However, $p = (-1/3, -1/3)$ does not follow from the Implicit Function Theorem.

Let us first do a few calculations:

$$\begin{aligned} f(x, y) &= x^3 + xy + y^3 + 1 \\ f(0, 0) &= 1 \\ f(1/3, 1/3) &= 32/27 \\ f(-1/3, -1/3) &= 28/27 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 0 & 0 \end{bmatrix} = Df &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 3x^2 + y & x + 3y^2 \end{bmatrix} \end{aligned}$$

The two points where Df has rank 0 (where Df does not have full rank) occur at $(0, 0)$ and $(-1/3, -1/3)$. Therefore, since f has full rank at all points $M = f^{-1}(32/37)$, we know that M is a codimension 1 submanifold of $\mathbb{R}^2$.

However, we do not yet know whether the other two sets are submanifolds or not. The above process merely guarantees a submanifold structure in certain situations; it does eliminate the possibility. In general, there is no standard method for showing that something is not a manifold, though usually you assume you have a submanifold and try to obtain some sort of contradiction. We investigate the two unknown situations from above more carefully.

First, let us consider, $f^{-1}f(-1/3, -1/3)$. Some sneaky algebra (it helps to use a computer) shows us that

$$\begin{aligned} x^3 + xy + y^3 + 1 &= 28/27 \\ x^3 + xy + y^3 - 1/27 &= 0 \\ \frac{1}{27}(3x + 3y - 1)(9x^2 + 3x - 9xy + 3y + 9y^2 + 1) &= 0 \end{aligned}$$

We know that $3x + 3y - 1 = 0$ is a submanifold of $\mathbb{R}^2$. Further investigation shows that the second factor

$$9x^2 + 3x - 9xy + 3y + 9y^2 + 1 \geq 0.$$

The above follows from a bit of Calc 3. We can show that this quadratic function has a critical point at $(-1/3, -1/3)$, $f(-1/3, 1/3) = 0$, and the Hessian is positive definite (f is "concave up" in every direction). Therefore,

$$\begin{aligned} f^{-1}f(-1/3,-1/3) &= \{(x,y)|3x+3y-1=0\} \cup (-1/3,-1/3) \\ &= \{(x,y)|3x+3y-1=0.\} \end{aligned}$$

So, this particular subset is a submanifold, though you cannot simply use the Implicit Function Theorem.

On the other hand, $f^{-1}f(0,0)$ is not a manifold because of the point $(0,0)$. Seeing this is a little trickier, and I won't expect you to do anything like this on the exam. I'll try to write up a clean solution to show this is not, but this is not the important part of the problem. It is, though, good to see that these things can happen.

2. Let M be a compact manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be a smooth map. Then f must have at least one critical point.

Proof. Suppose there are no critical points. Then, by the Inverse Function Theorem, for every $p \in M$, there exists an open neighborhood U of p such that $f_{|U}$ is a diffeomorphism. This implies that f is an open map (it takes open sets to open sets), since these open neighborhoods form a basis for the open sets in M. Therefore, $f(M) \subset \mathbb{R}^n$ is open subset. However, M is compact, and therefore the image $f(M) \subset \mathbb{R}^n$ is also compact. We have now reached a contradiction, as there are no open compact subsets of $\mathbb{R}^n$ (other than the empty set). The map f must have at least one critical point. $\square$

3. Let S^2 be the 2-sphere. Let $U_N = S^2 - \{N\}, U_S = S^2 - \{S\}$ be the open sets obtained by removing the "North Pole" and the "South Pole," respectively. On both U_N and U_S there exist standard stereographic projections to $\mathbb{R}^2$ (dealt with in a previous homework). These coordinate charts give a trivialization of the tangent bundle over each open set. QUESTION: Compute the transition functions for the tangent bundle on the overlap. i.e. if ϕ_N, ϕ_S are the local trivializations of TS^2 induced by the two stereographic projections, calculate $\phi_S \circ \phi_N^{-1}$.

