Remus Radu

Institute for Mathematical Science
Stony Brook University

office: Math Tower 4-103
phone: (631) 632-8266
e-mail: remus.radu@stonybrook.edu

MAT 324: Real Analysis
Fall 2016
Schedule & Homework

Schedule

The PDF version of the schedule is available for print **here**.

Date	Topic	Reading	Assignments
Aug 30	Introduction: basic definitions and notations Null sets	**1, 2.1**	
Sept 1	The Cantor middle-thirds set Outer measure	**2.1, 2.2**	**HW1** (Due Sept 8) **Solutions**
Sept 6	*no class (Labor day)*		
Sept 8	σ-algebras Properties of measurable sets	**2.3, 2.4**	**HW2** (Due Sept 20) **Solutions**
Sept 13	Lebesgue measure & properties	**2.4, 2.5, 2.7**	
Sept 15	Borel sets Regularity of Lebesgue measure	**2.5, 2.7**	
Sept 20	Example of a non measurable set Lebesgue-measurable functions	**Appendix, 3.1**	**HW3** (Due Sept 29) **Solutions**
Sept 22	Measurable functions	**3.1, 3.2**	
Sept 27	Properties of measurable functions & examples	**3.3, 3.4, 3.6**	
Sept 29	The Lebesgue integral	**4.1, 4.2, 4.8**	**HW4** (Due Oct 6) **Solutions**
Oct 4	Integrable functions Modes of convergence; **Egorov's Theorem**	**4.2, 4.3, 8.1**	
Oct 6	Fatou's Lemma The Monotone Convergence Theorem	**4.2, 4.8**	**HW5** (Due Oct 20) **Solutions**
Oct 11	The Dominated Convergence Theorem	**4.3, 4.4**	
	Relations to the Riemann integral		

Oct 13	Applications of the convergence theorems	**4.4, 4.5**	
Oct 18	Approximation of measurable functions	**4.6, 4.8**	
Oct 20	Spaces of integrable functions: introduction	**5.1**	**HW6** (Due Nov 3) **Solutions**
Oct 25	**Midterm** (10:00-11:20am) Covers: 1, 2.1-2.5, 2.7, 3.1-3.4, 3.6, 4.1-4.6, 4.8 -- **Solutions** Practice **Midterm FA2014** & **Solutions**.		
Oct 27	The space L^1; completeness	**5.2**	
Nov 1	Inner product spaces The Hilbert space L^2	**5.2, 5.5**	
Nov 3	L^p spaces	**5.3, 5.5**	**HW7** (Due Nov 10) **Solutions**
Nov 8	L^p spaces & examples The L^∞ space	**5.3**	
Nov 10	Multi-dimensional Lebesgue measure	**6.1, 6.2** **Video**	**HW8** (Due Nov 22) **Solutions**
Nov 15	Construction of the product measure	**6.3**	
Nov 17	Fubini's Theorem & applications	**6.4, 6.6**	
Nov 22	Abstract measure theory: absolutely continuous measures, singular measures, examples	**7.1, 7.2**	**HW9** (Due Nov 29) **Solutions**
Nov 24	*no class (Thanksgiving)*		
Nov 29	The Radon-Nikodym Theorem I	**7.2, 7.5**	**HW10** (Due Dec 8) **Solutions**
Dec 1	The Radon-Nikodym Theorem II	**7.2, 7.3**	
Dec 6	Lebesgue-Stieltjes measures & FTC	**7.2**	
Dec 8	Lebesgue decomposition theorem Review	**7.3**	
Dec 16	**Final Exam** (11:15am-1:45pm) -- cumulative, covers everything Practice **Final FA2014** & **Solutions**.		

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Stony Brook University

e-mail: rradu@math.stonybrook.edu

About me

From 2013 to 2017 I was a Milnor Lecturer at the **Institute for Mathematical Sciences** at Stony Brook University. I got my Ph.D. in Mathematics from **Cornell University** in 2013, under the supervision of **John H. Hubbard**.

I started my undergraduate studies at the **University of Bucharest** and after one year I transfered to **Jacobs University Bremen**, where I earned my B.S. degree in Mathematics in 2007. I got a M.S. in Computer Science from Cornell University in 2012.

Research Interests

My interests are in the areas of Dynamical Systems (in one or several complex variables), Analysis, Topology and the interplay between these fields.

My research is focused on the study of complex Hénon maps, which are a special class of polynomial automorphisms of $\mathbb{C}^2$ with chaotic behavior. I am interested in understanding the global topology of the Julia sets J, J^- and J^+ of a complex Hénon map and the dynamics of maps with partially hyperbolic behavior such as holomorphic germs of diffeomorphisms of $(\mathbb{C}^n, 0)$ with semi-neutral fixed points. Some specific topics that I work on include: relative stability of semi-parabolic Hénon maps and connectivity of the Julia set J, regularity properties of the boundary of a Siegel disk of a semi-Siegel Hénon map, local structure of non-linearizable germs of diffeomorphisms of $(\mathbb{C}^n, 0)$.

Other activities

I was organizer for the **Dynamics Seminar** at Stony Brook University.
I have also developed projects for MEC (Math Explorer's Club): **Mathematics of Web Search** and **Billiards & Puzzles**.

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MAT 324: Real Analysis
Fall 2016
Course Information

Synopsis

In this course we will discuss Lebesgue measure, Lebesgue integration, metric spaces (including compactness, connectedness, completeness, and continuity), aspects of Fourier series, function spaces, Hilbert spaces and Banach spaces. After developing the basic theory we will also give some applications to Probability. The main results will be the Monotone Convergence Theorem, the Dominated Convergence Theorem, the Radon-Nykodyn Theorem, and the Central Limit Theorem.

Click here to download a copy of the course syllabus. Please visit the **course website on Blackboard** to see your grades.

Lectures

Tuesdays & Thursdays 10:00-11:20am in Melville Library W4535

Instructor

Remus Radu
Office: Math Tower 4-103
Office hours: Wednesday 12:00-2:00pm in Math Tower 4-103
Monday 3-4pm in MLC, or by appointment

Teaching Assistant

El Mehdi Ainasse
Office: Physics D-107
Office hours: Wednesday 4-5pm in Physics D-107
Monday 4-4:30pm, Wednesday 10-11am, and Friday 12-12:30pm in MLC

Textbook

Marek Capinski & Ekkehard Kopp, *Measure, Integral and Probability*, 2nd ed., Springer-Verlag, Springer Undergraduate Mathematics Series, ISBN 1-85233-781-8.

Other resources:

- G. Folland, *Real Analysis: Modern Techniques and their Applications (2nd ed.)*, John Wiley, 1999.
- E. Stein and R. Shakarchi, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, 2005.
- Terrence Tao, *An introduction to measure theory*, Graduate Studies in Mathematics, vol. 126, 2011.
  The book is based on the notes posted on his **blog** (begin reading with Prologue & Note 1 -- bottom of the webpage).

Grading Policy

Grades will be computed using the following scheme:

- Homework – 25%
- Midterm – 30%

- Final – 45%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading.

Exams

There will be a midterm and a final exam, scheduled as follows:

- Midterm – Thursday, October 25, 10:00-11:20am, in Library W4535.
- Final Exam – Friday, December 16, 11:15am-1:45pm, room TBA.

Schedule & Homework

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MAT324: Real Analysis – Fall 2016
Assignment 1

Due Thursday, **September 8**, in class.

Problem 1: Let $\mathcal{C}$ be the Cantor middle-thirds set constructed in the textbook. Show that $\mathcal{C}$ is compact, uncountable, and a null set.

Problem 2: Let A be the subset of $[0, 1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

Problem 3: Let A be a null set. Show that $m^*(A \cup B) = m^*(B)$ for any set B.

Problem 4: Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets. Show that for all $A \subseteq \mathbb{R}$, we have

$$m^* \left(A \cap \left(\bigcup_{j=1}^{n} E_j \right) \right) = \sum_{j=1}^{n} m^* \left(A \cap E_j \right).$$

MAT324: Real Analysis – Fall 2016
Assignment 1 – Solutions

Problem 1: Let $\mathcal{C}$ be the Cantor middle-thirds set constructed in the textbook. Show that $\mathcal{C}$ is compact, uncountable, and a null set.

Solution. The textbook proves that $\mathcal{C}$ is a null set (page 19). To check that $\mathcal{C}$ is compact, notice that it is bounded, $\mathcal{C} \subset [0,1]$, and each $\mathcal{C}_n$ constructed in the definition of $\mathcal{C}$ is closed, so that

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n$$

is a closed set. Hence, by the Heine-Borel Theorem, $\mathcal{C}$ is closed. To prove that $\mathcal{C}$ is uncountable, consider for each $x \in \mathcal{C}$ its infinite ternary expansion[1]

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}.$$

As shown in the textbook, since $x \in \mathcal{C}$, $a_k = 0$ or 2, for each $k \in \mathbb{N}$. Suppose there is an enumeration of the Cantor set, $\mathcal{C} = \{x_1, x_2, \cdots, \}$, where

$$x_1 = \sum_{k=1}^{\infty} \frac{a_{1k}}{3^k}$$
$$x_2 = \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k}$$
$$\vdots$$

Then $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ where, $a_k = |2 - a_{kk}|$ is not on the list, and belongs to the Cantor middle-third set, hence $\mathcal{C}$ is uncountable (check that the a_k's are not eventually zero!). □

Problem 2: Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

Solution. Let $x = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$ be the infinite decimal representation of x. By a similar argument given in the construction of the Cantor middle-third set, one can construct A by the following procedure:

1. Let $A_0 = [0,1]$.
2. Define A_1 by removing from A_0 the set $(\frac{4}{10}, \frac{5}{10})$, i.e., all the numbers x whose infinite decimal representation is such that $x_1 = 4$.

[1] Recall that in such an expansion, if $a_k = 0$, for all $k > N$, for some $N \in \mathbb{N}$ and $a_N \neq 0$, we replace the a_k by $\overline{a}_k = 2$, if $K > N$, and a_N by $\overline{a}_N = a_N - 1$.