First, let us set up the following notation:

$$\begin{aligned} S^2 &= \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\} \\ U &= U_N, V = U_S \\ (u^1,u^2) &= (\frac{x}{1-z}, \frac{y}{1-z}) = u(x,y,z) \\ (v^1,v^2) &= (\frac{x}{1+z}, \frac{y}{1+z}) = v(x,y,z) \end{aligned}$$

Here, u and v are the standard coordinates induced from the stereographic projection, and on overlap $U \cap V$, we have functions $u \circ v^{-1}$ and $v \circ u^{-1}$ given by the relations

$$v^i = \frac{u^i}{(u^1)^2 + (u^2)^2}$$
$$u^i = \frac{v^i}{(v^1)^2 + (v^2)^2}$$

The two trivializations u, v both give rise to a trivialization of the tangent bundle over U and V, respectively. Explicitly, this is an equivalence of bundles ϕ_U

$$U \times \mathbb{R}^2 \xleftarrow{\phi_U} TU$$
$$(u, (a^1, a^2)) \longmapsto a^1 \frac{\partial}{\partial u^1} + a^2 \frac{\partial}{\partial u^2}$$

along with the similarly defined equivalence ϕ_V. The transition function $\phi_V \circ \phi_U^{-1}$ is then a bundle equivalence

$$\phi_V \circ \phi_U^{-1} : (U \cap V) \times \mathbb{R}^2 \to (U \cap V) \times \mathbb{R}^2.$$

Since $\phi_V \circ \phi_U^{-1}$ is a bundle map, we can consider this as a smooth map

$$\phi_V \circ \phi_U^{-1} : U \cap V \to Gl(2, \mathbb{R}).$$

Remembering that the same vector X can be expressed as

$$X = a^i \frac{\partial}{\partial u^i} = b^i \frac{\partial}{\partial v^i},$$

we obtain the standard relationship

$$b^j = \frac{\partial v^j}{\partial u^i} a^i.$$

(Remark: this can be seen directly from the chain rule, which states

$$\frac{\partial}{\partial v^j} = \frac{\partial u^i}{\partial v^j} \frac{\partial}{\partial u^i}. \quad)$$

Now, we see that the transition function $\phi_V \circ \phi_U^{-1}$, which maps

$$(a^1, a^2) \overset{\phi_V \circ \phi_U^{-1}}{\longmapsto} (b^1, b^2),$$

is given by the linear operators (matrices)

$$\phi_V \circ \phi_U^{-1}(p) = \left[\frac{\partial v^i}{\partial u^j} \right]_{|p} = D(v \circ u^{-1})_p.$$

A simple calculation now shows us that

$$
\begin{aligned}
\frac{\partial v^i}{\partial u^j} &= \frac{(\sum_i (u^i)^2)\delta^i_j - 2u^i u^j}{(\sum_i (u^i)^2)^2} \\
&= \left(\frac{1}{\sum_i (u^i)^2}\right)^2 \begin{bmatrix} -(u^1)^2 + (u^2)^2 & -2u^1u^2 \\ -2u^1u^2 & (u^1)^2 - (u^2)^2 \end{bmatrix} \\
&= \left(\frac{1}{1+z}\right)^2 \begin{bmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{bmatrix}
\end{aligned}
$$

So, we have the transition function $g_{VU} = \phi_V \circ \phi_U^{-1} : (U \cap V) \to Gl(2, \mathbb{R})$ is given by

$$
g_{VU} = \left(\frac{1}{\sum_i (u^i)^2}\right)^2 \begin{bmatrix} -(u^1)^2 + (u^2)^2 & -2u^1u^2 \\ -2u^1u^2 & (u^1)^2 - (u^2)^2 \end{bmatrix} = \left(\frac{1}{1+z}\right)^2 \begin{bmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{bmatrix},
$$

in u coordinates and (x, y, z) coordinates, respectively (one may be easier to use than the other). We can also compute $g_{UV} = g_{VU}^{-1} = \phi_U \circ \phi_V^{-1}$. This can be done by performing the above calculation for $\dfrac{\partial u^i}{\partial v^j}$, or by taking the inverse of the matrix g_{VU} above. Either way, we end up with

$$
g_{UV} = \left(\frac{1}{\sum_i (v^i)^2}\right)^2 \begin{bmatrix} -(v^1)^2 + (v^2)^2 & -2v^1v^2 \\ -2v^1v^2 & (v^1)^2 - (v^2)^2 \end{bmatrix} = \left(\frac{1}{1-z}\right)^2 \begin{bmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{bmatrix}
$$

4. The Klein bottle is not orientable.

Proof. The Klein bottle contains the open Mobius band as a submanifold. We know that if a manifold M contains a non-orientable submanifold, then the manifold M is not orientable.

Alternatively, we can think of the Klein bottle M as a quotient of the cylinder. In other words, $(\mathbb{R} \times S^1)/\mathbb{Z}$, where $\mathbb{Z}$ acts by the group action

$$(x, \theta) \sim (x + m, (-1)^m \theta), \quad m \in \mathbb{Z}.$$

The quotient map $\pi : \mathbb{R} \times S^1 \to M$ induces a map on the tangent bundles

$$T(\mathbb{R} \times S^1) \xrightarrow{\pi_*} TM,$$

and for any $m \in \mathbb{Z}$, we have the following commutative diagram

$$
\begin{tikzcd}
T(\mathbb{R} \times S^1) \arrow[rr, "g_*"] \arrow[dr, "\pi_*"'] & & T(\mathbb{R} \times S^1) \arrow[dl, "\pi_*"] \\
& TM &
\end{tikzcd}
$$

Now, we notice that the map

$$(x, \theta) \mapsto (x+1, -\theta),$$

which is the action of $1 \in \mathbb{Z}$, reverses the orientation of $\mathbb{R} \times S^1$. (This is an easy calculation.) Therefore, by the lemma below, M is not orientable. $\square$

Lemma 1. *Let X is a connected orientable manifold with map $g : X \to X$, where g is orientation-reversing. If there exists a local diffeomorphism $\pi : X \to M$ such that the following diagram commutes*

$$\begin{tikzcd} TX \arrow[rr, "g_*"] \arrow[dr, "\pi_*"'] & & TX \arrow[dl, "\pi_*"] \\ & TM & \end{tikzcd}$$

then M is not orientable.

In particular, if a discrete group G acts on X and there is some $g \in G$ that reverses the orientation, then X/G is non-orientable.

Proof. First, notice that if M is orientable, and X and M are both oriented, then π_* must either preserve orientation at every point, or it must reverse orientation at every point.

Now, let $x \in X$, $p = \pi(x)$, and choose an orientation on X. Notice that

$$\pi_{*|x} : T_x X \to T_p M$$

either preserves or reverses orientation. However,

$$(\pi_* \circ g_*)_{|x} : T_x X \to T_p M$$

does the opposite of π_{*x}. By the commutativity of the diagram, $\pi_{*|x} = (\pi_* \circ g_*)_{|x}$, and thus M cannot admit an orientation. $\square$

5. On $\mathbb{R}^3$ with standard coordinates, consider the vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$
$$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Then,

$$[X, Y] = Z, [Y, Z] = X, [Z, X] = Y.$$

This is a straight-forward calculation using the definition of the Lie bracket.

$$\begin{aligned}[X,Y] &= \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right) - \left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \\ &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = Z\end{aligned}$$

Notice that, since we know the answer will be vector field, we can ignore terms with higher-order derivatives. Of course, if you just write everything out, you will see that these cancel out. Similar calculations show $[Z,X] = Y, [Y,Z] = X$.