3. Define A_2 by removing from A_1 the sets $(\frac{4+10k}{100}, \frac{5+10k}{100})$, where $0 \leq k \leq 9$ thus removing all the numbers left in A_1 such that $x_2 = 4$.

4. Assume A_n has been defined. Define A_{n+1} removing from A_n all the intervals of the form
$$\left(\frac{4+10^n k}{10^{n+1}}, \frac{5+10^n k}{10^{n+1}}\right),$$
for $0 \leq k \leq 10^k - 1$, thus removing from A_n all the numbers left in A_1 such that $x_{n+1} = 4$.

Now this description is not optimal, since some of the intervals have already been removed in the previous steps. In fact, in the n-th step we remove 9^{n-1} disjoint intervals, each of them with lenght 10^{-n}. In addition, $A = \cap_{n\in\mathbb{N}} A_n$ and
$$m(A_n) = 1 - \sum_{k=1}^{n} \frac{9^{n-1}}{10^n}$$
Since the A_n form a descreasing sequence (i.e., $A_n \supset A_{n-1}$),
$$m(A) = m(\cap A_n) = \lim_{n\to\infty} m(A_n) = 1 - \sum_{k=1}^{\infty} \frac{9^{n-1}}{10^n} = 0$$
$\square$

Problem 3: Let A be a null set. Show that $m^*(A \cup B) = m^*(B)$ for any set B.

Solution. By monotonicity,
$$m^*(B) \leq m^*(A \cup B).$$
By subadditivity,
$$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B).$$
$\square$

Problem 4: Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets. Show that for all $A \subseteq \mathbb{R}$, we have
$$m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right) = \sum_{j=1}^{n} m^*\left(A \cap E_j\right).$$
Solution. Notice that since the E_j's are measurable,
$$m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right) = m^*\left(\left[A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right] \cap E_n\right) + m^*\left(\left[A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right] \cap (E_n)^c\right)$$
Now since the E_j are disjoint,
$$m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right) = m^*\left(A \cap E_n\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{n-1} E_j\right)\right)$$
Therefore the result can be proved by induction on n. $\square$

MAT324: Real Analysis – Fall 2016
ASSIGNMENT 2

Due Tuesday, **September 20**, in class.

Problem 1: Suppose $E_1, E_2 \subseteq \mathbb{R}$ are measurable sets. Show that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Problem 2: Construct a *Cantor-like* closed set $\mathcal{C} \subset [0,1]$ so that at the k^{th} stage of the construction one removes 2^{k-1} centrally situated open intervals each of length ℓ_k, with

$$\ell_1 + 2\ell_2 + \ldots + 2^{k-1}\ell_k < 1.$$

Suppose ℓ_k are chosen small enough so that $\sum\limits_{k=1}^{\infty} 2^{k-1}\ell_k < 1$.

a) Show that $m(\mathcal{C}) = 1 - \sum\limits_{k=1}^{\infty} 2^{k-1}\ell_k$ and conclude that $m(\mathcal{C}) > 0$.

b) Give an example of a sequence $(\ell_k)_{k\geq 1}$ that verifies the hypothesis.

Problem 3: Let $E_1, E_2, \ldots, E_{2016} \subset [0,1]$ be measurable sets such that $\sum\limits_{k=1}^{2016} m(E_k) > 2015$. Show that $m\left(\bigcap\limits_{k=1}^{2016} E_k\right) > 0$.

Problem 4: Suppose $A \in \mathcal{M}$ and $m(A\Delta B) = 0$. Show that $B \in \mathcal{M}$ and $m(A) = m(B)$.

Problem 5: Suppose $A \subset E \subset B$ where A and B are measurable sets of finite measure. Show that if $m(A) = m(B)$, then E is measurable.

Problem 6: Suppose $E \in \mathcal{M}$ and $m(E) > 0$. Prove that there exists an open interval I such that

$$m(E \cap I) > 0.99 \cdot m(I).$$

Hint: Argue by contradiction, using the regularity of Lebesgue measure. See Theorems 2.17, 2.29.

MAT324: Real Analysis – Fall 2016
Assignment 2 – Solutions

Problem 1: Suppose $E_1, E_2 \subseteq \mathbb{R}$ are measurable sets. Show that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Solution. Notice that $E_1 \cup E_2$ can be expressed as a union of disjoint measurable sets

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_1).$$

Additivity implies that

$$\begin{aligned} m(E_1 \cup E_2) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2) + m(E_2 \setminus E_1) \\ &= m(E_1 \cap E_2) + m(E_1 \cap (E_2)^c) + m(E_2 \cap (E_1)^c) \end{aligned}$$

Hence

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) &= [m(E_1 \cap E_2) + m(E_1 \cap (E_2)^c)] + [m(E_2 \cap (E_1)^c) + m(E_1 \cap E_2)] \\ &= m(E_1) + m(E_2) \end{aligned}$$

□

Problem 2: Construct a *Cantor-like* closed set $\mathcal{C} \subset [0,1]$ so that at the k^{th} stage of the construction one removes 2^{k-1} centrally situated open intervals each of length ℓ_k, with

$$\ell_1 + 2\ell_2 + \ldots + 2^{k-1}\ell_k < 1.$$

Suppose ℓ_k are chosen small enough so that $\sum\limits_{k=1}^{\infty} 2^{k-1}\ell_k < 1$.

a) Show that $m(\mathcal{C}) = 1 - \sum\limits_{k=1}^{\infty} 2^{k-1}\ell_k$ and conclude that $m(\mathcal{C}) > 0$.

b) Give an example of a sequence $(\ell_k)_{k\geq 1}$ that verifies the hypothesis.

Solution.

a) The intervals removed are disjoint. If C_n denotes what is left in $[0,1]$ after the n-th process, then $m(C_n) = 1 - \sum_{k=1}^{n} 2^{k-1} l_k$. Furthermore, by construction we have $C_{n+1} \subset C_n$. Apply Theorem 2.19.

b) Let $l_k = 4^{-k} = 2^{-2k}$. Then

$$\sum_{k=1}^{\infty} 2^{k-1} 2^{-2k} = \sum_{k=1}^{\infty} 2^{-1-k} = \frac{1}{2}$$ $\square$

Problem 3: Let $E_1, E_2, \ldots, E_{2014} \subset [0,1]$ be measurable sets such that $\sum\limits_{k=1}^{2014} m(E_k) > 2013$. Show that $m\left(\bigcap\limits_{k=1}^{2014} E_k\right) > 0$.

Solution. Let $F_n = [0,1] \setminus E_n$, for each $1 \leq n \leq 2014$. Notice that

$$m\left(\bigcup_{n=1}^{2014} F_n\right) = 1 - m\left(\bigcap_{n=1}^{2014} E_n\right).$$

Use subadditivity and the inequality provided to show that $m\left(\bigcup\limits_{n=1}^{2014} E_n\right) < 1$. Combined with the result in the previous paragraph, $m\left(\bigcup\limits_{n=1}^{2014} E_n\right) > 0$. $\square$

Problem 4: Suppose $A \in \mathcal{M}$ and $m(A \Delta B) = 0$. Show that $B \in \mathcal{M}$ and $m(A) = m(B)$.

Solution. See page 36 in the textbook. $\square$

Problem 5: Suppose $A \subset E \subset B$ where A and B are measurable sets of finite measure. Show that if $m(A) = m(B)$, then E is measurable.

Solution. Notice that

$$m(B) = m(A) + m(B \setminus A)$$

Since $m(A) = m(B) < \infty$, we can subtract this on both sides to get $m(B \setminus A) = 0$. Since $E \setminus A \subset B \setminus A$, completeness of the Lebesgue measure shows that $E \setminus A$ is measurable, but then so is $E = A \cup (E \setminus A)$. $\square$

Problem 6: Suppose $E \in \mathcal{M}$ and $m(E) > 0$. Prove that there exists an open interval I such that

$$m(E \cap I) > 0.99 \cdot m(I).$$

Hint: Argue by contradiction, using the regularity of Lebesgue measure. See Theorems 2.17, 2.29.

Solution. We'll show that in fact a more general result holds.

Claim 1 *If $E \in \mathcal{M}$ and $m(E) > 0$, then for any $0 < \alpha < 1$, there exists an interval I such that*

$$m(E \cap I) > \alpha \cdot m(I).$$

Proof of Claim 1. We'll use a slight modification of Theorem 2.29, which the reader can prove as an exercise. This is the

Lemma 1 *If $E \in \mathcal{M}$, then*

$$m(E) = \sup\{m(K) \mid K \subset E,\ K \text{ is compact}\}.$$

With this the reader can easily prove that if E has finite measure, we can find a finite union of disjoint open intervals $A = \bigcup_{n=1}^{N} I_n$ such that $m(E\Delta A) < \epsilon$ (consider a suitable open cover of K by open intervals, and extract a finite subcover).

Let $\epsilon = (1-\alpha)m(E)$, and ket A be the set given by the lemma. Since A is a measurable set,

$$\begin{aligned}
m(E) &= m(E \cap A) + m(E \cap A^c) \\
m(E) &\leq m(E \cap A) + m(E \Delta A^c) \\
m(E) &< m(E \cap A) + (1-\alpha)m(E) \\
\alpha m(E) &< m(E \cap A)
\end{aligned}$$

Since E is a measurable set,

$$\begin{aligned}
m(A) &= m(A \cap E) + m(A \cap E^c) \\
m(A) &\leq m(A \cap E) + (1-\alpha)m(E) \\
m(A) &< m(A \cap E) + \frac{1-\alpha}{\alpha} m(E \cap A) \\
m(A) &< \frac{1}{\alpha} m(E \cap A)
\end{aligned}$$

Now we notice that

$$\begin{aligned}
m(A) &= \sum_{n=1}^{N} m(I_n) \\
m(A \cap E) &= \sum_{n=1}^{N} m(E \cap I_n)
\end{aligned}$$

This yields,

$$\sum_{n=1}^{N} m(I_n) < \frac{1}{\alpha} \left(\sum_{n=1}^{N} m(E \cap I_n) \right)$$

And this proves the claim if $m(E) < \infty$ (argue by contradiction). Now if $m(E) = +\infty$, take $E' \subset E$ with $m(E') < \infty$ and proceed in the same way to get the result for E'. Apply monotonicity to get the claim in its general form. $\square$

MAT324: Real Analysis – Fall 2016
Assignment 3

Due Thursday, **September 29**, in class.