One might notice that this structure looks a bit like the cross-product structure on $\mathbb{R}^3$ (replacing X,Y,Z with i,j,k). In fact, this can be made explicit. The 3-dimensional vector space spanned by the vector fields $\{X,Y,Z\}$, together with the Lie bracket, form a Lie algebra. Likewise, $\mathbb{R}^3$ with the cross-product structure also forms a Lie algebra. These two Lie algebras are isomorphic under the map

$$aX + bY + cZ \longmapsto a\widehat{i} + b\widehat{j} + c\widehat{k}.$$

The above map is linear and invertible, and the two brackets commute with each other (check this).

Furthermore, it is not too difficult to show that the flow generated by $aX + bY + cZ$ is actually a 1-parameter subgroup of $SO(3,\mathbb{R})$. This can be seen through the work done in homework problem Ch. 5, number 6, as well as a bit of what we talked about in class concerning matrix groups.

Math 531 Midterm

Please abide by the following rules:

- Time Limit: 3 hours
- You are allowed 1 untimed break
- This test is open notes and open Spivak. You may not consult any additional sources (or people)
- You are to answer 2 of the first 3 questions and 1 of the last 2 questions.
- Exam due Friday, March 28 (by end of day).

The test is not meant to be confusing. Any extra information contained in the questions is designed to clarify and not obscure, and the questions are not designed to trick you. Please let me know if anything is unclear.

Math 531 Midterm

Solve any 2 of the following 3 problems:

1. In multivariable calculus, given a function $f : \mathbb{R}^n \to \mathbb{R}$, you define the gradient vector ∇f by

$$\nabla f = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

In other words, it is vector field with component functions $\dfrac{\partial f}{\partial x^i}$.

Consider the more general situation where M is a smooth manifold, and $f : M \to \mathbb{R}$ a smooth function.

a. Show that the functions $\dfrac{\partial f}{\partial x^i}$ (where x is a local coordinate system) do not form the component functions of a vector field. Show, however, that they do naturally form the component functions of a 1-form.

b. Suppose we have a tensor $g \in C^\infty(M, TM^* \otimes TM^*)$, and further suppose that at each point $x \in M$, the tensor g_x is symmetric and non-degenerate. In other words,

$$g_x(v_1, v_2) = g_x(v_2, v_1),$$

and the rewritten

$$g_x : T_x M^{**} \to T_x M^*$$

is invertible. Then, **show** that g can be used to change the components $\frac{\partial f}{\partial x^i}$ into components of a vector field.

2. Consider the polar coordinates on (subsets of) $\mathbb{R}^2$ defined by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Compute $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in polar coordinates .

3. In an ODE class, you often encounter the following spring equation:

$$\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt},$$

where $x(t)$ is the displacement of a mass at time t, and k, c are constants related to the strength of the spring force and resistance force, respectively.

a. Let $y = \dfrac{dx}{dt}$, and rewrite the above equation in terms of a vector field on $M = \mathbb{R}^2$.

b. Show that flow induced by the vector field in (a) is a 1-parameter subgroup of $Gl(2, \mathbb{R})$.

Solve 1 of the following 2 problems:

4. Implicit differentiation is a useful tool in calculus. Essentially, if we have

$$f(x, y) = c,$$

then we solve for $\dfrac{dy}{dx}$ by implicitly differentiating and obtaining

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Prove that this is mathematically rigorous.

5. Suppose M and N are connected, oriented manifolds and $f : M \to N$ is a local diffeomorphism. Show that if f_{*p} preserves orientation at some $p \in M$, then f_* preserves orientation at all points $p \in M$.

Math 531 Midterm Solutions

1. In multivariable calculus, given a function $f : \mathbb{R}^n \to \mathbb{R}$, you define the gradient vector ∇f by

$$\nabla f = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

In other words, it is vector field with component functions $\dfrac{\partial f}{\partial x^i}$.

Consider the more general situation where M is a smooth manifold, and $f : M \to \mathbb{R}$ a smooth function.

a. Show that the functions $\dfrac{\partial f}{\partial x^i}$ (where x is a local coordinate system) do not form the component functions of a vector field. Show, however, that they do naturally form the component functions of a 1-form.

b. Suppose we have a tensor $g \in C^\infty(M, TM^* \otimes TM^*)$, and further suppose that at each point $x \in M$, the tensor g_x is symmetric and non-degenerate. In other words,

$$g_x(v_1, v_2) = g_x(v_2, v_1),$$

and the rewritten

$$g_x : T_x M^{**} \to T_x M^*$$

is invertible. Then, **show** that g can be used to change the components $\frac{\partial f}{\partial x^i}$ into components of a vector field.

Solution:

Notice that we can define $df \in C^\infty(M, TM^*)$ in a coordinate-free way. That is, $df(X) = X(f)$, where X is a vector field. Then, a simple calculation shows that

$$df = \frac{\partial f}{\partial x^i} dx^i$$

in local coordinates. This implies $\frac{\partial f}{\partial x^i}$ are the components of a (non-trivial) 1-form, and hence not a vector field.

More explicitly, suppose a^i, b^i are the component functions of a vector field in x, y coordinates, respectively. Then these are related by the equation

$$b^j = a^i \frac{\partial y^j}{\partial x^i}.$$

However, we see that

$$\frac{\partial f}{\partial y^j} = \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial y^j}.$$

This does not satisfy the transformation property for components of vector fields, but instead satisfies the transformation property for differential forms.

In fact, we can see that that

$$\frac{\partial f}{\partial x^i} dx^i = \left(\frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^i} \right) \left(\frac{\partial x^i}{\partial y^j} dy^j \right) = \frac{\partial f}{\partial y^j} dy^j,$$

whereas

$$\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} = \frac{\partial f}{\partial y^j}\frac{\partial}{\partial y^j}\left(\frac{\partial x^i}{\partial y^j}\right)^2 \neq \frac{\partial f}{\partial y^j}\frac{\partial}{\partial y^j}.$$

Therefore, $\frac{\partial f}{\partial x^i}dx^i$ is a well-defined 1-form on M, but not a well-defined vector field.

Now, suppose we have the tensor g. Such a tensor g is a non-degenerate inner product on each tangent space and is called a metric (if g is positive-definite, then it is a Riemannian metric). (Remark: we don't need the symmetric property here, but it makes indices a little easier to deal with.)

If $g \in C^\infty(M, TM^* \otimes TM^*)$, then we can equivalently consider

$$g \in C^\infty(M, \mathrm{Hom}(TM, TM^*))$$

(using the generic natural equivalences $\mathrm{Hom}(V, W) \cong V^* \otimes W$ and $V^{**} \cong V$). The non-degeneracy of g implies that we have

$$g^{-1} \in C^\infty(M, \mathrm{Hom}(TM^*, TM)) \cong C^\infty(M, TM \otimes TM).$$

Therefore, we can construct $\nabla f = g^{-1}(df)$ by the following:

$$\begin{array}{ccccc} C^\infty(M) & \xrightarrow{d} & C^\infty(M, TM^*) & \xrightarrow{g^{-1}} & C^\infty(M, TM) \\ f & \longmapsto & df & \longmapsto & g^{-1}(df) = \nabla f \end{array}$$

For fun, let's write this in local coordinates. We know that

$$g = g_{ij}dx^i \otimes dx^j.$$

Since g is non-degenerate, the matrix g_{ij} is invertible. Hence, we also have the tensor

$$g^{-1} = g^{ji}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

We now see that we can use g^{-1} to transform the above df into a vector field. We define

$$\begin{aligned} \nabla f = g^{-1}(df) &= g^{kj}\frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}\left(\frac{\partial f}{\partial x^i}dx^i\right) \\ &= g^{kj}\frac{\partial}{\partial x^j}\delta^i_k\frac{\partial f}{\partial x^i} \\ &= g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} \end{aligned}$$