Problem 1: Suppose that $E_1, E_2, \ldots$ are measurable subsets of $\mathbb{R}$. Show that if $E_k \supset E_{k+1}$ for all $k \geq 1$ and $m(E_{2016}) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k).$$

Problem 2: Let $\mathcal{N} \subset [0,1]$ be a non-measurable set. Determine whether the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -x & \text{if} \quad x \in \mathcal{N} \\ x & \text{if} \quad x \notin \mathcal{N} \end{cases}$$

is measurable. Explain.

Problem 3: Suppose that, for each rational number q, the set $\{x \mid f(x) > q\}$ is measurable. Can we conclude that f is measurable?

Problem 4: Suppose $f, g : E \to \mathbb{R}$ are measurable functions on $E \in \mathcal{M}$. Show that $h : E \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \dfrac{f(x)}{g(x)} & \text{if} \quad g(x) \neq 0 \\ 0 & \text{if} \quad g(x) = 0 \end{cases}$$

is measurable.

Problem 5: Let $f : (a,b) \to \mathbb{R}$. If f has a finite derivative at all points then show that f' is measurable.

Problem 6: (Extra Credit - 5p) Prove that any measurable set E with $m(E) > 0$ has a nonmeasurable subset $\mathcal{N} \subset E$.

MAT324: Real Analysis – Fall 2016

Assignment 3 – Solutions

Problem 1: Suppose that $E_1, E_2, \ldots$ are measurable subsets of $\mathbb{R}$. Show that if $E_k \supset E_{k+1}$ for all $k \geq 1$ and $m(E_{2016}) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k).$$

Solution. Let $A_1 = \bigcap_{k=1}^{2015} E_k$, and $A_n = E_n$, if $n \geq 2016$. Then the collection A_n satisfies the hypothesis of Theorem 2.13 (ii) (please see the proof there). $\square$

Problem 2: Let $\mathcal{N} \subset [0,1]$ be a non-measurable set. Determine whether the function

$$f(x) = \begin{cases} -x & \text{if} \quad x \in \mathcal{N} \\ x & \text{if} \quad x \notin \mathcal{N} \end{cases}$$

is measurable. Explain.

Solution. The function f is not measurable. Indeed,

$$f^{-1}((-1,0)) \cap (0,1) = \mathcal{N}$$

is a nonmeasurable set, so $f^{-1}((-1,0))$ is nonmeasurable. $\square$

Problem 3: Suppose that, for each rational number q, the set $\{x \mid f(x) > q\}$ is measurable. Can we conclude that f is measurable?

Solution. Yes, it is true. Use the density of $\mathbb{Q}$ in $\mathbb{R}$ to show that for any $a \in \mathbb{R}$,

$$f^{-1}((a,+\infty)) = \bigcup_{\{r\in\mathbb{Q} | r>a\}} f^{-1}((r,+\infty))$$

$\square$

Problem 4: Suppose $f, g : E \to \mathbb{R}$ are measurable functions on $E \in \mathcal{M}$. Show that $h : E \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \dfrac{f(x)}{g(x)} & \text{if} \quad g(x) \neq 0 \\ 0 & \text{if} \quad g(x) = 0 \end{cases}$$

is measurable.

Solution. It suffices to show that if g is a measurable function, then so is the function

$$\left(\frac{1}{g}\right)(x) = \begin{cases} \dfrac{1}{g(x)} & \text{if} \quad g(x) \neq 0 \\ 0 & \text{if} \quad g(x) = 0 \end{cases}$$

If this is proven, then the result follows from the fact that the measurable functions are closed under products. Notice that

1. If $a > 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a,+\infty)) = \left\{x|\frac{1}{g(x)} > a\right\} = \left\{x|g(x) < \frac{1}{a}\right\}$$
2. If $a = 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a,+\infty)) = \{x|g(x) > 0\}$$
3. If $a < 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a,+\infty)) = \left\{x|\frac{1}{g(x)} < a\right\} \cup \{x|g(x) \geq 0\}$$

In each case, the sets are measurable, hence the result follows. □

Problem 5: Let $f : (a,b) \to \mathbb{R}$. If f has a finite derivative at all points then show that f' is measurable.

SOLUTION. For each $n \in \mathbb{N}$, define $f_n : (a,b) \to \mathbb{R}$

$$f_n(x) = \begin{cases} \left(f\left(x + \dfrac{1}{n}\right) - f(x)\right)n, & \text{if} \quad x + \frac{1}{n} \in (a,b) \\ 0, & \text{otherwise.} \end{cases}$$

For any $x \in (a,b)$, there exists $N = N(x)$ such that $n > N$ implies $x + \frac{1}{n} \in (a,b)$, hence for every $x \in (a,b)$, $f_n(x) \to f'(x)$. Furthermore, each $f_n(x)$ is measurable (check!). Now use Corollary 3.8 from the textbook. □

MAT324: Real Analysis – Fall 2014
Assignment 4

Due Thursday, **October 6**, in class.

Problem 1: Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$. Show that the sets

$$\begin{aligned} E_1 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\} \\ E_2 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = \infty\} \\ E_3 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = -\infty\} \end{aligned}$$

are measurable.

Problem 2: Let $\mathcal{C} \subset [0,1]$ be the Cantor middle-thirds set. Suppose that $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = 0$ for $x \in \mathcal{C}$ and $f(x) = k$ for all x in each interval of length 3^{-k} which has been removed from $[0,1]$ at the k^{th} step of the construction of the Cantor set. Show that f is measurable and calculate $\int_{[0,1]} f dm$.

Problem 3: Let E be a measurable set. For a function $f : E \to \mathbb{R}$ we define the *positive part* $f^+ : E \to \mathbb{R}$, $f^+(x) = \max(f(x), 0)$, and the *negative part* $f^- : E \to \mathbb{R}$, $f^-(x) = \min(f(x), 0)$. Prove that f is measurable if and only if both f^+ and f^- are measurable.

Problem 4: Prove that if f is integrable on $\mathbb{R}$ and $\int_E f(x)dm \geq 0$ for every measurable set E, then $f(x) \geq 0$ a.e. x.
Hint: Show that the set $F = \{x \mid f(x) < 0\}$ is null.

Problem 5: Let E be a measurable set. Suppose $f \geq 0$ and let $E_k = \{x \in E \mid 2^k < f(x) \leq 2^{k+1}\}$ for any integer k. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} E_k = \{x \in E \mid f(x) > 0\},$$

and the sets E_k are disjoint.

(a) Prove that f is integrable if and only if $\sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

(b) Let $a > 0$ and consider the function

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use part a) to show that f is integrable on $\mathbb{R}$ if and only if $a < 1$.

MAT324: Real Analysis – Fall 2016
Assignment 4 – Solutions

Problem 4: Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$. Show that the sets

$$\begin{aligned} E_1 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\} \\ E_2 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = \infty\} \\ E_3 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = -\infty\} \end{aligned}$$

are measurable.

Solution. Theorem 3.5 of the textbook says that if $\{f_n\}$ is a sequence of measurable functions, then the functions $g = \liminf_n f_n$ and $h = \limsup_n f_n$ are measurable.

Notice that $\lim_{n\to\infty} f_n(x) = \infty$, if and only if $\liminf_n f_n(x) = \infty$. Hence,

$$E_2 = \{x | g(x) = \infty\} = \bigcap_{k\in\mathbb{N}} \{x | g(x) > k\}$$

is measurable. Likewise,

$$E_3 = \{x | h(x) = -\infty\} = \bigcap_{k\in\mathbb{N}} \{x | h(x) < -k\}$$

is measurable.

Further, notice that $E_1 = \{x \in | g(x) = h(x)\} \setminus (E_2 \cup E_3)$, hence E_1 is also measurable. □

Problem 2: Let $\mathcal{C} \subset [0, 1]$ be the Cantor middle-thirds set. Suppose that $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = 0$ for $x \in \mathcal{C}$ and $f(x) = k$ for all x in each interval of length 3^{-k} which has been removed from $[0, 1]$ at the k^{th} step of the construction of the Cantor set. Show that f is measurable and calculate $\int_{[0,1]} f dm$.

Solution. Denote by $f_n : [0, 1] \to \mathbb{R}$ the function constructed following way: If $\mathcal{C}_k$ denotes the union of the intervals of lenght 3^{-k} removed in the k-th step of the construction of the Cantor middle-third set, let $f_n(x) = k$ for $x \in \mathcal{C}_k$, and zero elsewhere. Then f_n is a simple function (it only takes $(n+1)$ values). Furthermore, it is easy to see that $f_n \to f$ pointwise, hence f is a measurable function. In addition, the sequence f_n is increasing to f, hence the Monotone Convergence Theorem gives us

$$\int_{[0,1]} f dm = \lim_{n\to\infty} \int_{[0,1]} f_n dm = \lim_{n\to\infty} \left(\sum_{k=1}^{n} k 2^{k-1} 3^{-k} \right) = \frac{1}{3} \lim_{n\to\infty} \left[\sum_{k=1}^{n} k \left(\frac{2}{3}\right)^{k-1} \right]$$

The answer up to this point is fine. With a little more effort, one can get the answer 3. This uses the following relation:

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \quad \text{if } 0 < |x| < 1$$

There are a number of ways one can use to prove this fact, including Riemman sums and Taylor's formula. ☐

Problem 3: Let E be a measurable set. For a function $f : E \to \mathbb{R}$ we define the *positive part* $f^+ : E \to \mathbb{R}$, $f^+(x) = \max(f(x), 0)$, and the *negative part* $f^- : E \to \mathbb{R}$, $f^-(x) = \min(f(x), 0)$. Prove that f is measurable if and only if both f^+ and f^- are measurable.