It is simple to check that ∇f is now a well-defined vector field. Suppose that

$$g^{-1} = g^{\beta\alpha}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}.$$

Then, we see that

$$\begin{aligned} g^{\alpha\beta}\frac{\partial f}{\partial y^\alpha}\frac{\partial}{\partial y^\beta} &= \left(g^{ij}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\right)\left(\frac{\partial f}{\partial x^k}\frac{\partial x^k}{\partial y^\alpha}\right)\left(\frac{\partial x^l}{\partial y^\beta}\frac{\partial}{\partial x^l}\right) \\ &= g^{ij}\delta_i^k\delta_j^l\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^l} \\ &= g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} \end{aligned}$$

Therefore, we see that ∇f is a well-defined vector field.

The above demonstrates the general principle that if some quantity is written in local coordinates, each index should appear "up" and "down" an equal number of times.

In summary, we see that we can use the tensor g to transform the functions $\frac{\partial f}{\partial x^i}$ into components of a vector field. In the situation where $M = \mathbb{R}^n$ with standard coordinate x, and $g_{ij} = \delta_{ij}$ is the usual Euclidean metric, we see that

$$\nabla f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \delta^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}.$$

2. Consider the polar coordinates on (subsets of) $\mathbb{R}^2$ defined by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Compute $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in polar coordinates .

Solution:
We know that

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \end{aligned}$$

Letting u and p denote the Euclidean and polar coordinate systems, respectively, we know that

$$D(p\circ u^{-1}) = D((u\circ p^{-1})^{-1}) = [D(u\circ p^{-1})]^{-1}].$$

As easy calculation shows us that

$$D(u\circ p^{-1}) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

Therefore,

$$\begin{aligned} D(p\circ u^{-1}) &= \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \end{aligned}$$

Therefore, we see that

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}$$

3. In an ODE class, you often encounter the following spring equation:

$$\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt},$$

where $x(t)$ is the displacement of a mass at time t, and k, c are constants related to the strength of the spring force and resistance force, respectively.

a. Let $y = \dfrac{dx}{dt}$, and rewrite the above equation in terms of a vector field on $M = \mathbb{R}^2$.

b. Show that flow induced by the vector field in (a) is a 1-parameter subgroup of $Gl(2, \mathbb{R})$.

Solution:

First, let $y = \frac{dx}{dt}$. The 2nd-order differential equation now becomes a system of first-order equations

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -kx - cy \end{cases}$$

Remembering that we are solving for functions $(x(t), y(t))$, we think about solutions to the above equation as paths $\gamma = (\gamma^x, \gamma^y) : (-\epsilon, \epsilon) \to \mathbb{R}^2$. The equation is then equivalent to

$$\begin{cases} \frac{d\gamma^x}{dt}\frac{\partial}{\partial x} = y\frac{\partial}{\partial x} \\ \frac{d\gamma^y}{dt}\frac{\partial}{\partial y} = (-kx - cy)\frac{\partial}{\partial y} \end{cases}$$

In other words, we want

$$\gamma_*(\frac{\partial}{\partial t}) = \frac{d\gamma}{dt} = \frac{d\gamma^x}{dt}\frac{\partial}{\partial x} + \frac{d\gamma^y}{dt}\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} + (-kx - cy)\frac{\partial}{\partial y} = X_{\gamma(t)}.$$