PROOF. One could directly apply the definition of a measurable function or use Theorem 3.5 for the maximum/minimum of two functions $f(x)$ and $g(x) = 0$. ☐

Problem 4: Prove that if f is integrable on $\mathbb{R}$ and $\int_E f(x)dm \geq 0$ for every measurable set E, then $f(x) \geq 0$ a.e. x.

SOLUTION. Since f is integrable, it is in particular measurable. Let E be the measurable set $E = \{x | f(x) < 0\}$. By hypothesis, and using monotonicity of the integral

$$0 \leq \int_E f(x)dm \leq \int_E 0dm = 0 \Rightarrow \int_E f(x)dm = 0$$

Notice that $-f$ is a positive function on E, and

$$\int_E (-f(x))dm = 0.$$

Now Theorem 4.4 implies that $-f$ is zero almost everywhere. By the definition of E, this happens if and only if E has zero measure. ☐

Problem 5: Let E be a measurable set. Suppose $f \geq 0$ and let $E_k = \{x \in E \mid 2^k < f(x) \leq 2^{k+1}\}$ for any integer k. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} E_k = \{x \in E \mid f(x) > 0\},$$

and the sets E_k are disjoint.

(a) Prove that f is integrable if and only if $\sum\limits_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

(b) Let $a > 0$ and consider the function

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use part a) to show that f is integrable on $\mathbb{R}$ if and only if $a < 1$.

SOLUTION.

(a) Suppose f is integrable. Since $f(x) > 2^k$ on E_k, we have

$$\int_{E_k} f dm \geq \int_{E_k} 2^k dm = 2^k m(E_k)$$

Therefore, by the comparison test,

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) \leq \sum_{k=-\infty}^{\infty} \int_{E_k} f dm = \int_{\mathbb{R}} f dm < \infty$$

Next suppose $\sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$. Then $2\left(\sum_{k=-\infty}^{\infty} 2^k m(E_k)\right) = \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty$. Since $f(x) \leq 2^{k+1}$ on E_k, we have

$$\int_{E_k} f dm \leq \int_{E_k} 2^{k+1} dm = 2^{k+1} m(E_k).$$

Then

$$\int_{\mathbb{R}} f dm = \sum_{k=-\infty}^{\infty} \int_{E_k} f dm \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty,$$

and f is integrable.

(b) Following part a), we need to find the measure of the sets E_k. If $K \geq 0$, then

$$2^k < |x|^{-a} \leq 2^{k+1},$$

and

$$2^{-k} > |x|^a \geq 2^{-k-1}$$

$$2^{\frac{-k}{a}} > |x| \geq 2^{\frac{-k-1}{a}}$$

Then $m(E_k) = 2 \cdot 2^{\frac{-k-1}{a}}(2^{\frac{1}{a}} - 1)$. If $k < 0$, then $2^k < |x|^{-a} \leq 2^{k+1}$ implies $|x| \geq 1$, hence $m(E_k) = 0$, if $k < 0$. Thus,

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) = \sum_{k=-0}^{\infty} 2^{k+1} \cdot 2^{\frac{-k-1}{a}} (2^{\frac{1}{a}} - 1)$$

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) = (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} 2^{\frac{(k+1)(a-1)}{a}}$$

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) = (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} \left[2^{\frac{(a-1)}{a}}\right]^{(k+1)}$$

Notice that this geometric series converge if and only if $2^{\frac{(a-1)}{a}} < 1$, and this happens if and only if $a < 1$. $\square$

MAT324: Real Analysis – Fall 2016
Assignment 5

Due Thursday, **October 20**, in class.

Problem 1: Suppose $\int_E f\,dm = \int_E g\,dm$ for every measurable set $E \in \mathcal{M}$. Show that $f = g$ almost everywhere.

Problem 2: Suppose $(f_n)_{n\geq 1}$ is a sequence of non-negative measurable functions on $E \in \mathcal{M}$. If f_n decreases to f almost everywhere and $\int_E f_1 dm < \infty$, then show that

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

Hint: Look at the sequence $g_n = f_1 - f_n$.

Problem 3: Suppose $(f_n)_{n\geq 1}$ is a sequence of non-negative measurable functions. Show that

$$\int \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int f_n dm.$$

Problem 4: Compute the following limits if they exist and justify the calculations:

a) $\displaystyle\lim_{n\to\infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx$

b) $\displaystyle\lim_{n\to\infty} \int_0^{\infty} \frac{n^2 x e^{-n^2 x^2}}{1+x} dx.$

c) $\displaystyle\lim_{n\to\infty} \int_1^{\infty} \frac{n^2 x e^{-n^2 x^2}}{1+\sqrt[n]{x}} dx.$

Problem 5: Suppose $E \in \mathcal{M}$. Let (g_n) be a sequence of integrable functions which converges a.e. to an integrable function g. Let (f_n) be a sequence of measurable functions which converge a.e. to a measurable function f. Suppose further that $|f_n| \leq g_n$ a.e. on E for all $n \geq 1$. Show that if $\int_E g\,dm = \lim_{n\to\infty} \int_E g_n\,dm$, then $\int_E f\,dm = \lim_{n\to\infty} \int_E f_n\,dm$.
Hint: Rework the proof of the Dominated Convergence Theorem.

Problem 6: Let $E \in \mathcal{M}$. Let (f_n) be a sequence of integrable functions which converges a.e. to an integrable function f. Show that $\int_E |f_n - f|\,dm \to 0$ as $n \to \infty$ if and only if $\int_E |f_n|\,dm \to \int_E |f|\,dm$ as $n \to \infty$.

Problem 7: Consider two functions $f, g : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if} \quad x = \frac{p}{q} \in \mathbb{Q}, \text{ where } p \text{ and } q \text{ are relatively prime} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Show that f is Riemann integrable on $[0,1]$, but g is not Riemann integrable on $[0,1]$.

Problem 8: Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x > 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Show that f has an improper Riemann integral over the interval $[0,\infty)$, but f is not Lebesgue integrable.

MAT324: Real Analysis – Fall 2016

Assignment 5 – Solutions

Problem 1: Suppose $\int_E f\, dm = \int_E g\, dm$ for every measurable set $E \in \mathcal{M}$. Show that $f = g$ almost everywhere.

Solution. See the proof of Theorem 4.22. $\square$

Problem 2: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions on $E \in \mathcal{M}$. If $\{f_n\}$ decreases to f almost everywhere and $\int_E f_1 dm < \infty$, then show that

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

Hint: Look at the sequence $g_n = f_1 - f_n$.

Solution. Consider the sequence of measurable functions $g_n = f_1 - f_n$. Since $\{f_n\}$ is a decreasing sequence, the sequence $\{g_n\}$ is an increasing sequence of nonnegative measurable functions converging to $g = (f_1 - f)$. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_E g_n dm = \int_E g dm.$$

On the other hand, since $\int_E f_1 dm < \infty$, and the f_n's decrease, monotonicity gives us $\int_E f_n dm < \infty$. Then, for each $n \in \mathbb{N}$, we have

$$\int_E g_n dm = \int_E (f_1 - f_n) dm = \int_E f_1 dm - \int_E f_n dm.$$

Likewise,

$$\int_E g dm = \int_E (f_1 - f) dm = \int_E f_1 dm - \int_E f dm.$$

The result now follows from cancellation (notice that it is necessary to assume $\int_E f_1 dm < \infty$ for this). $\square$

Problem 3: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. Show that

$$\int \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int f_n dm.$$

Solution. Consider the sequence of measurable functions $g_n = \sum_{k=1}^{n} f_k$. This sequence is clearly increasing since $f_k \geq 0$ for all $k \geq 1$, and $g_n(x) \to g(x) = \sum_{k=1}^{\infty} f_k(x)$, pointwise for every x. Convergence follows from the Monotone Convergence Theorem. $\square$

Problem 4: Compute the following limits if they exist and justify the calculations:

a) $\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx$

b) $\lim_{n\to\infty} \int_0^\infty \frac{n^2 x e^{-n^2x^2}}{1+x} dx.$

c) $\lim_{n\to\infty} \int_1^\infty \frac{n^2 x e^{-n^2x^2}}{1+\sqrt[n]{x}} dx.$

SOLUTION.

a) The integrand is dominated by

$$g(x) = \frac{1}{(1+\frac{x}{2})^2},$$

which is integrable (check!). In addition, the integrand goes to zero for every x. The Dominated Convergence Theorem gives that the limit is zero.

b) See the worked out example example online and make the change of variables $y = nx$. The computations are exactly the same. The limit exists and is $\frac{1}{2}$.

c) The limit exists and is 0. Check that the integrands are dominated by the same L^1 function as before. □

Problem 5: Suppose $E \in \mathcal{M}$. Let (g_n) be a sequence of integrable functions which converges a.e. to an integrable function g. Let (f_n) be a sequence of measurable functions which converge a.e. to a measurable function f. Suppose further that $|f_n| \le g_n$ a.e. on E for all $n \ge 1$. Show that if $\int_E g\, dm = \lim_{n\to\infty} \int_E g_n\, dm$, then $\int_E f\, dm = \lim_{n\to\infty} \int_E f_n\, dm$.

Hint: Rework the proof of the Dominated Convergence Theorem.

SOLUTION. Since $\pm f_n \le |f_n| \le g_n$ we get that $g_n \pm f_n \ge 0$. Apply Fatou's Lemma to the sequences $g_n - f_n$ and $g_n + f_n$ and carefully work out the details. □

Problem 6: Let $E \in \mathcal{M}$. Let (f_n) be a sequence of integrable functions which converges a.e. to an integrable function f. Show that $\int_E |f_n - f|\, dm \to 0$ as $n \to \infty$ if and only if $\int_E |f_n|\, dm \to \int_E |f|\, dm$ as $n \to \infty$.

SOLUTION. First, suppose (f_n) is a sequence of integrable functions which converges a.e. to an integrable function f and $\int_E |f_n - f|\, dm \to 0$ as $n \to \infty$. We need to show that $\int_E |f_n|\, dm \to \int_E |f|\, dm$. The reverse triangle inequality tells us that

$$||f_n(x)| - |f(x)|| \le |f_n(x) - f(x)|$$

We integrate both sides and pass to the limit as $n \to \infty$.

For the converse, apply the previous problem to the sequence $|f_n - f|$ which is dominated by the sequence of functions $g_n = |f_n| + |f|$. □

Problem 7: Consider two functions $f, g : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if} \quad x = \frac{p}{q} \in \mathbb{Q}, \text{ where } p \text{ and } q \text{ are relatively prime} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Show that f is Riemann integrable on $[0, 1]$, but g is not Riemann integrable on $[0, 1]$.

Solution. For f, see Example 4.6 from the textbook. The function f is continuous at all irrational points, hence almost everywhere. For g, notice that the set of discontinuities is the whole interval $[0, 1]$. Then use Theorem 4.23, part (i). $\square$

Problem 8: Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x > 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Show that f has an improper Riemann integral over the interval $[0, \infty)$, but f is not Lebesgue integrable.

Solution. Notice that if $a, A > 0$,

$$\int_a^A \frac{\sin x}{x} dx = \frac{\cos a}{a} - \frac{\cos A}{A} - \int_a^A \frac{\cos x}{x^2}$$

The integral on the right-hand side is convergent, whereas the diference also is, hence the Riemann integral exists. However, f is not Lebesgue integrable. Indeed, $|f|$ is not integrable since

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx = \frac{2}{(k+1)\pi}$$

and therefore

$$\lim_{n\ to \infty} \int_1^n \frac{|\sin x|}{x} dx \geq \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{1}{k} = \infty$$

diverges. $\square$

MAT324: Real Analysis – Fall 2016
Assignment 6

Due Thursday, **November 3**, in class.

Problem 1: Let $f : E \to [0, \infty)$ be a Lebesgue integrable function and suppose $\int_E f\, dm = C$ and $0 < C < \infty$. Prove that

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \left(\frac{f(x)}{n}\right)^{\alpha}\right) dm = \begin{cases} \infty, & \text{for } \alpha \in (0,1) \\ C, & \text{for } \alpha = 1 \\ 0, & \text{for } 1 < \alpha < \infty. \end{cases}$$

Hint: For $\alpha = 1$, use the inequality $e^x \geq x + 1$, for all $x \geq 0$. For $\alpha > 1$, use $(1 + x)^{\alpha} \geq 1 + x^{\alpha}$. DCT and the Fatou Lemma might prove useful.

Problem 2: Consider the sequence of functions

$$f_n(x) = \frac{1}{\sqrt{x}} \chi_{(0, \frac{1}{n}]}(x), \quad n \geq 1.$$

a) Is f_n in $L^1(0, 1]$?

b) Is the sequence Cauchy in $L^1(0, 1]$?

c) Is f_n in $L^p(0, 1]$ for $p \geq 4$?

Problem 3: Consider the sequence $f_n = n\chi_{[n+\frac{1}{n^3}, n+\frac{2}{n^3}]}$, $n \geq 1$. Determine whether the following are true or false and explain your answers.

a) $(f_n)_{n\geq 1}$ is Cauchy as a sequence of $L^1(0, \infty)$.

b) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^1(\mathbb{R})$.

b) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^2(\mathbb{R})$.

c) $f_n \in L^2(\mathbb{R})$ for each $n \geq 1$.

Problem 4: Let $(X, \|\cdot\|)$ be a normed vector space. Show that X is complete if and only if whenever $\sum_{j=1}^{\infty} \|x_j\| < \infty$, then $\sum_{j=1}^{\infty} x_j$ converges to an element $x^* \in X$.
Hint: Rework the proof of the completeness theorem for L^1.

MAT324: Real Analysis – Fall 2016
ASSIGNMENT 6 – SOLUTIONS

Problem 1: Let $f : E \to [0, \infty)$ be a Lebesgue integrable function and suppose $\int_E f\, dm = C$ and $0 < C < \infty$. Prove that

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) dm = \begin{cases} \infty, & \text{for } \alpha \in (0,1) \\ C, & \text{for } \alpha = 1 \\ 0, & \text{for } 1 < \alpha < \infty. \end{cases}$$

Hint: For $\alpha = 1$, use the inequality $e^x \geq x + 1$, for all $x \geq 0$. For $\alpha > 1$, use $(1 + x)^\alpha \geq 1 + x^\alpha$. DCT and the Fatou Lemma might prove useful.

SOLUTION. There are three cases to consider. First, suppose that $0 < \alpha < 1$. Then, by Fatou's Lemma, we get

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) dm \geq \int_E \lim_{n\to\infty} n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) dm = \infty,$$

because

$$\lim_{n\to\infty} n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) = \lim_{n\to\infty} n^{1-\alpha} f(x)^\alpha = \infty.$$

Suppose $\alpha = 1$. Then $n \ln\left(1 + \frac{f(x)}{n}\right) \leq n\frac{f(x)}{n} = f(x)$, which is integrable. We can apply DCT and get

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \frac{f(x)}{n}\right) dm = \int_E \lim_{n\to\infty} n \ln\left(1 + \frac{f(x)}{n}\right) dm = \int_E f(x) dx = C.$$

Finally, suppose $\alpha > 1$. Then, using the inequality $(1 + x)^\alpha \geq 1 + x^\alpha$, we get

$$n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) \leq \alpha n \ln\left(1 + \frac{f(x)}{n}\right) \leq \alpha f(x),$$

which is integrable. The last inequality follows from the fact that the sequence $\left(1 + \frac{f(x)}{n}\right)^n$ is increasing to $e^{f(x)}$ so $\ln\left(1 + \frac{f(x)}{n}\right) \leq f(x)$. We can therefore apply DCT and interchange the integral and the limit. We get that the limit is 0 because

$$\lim_{n\to\infty} n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) = \lim_{n\to\infty} n^{-(\alpha-1)} f(x)^\alpha = 0.$$

□

Problem 2: Consider the sequence of functions

$$f_n(x) = \frac{1}{\sqrt{x}} \chi_{(0, \frac{1}{n}]}(x), \quad n \geq 1.$$

a) Is f_n in $L^1(0, 1]$?

b) Is the sequence Cauchy in $L^1(0,1]$?

c) Is f_n in $L^p(0,1]$ for $p \geq 4$?

Solution.

a) Yes, it is. Notice that $\|f_n\|_1 = \frac{1}{2\sqrt{n}}$.

b) Yes, it is. In fact, it converges to the zero function.

c) No. If $p \geq 4$, then

$$\int_{(0,1)} |f_n(x)|^p dx = \int_{(0,\frac{1}{n}]} x^{-\frac{p}{2}} dx \geq \int_{(0,\frac{1}{n}]} x^{-2} dx = +\infty$$

□

Problem 3: Consider the sequence $f_n = n\chi_{[n+\frac{1}{n^3}, n+\frac{2}{n^3}]}$, $n \geq 1$. Determine whether the following are true or false and explain your answers.

a) $(f_n)_{n\geq 1}$ is Cauchy as a sequence of $L^1(0,\infty)$.

b) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^1(\mathbb{R})$.

c) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^2(\mathbb{R})$.

d) $f_n \in L^2(\mathbb{R})$ for each $n \geq 1$.

Solution.

a) Yes, the sequence is Cauchy. Note that $\|f_n\|_1 = \frac{1}{n^2}$.

b) Yes, $f \in L^1(\mathbb{R})$. Note that $\|f\|_1$ behaves as $\sum \frac{1}{n^2}$, which converges.

c) No, $f \notin L^2(\mathbb{R})$. Note that $\|f\|_2$ behaves as $\sum \frac{1}{n}$, which diverges.

d) Yes, each f_n belongs to $L^2(\mathbb{R})$.

□

Problem 4: Let $(X, \|\cdot\|)$ be a normed vector space. Show that X is complete if and only if whenever $\sum\limits_{j=1}^{\infty} \|x_j\| < \infty$, then $\sum\limits_{j=1}^{\infty} x_j$ converges to an element $x^* \in X$.

Hint: Rework the proof of the completeness theorem for L^1.

Solution. Suppose that X is complete and $\sum\limits_{j=1}^{\infty} \|x_j\| < \infty$. Let $y_n = \sum\limits_{j=1}^{n} x_j$. Then $(y_n)_{n\geq 1}$ is Cauchy because the tail $\sum\limits_{j=n}^{\infty} \|x_j\|$ can be made arbitrarily small. Since X is complete, we get that y_n converges to, say, x^* and x^* belongs to X, which proves the claim. The converse implication is the same as the proof of Theorem 5.5 from the textbook. □

MAT 324 – Real Analysis
FALL 2016

Midterm – October 25, 2016
SOLUTIONS

NAME: ____________________________

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Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

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PROBLEM	SCORE
1	
2	
3	
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5	
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Problem 1: (21 points) Let E a null subset of $\mathbb{R}$.

a) Use the definition of a null set to show that the set $-E = \{-x \mid x \in E\}$ is null.

SOLUTION. The set E is null, so for every $\epsilon > 0$ there exists a cover $E \subset \cup_{j=1}^{\infty} I_j$, of open intervals I_j, such that $\sum_{j=1}^{\infty} m(I_j) < \epsilon$. Note that $\bigcup_{j=1}^{\infty}(-I_j)$ is a cover for the set $-E$. This follows directly from the definition of the set $-E$. One can easily check that $m(I_j) = m(-I_j)$, so $\sum_{j=1}^{\infty} m(-I_j) < \epsilon$. This shows that $-E$ is null. $\square$

b) Consider $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Is $f^{-1}(E)$ measurable? Explain.

SOLUTION. Note that f is bijective and $f^{-1}(x) = x^2$, $f^{-1} : [0, \infty) \to [0, \infty)$. The function f is continuous, so it is measurable. The set E is null, but it may not be Borel (note that not all null sets are Borel!). So we cannot draw the conclusion that $f^{-1}(E)$ is measurable from these observations. However, by definition,

$$f^{-1}(E) = \{x^2 \mid x \in E, x \geq 0\},$$

so $f^{-1}(E) \subset F$, where F is defined in part c). By part c) F is null, which implies that $f^{-1}(E)$ is null and hence Lebesgue measurable. We are left to prove part c).