Solutions to the system of ODEs are equivalent to integral curves to the vector field

$$X = y\frac{\partial}{\partial x} + (-kx - cy)\frac{\partial}{\partial y}.$$

The above equation is system of linear equations with constant coefficients. In homework problem number...., we saw that if this is rewritten as

$$\frac{d\gamma}{dt} = A\gamma,$$

where A has constant coefficients, then the integral curve to X with initial point $\gamma(0) = \gamma_0$ is given by

$$\gamma(t) = e^{At}\gamma_0.$$

Therefore, we have that (in standard components)

$$\gamma'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Therefore, the integral curves, with $\gamma(0) = (x_0, y_0)$, will be of the form

$$\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} t\right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is then given by

$$\phi_t = e^{At} \in Gl(2, \mathbb{R}).$$

Furthermore, ϕ is a homomorphism $\mathbb{R} \to Gl(2, \mathbb{R})$ (where $\mathbb{R}$ is the additive group), showing that the flow is a 1-parameter subgroup of $Gl(2, \mathbb{R})$. Equivalently, we can explicitly see that

$$\phi_0 = e^{0A} = I, \quad \phi_t \circ \phi_s = e^{A(s+t)}.$$

Solve 1 of the following 2 problems:
4. Implicit differentiation is a useful tool in calculus. Essentially, if we have

$$f(x,y) = c,$$

then we solve for $\dfrac{dy}{dx}$ by implicitly differentiating and obtaining

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Prove that this is mathematically rigorous.

Solution: Suppose we wish to calculate $\frac{dy}{dx}$ at a point (x_0, y_0), and assume that $\frac{\partial f}{\partial y} \neq 0$ at (x_0, y_0). Then, by the implicit function theorem, $f(x,y) = c$ implicitly defines y as a function of x. In other words, in a neighborhood of (x_0, y_0), there exists $g(x)$ (=$y(x)$)such that

$$f(x, g(x)) = c.$$

Hence, we calculate

$$\begin{aligned} \frac{d}{dx}|_{x_0} \left(f(x, g(x))\right) &= \frac{d}{dx}|_{x_0} \left(c\right) \\ D_1 \left(f(x_0, g(x_0))\right) \frac{dx}{dx}|_{x_0} + D_2 \left(f(x_0, g(x_0))\right) \frac{dg}{dx}|_{x_0} &= 0 \end{aligned}$$

Then, using the shorthand of $y(x) = g(x)$, we obtain the standard expression

$$\begin{aligned} \frac{\partial f}{\partial x}|_{(x_0,y_0)} + \frac{\partial f}{\partial y}|_{(x_0,y_0)} \frac{dy}{dx}|_{(x_0,y_0)} &= 0 \\ -\frac{\partial f/\partial x}{\partial f/\partial y}|_{(x_0, g(x_0)} = \frac{dy}{dx}|_{(x_0,y_0)} &= \frac{dg}{dx}|_{x_0} \end{aligned}$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Notice that since wish to divide by $\frac{\partial f}{\partial y}$, we would have wanted to require $\frac{\partial f}{\partial y}$ to be non-zero.

5. Suppose M and N are connected, oriented manifolds and $f : M \to N$ is a local diffeomorphism. Show that if f_{*p} preserves orientation at some $p \in M$, then f_* preserves orientation at all points $p \in M$.

Solution: First, f is a local diffeomorphism implies that f_* is an isomorphism at every point. An isomorphism of vector spaces gives a bijection between the set of orientations on each vector space. Hence, it makes sense to talk about f_* preserving or switching orientation at each point.

The space of orientations at each point $p \in M$ is a discrete space. In fact, the space of orientations naturally forms a double cover, which we shall call $Or(M)$. That is, there is a topological space $Or(M)$ with a free $\mathbb{Z}/2$ action such that $Or(M)/\mathbb{Z}/2 = M$. If M, N are oriented, then the covers

$Or(M)$ and $Or(N)$ are disconnected. The choice of orientations M and N give us a choice of connected components in $Or(M), Or(N)$. We therefore have a continuous map

$$f_* : Or(M) \to Or(N).$$

Because of continuity, f_* cannot map a connected component to two different connected components. Therefore, f_* will preserve orientation at all points.