One could also prove directly that $f^{-1}(E)$ is null using the definition of a null set as in part a); the proof would be exactly the same as in part c) below. $\square$

c) Let $F = \{x^2 \mid x \in E\}$. Show that F is null.

SOLUTION. Suppose $F \subset [-N, N]$ for some large N, say $N > 100$. Observe that $|x| \leq N$ for all $x \in F$. Let $\epsilon > 0$ be small enough, say $\epsilon < N/2$. Let $E \subset \bigcup_{j=1}^{\infty} I_j$ be an open cover of the null set E, as in part a), such that $\sum_{j=1}^{\infty} m(I_j) < \epsilon$. Then $F \subset \bigcup_{j=1}^{\infty} I_j^2$. If $I_j = (a_j, b_j)$, then $I_j^2 = (a_j^2, b_j^2)$ or $I_j^2 = (b_j^2, a_j^2)$, depending whether a_j, b_j are positive or negative. In any case I_j^2 is an interval. Suppose without loss of generality that $0 < a_j < b_j$ (the other cases are treated similarly). Then

$$m(I_j^2) = b_j^2 - a_j^2 = (b_j - a_j)(b_j + a_j) < 4N m(I_j).$$

Note that if I_j is an interval that is part of the cover for E, then $I_j \subset [-\sqrt{N} - \epsilon, \sqrt{N} + \epsilon]$. So $|a_j|$ and $|b_j|$ are $\leq 2\sqrt{N} + 2\epsilon < 2N$, which justifies the inequality above. These inequalities are not optimal and other similar bounds work. It follows that

$$\sum_{j=1}^{\infty} m(I_j^2) < 4N \sum_{j=1}^{\infty} m(I_j) < 4N\epsilon,$$

which can be made arbitrarily small. This shows that F is null.

If F is unbounded then we write $F = \bigcup_{N=100}^{\infty} (F \cap [-N, N])$. This is an increasing union of bounded null sets, so it is null. $\square$

Problem 2: (20 points) Does there exist a Lebesgue measurable set $E \subset \mathbb{R}$ such that

$$m(E \cap I) \geq 0.99 \cdot m(E^c \cap I),$$

for every interval I? Either give an example of such a set, or prove that it does not exist.

SOLUTION. Any measurable set E with $m(E^c) = 0$ verifies this inequality. For example, one could take $E = \mathbb{R}$, for which $E^c = \emptyset$. We explain below why these examples arise naturally.

From the definition of a measurable set, we have $m(I) = m(E \cap I) + m(E^c \cap I)$ for every interval I. Using the inequality from the hypothesis, we find

$$m(I) = m(E \cap I) + m(E^c \cap I) \geq 1.99m(E^c \cap I).$$

However, if $m(E^c) > 0$ then, we know from HW that for every constant $0 < \alpha < 1$ there exists an interval I such that $m(E^c \cap I) \geq \alpha m(I)$. In particular, for $\alpha = 0.99$, there exists an interval I such that $m(E^c \cap I) > 0.99m(I)$. But then

$$m(I) \geq 1.99m(E^c \cap I) > 1.99 \cdot 0.99m(I) > 1.5m(I),$$

which is a contradiction. So we are left with the case $m(E^c) = 0$, for which the inequality is satisfied. $\square$

Problem 3: (25 points) Let E be a measurable set with $m(E) < \infty$. Let $f : E \to [0,1]$ be an integrable nonnegative function. For each $n \geq 0$ define the sets

$$E_{2^n,j} = \left\{ x \in E \mid \frac{j}{2^n} < f(x) \leq \frac{j+1}{2^n} \right\}, \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1.$$

and the functions

$$f_n(x) = \sum_{j=0}^{2^n-1} \frac{j+1}{2^n} \chi_{E_{2^n,j}}(x).$$

a) Explain why f_n is a measurable function and why $f_n(x) \geq f(x)$ for all $x \in E$.

Solution. The function f is integrablem, so it is measurable. Therefore each set $E_{2^n,j}$ is measurable. The function f_n is a finite sum of measurable functions, so it is measurable (in fact, it is a simple function). By construction, on the set $E_{2^n,j}$, we have $f(x) \leq \frac{j+1}{2^n} = f_n(x)$. Also, for fixed n, the sets $E_{2^n,j}$, for $j = 0, 1, \ldots, 2^n - 1$, are pairwise disjoint. Let $E_0 = \{x \in E \mid f(x) = 0\}$. The $E - E_0 = \bigcup_{j=0}^{2^n-1} E_{2^n,j}$. On the set E_0 we have $f_n(x) = f(x) = 0$. In conclusion, $f_n(x) \geq f(x)$ for all $x \in E$. $\square$

b) Prove that $(f_n)_{n\geq 1}$ is decreasing, that is, $f_{n+1}(x) \leq f_n(x)$ pointwise for every $x \in E$.

Solution. From the observations in part a), it is enough to prove that $f_{n+1}(x) \leq f_n(x)$ for $x \in E_{2^n,j}$, for $j = 0, 1, \ldots, 2^n - 1$. Note that

$$E_{2^n,j} = E_{2^{n+1},2j} \cup E_{2^{n+1},2j+1}.$$

If $x \in E_{2^{n+1},2j}$ then $f_{n+1}(x) = \frac{2j+1}{2^{n+1}} = \frac{j+1/2}{2^n}$ and $f_n(x) = \frac{j+1}{2^n}$, from the definition of the functions f_n and f_{n+1}. We have $f_{n+1}(x) < f_n(x)$. If $x \in E_{2^{n+1},2j+1}$ then $f_{n+1}(x) = \frac{2j+2}{2^{n+1}} = \frac{j+1}{2^n} = f_n(x)$. It follows that $f_{n+1}(x) \leq f_n(x)$ for $x \in E$. $\square$

(Problem 1 continued)

c) Prove that
$$\int_E f\,dm = \lim_{n\to\infty} \sum_{j=0}^{2^n-1} \frac{j+1}{2^n} m(E_{2^n,j}).$$

SOLUTION. First note that $(f_n)_{n\geq 1}$ is a decreasing sequence of nonnegative measurable functions. By construction of the function f_n, observe that $0 \leq f_n - f(x) \leq \frac{1}{n}$, so $f_n(x) \to f(x)$ as $n \to \infty$. The first term of the sequence is integrable. Indeed,
$$f_1(x) = \frac{1}{2}\chi_{E_{2,0}}(x) + \chi_{E_{2,1}}(x)$$
and $\int_E f_1\,dm = \frac{1}{2}m(E_{2,0}) + m(E_{2,1}) < 2m(E) < \infty$.

We can apply MCT (for a decreasing sequence, as in the HW) and obtain
$$\int_E f\,dm = \int_E \lim_{n\to\infty} f_n(x)\,dm = \lim_{n\to\infty} \int_E f_n\,dm = \lim_{n\to\infty} \sum_{j=0}^{2^n-1} \frac{j+1}{2^n} m(E_{2^n,j}).$$

The last equality is just the definition of the integral of a simple function. $\square$

Problem 4: (22 points) Compute the following limit if it exists and justify the calculations. If the limit does not exist explain why it does not exist.

a) $\displaystyle\lim_{n\to\infty}\int_0^{100}\left(1-\frac{n\cos x}{1+n^2\sqrt{x}}\right)dx$

Solution. Note that $1+n^2\sqrt{x} > n\sqrt{x}$ for all $n \geq 1$. We have

$$\left|1-\frac{n\cos x}{1+n^2\sqrt{x}}\right| \leq 1+\frac{n}{1+n^2\sqrt{x}} < 1+\frac{n}{n\sqrt{x}} = 1+x^{-1/2}.$$

The function $1+x^{-1/2}$ is integrable on $[0,100]$ since

$$\int_0^{100} 1+x^{-1/2}\,dx = (x+2x^{1/2})\Big|_0^{100} = 120.$$

We can therefore apply DCT and get

$$\lim_{n\to\infty}\int_0^{100}\left(1-\frac{n\cos x}{1+n^2\sqrt{x}}\right)dx = \int_0^{100}\lim_{n\to\infty}\left(1-\frac{n\cos x}{1+n^2\sqrt{x}}\right)dx = 100.$$

□

b) $\displaystyle\lim_{n\to\infty}\int_0^{100}\frac{ne^{-nx}}{1+x^n}\,dx$

Solution. We make a change of variables $y = nx$ and get

$$\int_0^{100}\frac{ne^{-nx}}{1+x^n}\,dx = \int_0^{100n}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\,dy = \int_0^{\infty}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\chi_{[0,100n]}(y)\,dy.$$

The sequence of functions $f_n(y) = \dfrac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\chi_{[0,100n]}(y)$, $n \geq 1$, is bounded above by $g(y) = e^{-y}\chi_{[0,\infty)}$, which is nonnegative and Riemann integrable since

$$\int_0^{\infty} e^{-y}\,dy = -e^{-y}\Big|_0^{\infty} = 1.$$

Hence g is also Lebesgue integrable and we can apply DCT. We have

$$\lim_{n\to\infty}\int_0^{\infty} f_n(y)\,dy = \int_0^{\infty}\lim_{n\to\infty}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\,dy = \int_0^1 e^{-y}\,dy = 1-e^{-1}.$$

We have use the fact that $\lim_{n\to\infty} y^n = \infty$ if $y > 1$, but $\lim_{n\to\infty} y^n = 0$, if $0 \leq y < 1$. Also, recall that $\lim_{n\to\infty} n^n = 1$. This gives

$$\lim_{n\to\infty}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n} = \begin{cases} 0 & \text{if } y > 1 \\ e^{-y}/2 & \text{if } y = 1 \\ e^{-y} & \text{if } 0 \leq y < 1. \end{cases}$$

In conclusion, $\displaystyle\lim_{n\to\infty}\int_0^{100}\frac{ne^{-nx}}{1+x^n}\,dx = 1-e^{-1}$. □

Problem 5: (12 points) Let $f : [0,1] \to \mathbb{R}$ be a function which is continuous everywhere except on a null set $E \subset [\frac{1}{4}, \frac{3}{4}]$. Compute the following limit and justify the calculations:

$$\lim_{n\to\infty} \int_0^1 f(x^n)\,dx.$$

Solution. The function is continuous a.e. so it is Riemann integrable and bounded. Moreover, it is Lebesgue integrable, measurable, and the Riemann and Lebesgue integrals agree. Let $f_n(x) = f(x^n)$. Since f is bounded we have $|f_n(x)| < M$, for all $x \in [0,1]$ and all $n \geq 1$. Each f_n is continuous a.e. (as a composition of f, which is continuous a.e., and the continuous function x^n), so it is measurable. We can apply DCT (or the uniform boundedness principle) and conclude that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\,dx = \int_0^1 \lim_{n\to\infty} f(x^n)\,dx = f(0).$$

Note that f is continuous at $x = 0$.

The sequence $(x^n)_{n\geq 1}$ is decreasing on the interval $[0,1)$, but we don't know whether $f_n(x) = f(x^n)$ is monotone (either increasing or decreasing). Also we don't know whether f is nonnegative or not, so we cannot apply MCT directly. $\square$

MAT 324 – Real Analysis
FALL 2014

Midterm – October 23, 2014

NAME: ______________________________

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PROBLEM	**SCORE**
1	
2	
3	
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TOTAL	

Problem 1: (25 points) Let $E_1, E_2, \ldots, E_{2014}$ be measurable subsets of $[0,1]$.

a) Suppose $m(E_k) > 1 - \dfrac{1}{2^k}$ for each $1 \le k \le 2014$. Show that $m\left(\bigcap_{k=1}^{2014} E_k\right) > 0$.

b) Suppose almost every x from the interval $[0,1]$ belongs to at least 3 of these subsets. Prove that there exists at least one set E_k with $1 \le k \le 2014$ such that $m(E_k) \ge \frac{3}{2014}$.

Hint: The function $f(x) = \sum_{k=1}^{2014} \chi_{E_k}(x)$ has the property that $f(x) \ge 3$ a.e.

Problem 2: (20 points) Does there exist a Lebesgue measurable subset E of $\mathbb{R}$ such that for every interval (a, b) we have

$$m\,(E \cap (a, b)) = \frac{b-a}{2}\ ?$$

Either construct such a set or prove it does not exist.

Problem 3: (25 points) Let E be a measurable set and $f : E \to \mathbb{R}$ Lebesgue integrable on E. Define $E_k = \left\{x \in E \mid |f(x)| < \frac{1}{k}\right\}$ for $k \geq 1$.

a) Show that each E_k is a measurable set.

b) Determine whether $\{E_k\}$ is an increasing or decreasing collection of sets.

c) Show that $\lim_{n\to\infty} \int_{E_k} |f|\, dm = 0$.

Problem 4: (30 points) Compute the following limit if it exists and justify the calculations. If the limit does not exist explain why it does not exist.

a) $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{\sqrt{n}}{\sqrt{x}} \cdot \chi_{[0,\frac{1}{n}]}\, dx$

b) $\lim_{n\to\infty} \int_a^\infty \frac{n\sin(\sqrt{x})}{1+n^2x^2}\, dx,$ where $a > 0$

MAT 324 – Real Analysis
Fall 2014

Midterm – October 23, 2014

NAME: ______________________________

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PROBLEM	SCORE
1	
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TOTAL	

Problem 1: (25 points) Let $E_1, E_2, \dots, E_{2014}$ be measurable subsets of $[0,1]$.

a) Suppose $m(E_k) > 1 - \dfrac{1}{2^k}$ for each $1 \le k \le 2014$. Show that $m\left(\bigcap_{k=1}^{2014} E_k\right) > 0$.

Solution. Note that

$$m\left(\bigcup_{k=1}^{2014} E_k^c\right) \le \sum_{k=1}^{2014} m(E_k^c) = \sum_{k=1}^{2014}(1 - m(E_k)) = 2014 - \sum_{k=1}^{2014} m(E_k),$$

by subadditivity and taking complements. Therefore

$$m\left(\bigcap_{k=1}^{2014} E_k\right) = 1 - m\left(\bigcup_{k=1}^{2014} E_k^c\right) \ge 1 - 2014 + \sum_{k=1}^{2014} m(E_k) = \sum_{k=1}^{2014} m(E_k) - 2013 > \frac{1}{2^{2015}} > 0.$$

The last inequality follows from the fact that for each E_k we have $m(E_k) > 1 - \frac{1}{2^k}$ so

$$\sum_{k=1}^{2014} m(E_k) > \sum_{k=1}^{2014}\left(1 - \frac{1}{2^k}\right) = 2013 + \frac{1}{2^{2015}}.$$

□

b) Suppose almost every x from the interval $[0,1]$ belongs to at least 3 of these subsets. Prove that there exists at least one set E_k with $1 \le k \le 2014$ such that $m(E_k) \ge \frac{3}{2014}$.
Hint: The function $f(x) = \sum_{k=1}^{2014} \chi_{E_k}(x)$ has the property that $f(x) \ge 3$ a.e.

Solution. We follow the hint and set $f(x) = \sum_{k=1}^{2014} \chi_{E_k}(x)$. Since almost every x from the interval $[0,1]$ belongs to at least 3 sets E_k we have that $f(x) \ge 3$ almost everywhere. The function f is a simple function with finite support, hence it is measurable and integrable and $\int_{[0,1]} f\, dm = \sum_{k=1}^{2014} m(E_k)$. Suppose that $m(E_k) < \frac{3}{2014}$. Then

$$3 \le \int_{[0,1]} f\, dm = \sum_{k=1}^{2014} m(E_k) < 2014 \cdot \frac{3}{2014} = 3,$$

so $3 < 3$, which is a contradiction. Hence there exists at least one set E_k such that $m(E_k) \ge \frac{3}{2014}$.

□

Problem 2: (20 points) Does there exist a Lebesgue measurable subset E of $\mathbb{R}$ such that for every interval (a, b) we have

$$m\left(E \cap (a,b)\right) = \frac{b-a}{2}\ ?$$

Either construct such a set or prove it does not exist.

Solution 1. Suppose such a set E exists. Then $E \cap (0,2)$ is a bounded measurable set with $m(E \cap (0,2)) = 1$. Let $0 < \epsilon < 1$. By the outer regularity property applied to $E \cap (0,2)$ there exists an open set O such that $E \cap (0,2) \subset O$ and $m(O - E \cap (0,2)) < \epsilon$. It follows that $m(O) < m(E \cap (0,2)) + \epsilon = 1 + \epsilon$. The set O is open so it can be written as a disjoint union of intervals $O = \bigcup_{k=1}^{\infty} I_k$, with $I_k = (a_k, b_k)$.

By hypothesis we have that $m(E \cap I_k) = \frac{m(I_k)}{2}$ for all $k \geq 1$. Hence

$$m(E \cap O) = m\left(E \cap \bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} m(E \cap I_k) = \frac{1}{2}\sum_{k=1}^{\infty} m(I_k) = \frac{m(O)}{2} < \frac{1+\epsilon}{2}.$$

However $E \cap (0,2) \subset O$ so $E \cap (0,2) \subset E \cap O$ and $1 = m(E \cap (0,2)) \leq m(E \cap O) < \frac{1+\epsilon}{2}$. We have obtained that $1 < \frac{1+\epsilon}{2}$ which gives $1 < \epsilon$. Contradiction! So there is no set E which verifies the hypothesis. □

Solution 2. This solution reduces the problem to the discussion from class. From the hypothesis we know that $m\left(E \cap (-n, n)\right) = \frac{n-(-n)}{2} = n \to \infty$ as $n \to \infty$. So the measure $m(E \cap \mathbb{R}) = m(E) = \infty$, hence E has to be unbounded.

However, $E_n = E \cap (-n, n)$ is bounded for every n. We claim that

$$m\left(E_n \cap (a,b)\right) \leq \frac{b-a}{2}$$

for every interval (a, b). There are four cases to consider, depending on how the interval (a, b) intersects the interval $(-n, n)$. If $(a, b) \subset (-n, n)$ then $E \cap (-n, n) \cap (a, b) = E \cap (a, b)$ and $m(E \cap (a, b)) = \frac{b-a}{2}$. If $(a, b) \cap (-n, n) = (n, b)$ (this is possible if $a < n < b$) then $m(E \cap (-n, n) \cap (a, b)) = m(E \cap (n, b)) = \frac{b-n}{2} \leq \frac{b-a}{2}$. Similarly if $(a, b) \cap (-n, n) = (a, -n)$ then $m(E \cap (-n, n) \cap (a, b)) = m(E \cap (a, -n)) = \frac{-n-a}{2} \leq \frac{b-a}{2}$ since $-n < b$. Finally, if $(-n, n) \subset (a, b)$ then $m(E \cap (-n, n) \cap (a, b)) = m(E \cap (-n, n)) = n \leq \frac{b-a}{2}$ since the length of (a, b) is greater than the length of $(-n, n)$, which is $2n$.

It now follows that $m(E_n) = 0$. Since n was arbitrary we get that E is null. Contradiction! So there is no such set E with this property. □

Recall that in class we have shown the following (but the proof was not required):

Fact : Let $0 < a < 1$ and suppose E is a bounded measurable set such that

$$m(E \cap I) \leq a m(I)$$

for every interval I. Then $m(E) = 0$. A particular case is $a = \frac{1}{2}$.

Problem 3: (25 points) Let E be a measurable set and $f : E \to \mathbb{R}$ Lebesgue integrable on E. Define $E_k = \left\{x \in E \mid |f(x)| < \frac{1}{k}\right\}$ for $k \geq 1$.

a) Show that each E_k is a measurable set.