Alternatively, we can let $x_0 \in M$ be an arbitrary point. Assume $f_* : T_{x_0}M \to T_{f(x_0)}N$ preserves orientation. Then, let $x \in M$ be any other point. Since M is connected (and connectedness is equivalent to path connectedness for manifolds), we can choose some path

$$\begin{aligned} \gamma : [0,1] &\to M \\ 0 &\mapsto x_0 \\ 1 &\mapsto x \end{aligned}$$

connecting x_0 and x. The tangent spaces of M and N are continuously oriented, and f is a smooth map. This implies that if f_{*p} preserves orientation, then f_* preserves orientation in some neighborhood of p. We then take open cover containing the image of γ and see that f_x must preserve orientation.

HW Due Thursday 4/24

1. Let G be a Lie group. A form $\omega \in \Omega^k(G)$ is (left) invariant if

$$L_g^*\omega = \omega \quad \forall g \in G.$$

(Above, L_g is left-multiplication by g.)

a. Show that the restriction of

$$0 \to C^\infty(G) \xrightarrow{d} \Omega^1(G) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(G) \to 0$$

to the subspaces of left-invariant forms is also a cochain complex. In other words, show that the exterior derivative of an invariant form is also invariant.

b. If $\mathfrak{g}$ is the Lie algebra of left-invariant vector fields, show that the cochain complex of left-invariant forms is the finite-dimensional complex

$$0 \to \mathbb{R} \xrightarrow{d} \mathfrak{g}^* \xrightarrow{d} \Lambda^2\mathfrak{g}^* \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n\mathfrak{g}^* \to 0$$

Use the invariant description of d to give a description of d above using only the Lie algebra structure.

c. Consider the cohomology $H^*(\mathfrak{g}, d)$ of the finite-dimensional complex from part (b) (this is called Lie algebra cohomology). Prove that

$$H^1(\mathfrak{g}) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*.$$

What is $H^1(\mathfrak{gl}(n, \mathbb{R})$, where $\mathfrak{gl}(n, \mathbb{R})$ denotes the Lie algebra of all $n \times n$ real matrices?

Math 531 Final
May 15, 2008

Solve any 2 of the following 3 problems:

1. Show directly that $\mathbb{R}P^n$ is a smooth manifold by giving local coordinate patches and calculating the transition functions (Hint: use homogeneous coordinates).

2. Let $M = \mathbb{R}^3$, and consider the 2-dimensional sub-bundle of $T\mathbb{R}^3$ spanned by the vector fields

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + f(x,y)\frac{\partial}{\partial z} \text{ and } \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + g(x,y)\frac{\partial}{\partial z}.$$

For what functions f, g will this be an integrable distribution?

3. Let $M = \mathbb{R}^2$. Apply Stokes Theorem to integrals of 1-forms (in local coordinates) to obtain two standard results from multi-variable calculus: The Fundamental Theorem of Line Integrals and Green's Theorem. (You don't need to remember what these theorems are to answer this question.)

Solve any 3 of the following 4 problems: (In the following, $H^k(M)$ denotes de Rham cohomology.)

4. Let M be a connected manifold. Show that $\pi_1(M) = 0$ implies that $H^1(M) = 0$. (Hint: Show that the integral of a closed 1-form over a closed 1-manifold is 0.)

5. Let M^n be a compact, connected, oriented n-manifold. Let D^n be a closed disc around a point $x_0 \in M$. Calculate the de Rham cohomology of $M^n - D^n$.

6. Let M, N be connected, compact, oriented n-manifolds. Prove that if $H^k(N) \neq 0$ and $H^k(M) = 0$ for some $k \neq 0$, then any map $f : M \to N$ must have degree 0. (Hint: Use Poincare duality.)

7. Let M be a compact, connected, oriented, n-manifold. What is the Euler class of the vector bundle $\Lambda^n TM^*$. Specifically, in what cohomology group does it live, and is it non-zero?

Solve the next problem:

8. Make up a good exam question and answer it.