SOLUTION. The function f is measurable (since it is integrable) and

$$E_k = f^{-1}((-\infty, \frac{1}{k})) \cap f^{-1}((-\frac{1}{k}, \infty))$$

is a measurable set. By definition, since f is measurable $f^{-1}((-\infty, \frac{1}{k}))$ and $f^{-1}((-\frac{1}{k}, \infty))$ are both measurable sets. $\square$

b) Determine whether $\{E_k\}$ is an increasing or decreasing collection of sets.

SOLUTION. Since $|f(x)| < \frac{1}{k+1}$ implies $|f(x)| < \frac{1}{k}$ we have that $E_{k+1} \subset E_k$, so $\{E_k\}$ is an increasing collection of sets. $\square$

c) Show that $\lim_{k\to\infty} \int_{E_k} |f|\, dm = 0$.

SOLUTION. Let $f_k = |f| \cdot \chi_{E_k}$ for $k \geq 1$. Then f_k is a decreasing sequence of non-negative measurable functions (product of two measurable functions) from part b). Note that each characteristic function χ_{E_k} is measurable from part a). Also, if f is measurable, then $|f|$ is measurable. We have $\int_E |f|\, dm < \infty$ since f is integrable. Since $E_k \subset E$ this gives that $\int_{E_1} f_1\, dm < \infty$. By the Monotone Convergence Theorem (the decreasing version from homework),

$$\lim_{k\to\infty} \int_{E_k} |f|\, dm = \lim_{k\to\infty} \int_E |f| \cdot \chi_{E_k}\, dm = \lim_{k\to\infty} \int_E f_k\, dm = \int_E \lim_{k\to\infty} f_k\, dm = 0,$$

since $f_k \searrow 0$ as $k \to \infty$. This is true because $f(x) = 0$ for $x \in \bigcap_{k=1}^{\infty} E_k$. $\square$

Problem 4: (30 points) Compute the following limit if it exists and justify the calculations. If the limit does not exist explain why it does not exist.

a) $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{\sqrt{n}}{\sqrt{x}}\cdot\chi_{[0,\frac{1}{n}]}\,dx$

SOLUTION. The function is nonnegative so the Lebesgue and Riemann integrals are the same, provided that the Riemann integral exists (as an improper integral in this case). We have

$$\int_0^1 \frac{\sqrt{n}}{\sqrt{x}}\cdot\chi_{[0,\frac{1}{n}]}\,dx=\int_0^{\frac{1}{n}}\frac{\sqrt{n}}{\sqrt{x}}\,dx=2\sqrt{n}\sqrt{x}\Big|_0^{\frac{1}{n}}=2\frac{\sqrt{n}}{\sqrt{n}}=2.$$

Hence the limit is 2. □

b) $\displaystyle\lim_{n\to\infty}\int_a^\infty \frac{n\sin(\sqrt{x})}{1+n^2x^2}\,dx$, where $a>0$

SOLUTION. We have

$$\left|\frac{n\sin(\sqrt{x})}{1+n^2x^2}\right|\le\left|\frac{n}{1+n^2x^2}\right|<\left|\frac{n}{n^2x^2}\right|=\frac{1}{nx^2}\le\frac{1}{x^2},$$

since $n\ge 1$ and $1+n^2x^2>n^2x^2$. The function $\frac{1}{x^2}$ is Lebesgue integrable on $[a,\infty)$ for $a>0$ since

$$\int_a^\infty \frac{1}{x^2}\,dx=\frac{-1}{x}\Big|_a^\infty=\frac{1}{a}<\infty.$$

As before, the function we are integrating is nonnegative so the Lebesgue and Riemann integrals are the same. Let $f_n(x)=\frac{n\sin(\sqrt{x})}{1+n^2x^2}$ for $n\ge 1$. We have shown that $|f_n(x)|<\frac{1}{x^2}$ and $\frac{1}{x^2}$ is Lebesgue integrable. Moreover, $f_n(x)\to 0$ as $n\to\infty$ pointwise for every x. By the Dominated Convergence Theorem it follows that

$$\lim_{n\to\infty}\int_a^\infty \frac{n\sin(\sqrt{x})}{1+n^2x^2}\,dx=0.$$

□

MAT324: Real Analysis – Fall 2016
Assignment 7

Due Thursday, **November 10**, in class.

Problem 1: Which of the following statements are true and which are false? Explain.

a) $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$

b) $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$

c) $L^1[3,5] \subset L^2[3,5]$

d) $L^2[3,5] \subset L^1[3,5]$

Problem 2: Let f be a positive measurable function defined on a measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$. Prove that

$$\left(\int_E f\,dm\right)\left(\int_E \frac{1}{f}\,dm\right) \geq m(E)^2.$$

Hint: Apply Cauchy-Schwarz inequality.

Problem 3: Let $f_n \in L^1(0,1) \cap L^2(0,1)$ for all $n \geq 1$. Prove or disprove the following:

a) If $\|f_n\|_1 \to 0$ then $\|f_n\|_2 \to 0$.

b) If $\|f_n\|_2 \to 0$ then $\|f_n\|_1 \to 0$.

Problem 4: Show that it is impossible to define an inner product on the space $\mathcal{C}([0,1])$ of continuous function $f : [0,1] \to \mathbb{R}$ which will induce the sup norm $\|f\|_{\text{sup}} = \sup\{|f(x)| : x \in [0,1]\}$.

Problem 5: Decide whether each of the following is Cauchy as a sequence in $L^2(0,\infty)$.

a) $f_n = \dfrac{1}{n^2}\chi_{[0,\frac{1}{n^3}]}$

b) $f_n = \dfrac{1}{x^2}\chi_{(n,\infty)}$.

MAT324: Real Analysis – Fall 2016
Assignment 7 – Solutions

Problem 1: Which of the following statements are true and which are false? Explain.

a) $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$

b) $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$

c) $L^1[3,5] \subset L^2[3,5]$

d) $L^2[3,5] \subset L^1[3,5]$

Solution.

a) The statement is false. The function

$$f(x) = \sum_{n=2}^{\infty} n\chi_{[n+\frac{1}{n^3}, n+\frac{2}{n^3}]}$$

belongs to $L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$

b) The statement is false. Indeed, the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\chi_{[n,n+1)}$$

belongs to $L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$.

c) The statement is false. To prove it, modify the function on part (a) in the following way. Let $C = \sum_{n=1}^{\infty} n^{-3}$. Define $s_0 = 3$, $s_n = s_{n-1} + \frac{1}{Cn^3}$, and $E_n = [s_{n-1}, s_n]$, for $n \in \mathbb{N}$ (notice that the E_n intersect only at the endpoints). Then

$$g(x) = \sum_{k=1}^{\infty} k\chi_{E_k} \in L^1[3,5] \setminus L^2[3,5]$$

d) It is true. Check proposition 5.3 on textbook. □

Problem 2: Let f be a positive measurable function defined on a measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$. Prove that

$$\left(\int_E f\,dm\right)\left(\int_E \frac{1}{f}\,dm\right) \geq m(E)^2.$$

Hint: Apply Cauchy-Schwarz inequality.

Solution. Follow the hint, if $f > 0$ then:

$$\left(\int_E (\sqrt{f})^2\,dm\right)^{\frac{1}{2}} \left(\int_E \frac{1}{(\sqrt{f})^2}\,dm\right)^{\frac{1}{2}} \geq \left(\int_E 1\,dm\right) = m(E).$$ □

Problem 3: Let $f_n \in L^1(0,1) \cap L^2(0,1)$ for all $n \geq 1$. Prove or disprove the following:

a) If $\|f_n\|_1 \to 0$ then $\|f_n\|_2 \to 0$.

b) If $\|f_n\|_2 \to 0$ then $\|f_n\|_1 \to 0$.

SOLUTION.

a) Consider the sequence of functions $f_n(x) = n^2 \chi_{[0,\frac{1}{n^3}]}$. Then $\|f_n\|_1 = n^{-1}$, so $\|f_n\|_1 \to 0$. On the other hand, $\|f_n\|_2 = \sqrt{n} \to \infty$.

b) Cauchy-Schwarz inequality gives $\|f_n\|_1 \leq \sqrt{\|f_n\|_2}$. This proves the statement. □

Problem 4: Show that it is impossible to define an inner product on the space $\mathcal{C}([0,1])$ of continuous function $f : [0,1] \to \mathbb{R}$ which will induce the sup norm $\|f\|_{\text{sup}} = \sup\{|f(x) : x \in [0,1]\}$.

SOLUTION. Recall that if a norm $\|\cdot\|$ on a space E is induced by an inner product, then it satisfies the parallelogram law:

$$\|x-y\|^2 + \|x+y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall \;\; x, y \in E$$

Consider the functions $f, g, \in \mathcal{C}([0,1])$, defined by

$$f(x) = \begin{cases} 1-2x & \text{if} \quad x \in [0, \frac{1}{2}] \\ 0 & \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \frac{1}{2}] \\ 2x-1 & \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

Check that the parallelogram law is not satisfied. □

Problem 5: Decide whether each of the following is Cauchy as a sequence in $L^2(0,\infty)$.

a) $f_n = \dfrac{1}{n^2} \chi_{[0,\frac{1}{n^3}]}$

b) $f_n = \dfrac{1}{x^2} \chi_{(n,\infty)}$.

SOLUTION. See Exercise 5.3 from the textbook. □

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MAT324: Real Analysis – Fall 2016
Assignment 8

Due Tuesday, **November 22**, in class.

Problem 1: Suppose $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$. Prove that f also belongs to $L^3(\mathbb{R})$.

Problem 2: Determine if the following functions belong to $L^\infty(\mathbb{R})$.

a) $f(x) = \frac{1}{x^2}\chi_{(0,n]}$ for some $n > 0$.

b) $f(x) = \frac{1}{\sqrt{x}}\chi_{[n,n^2]}$ for some $n > 0$.

Problem 3: Consider the function

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the sets $E_k = \{(x,y) \in [0,1] \times [0,1] : f(x,y) \in [k, k+1)\}$. Consider non-negative simple functions $\varphi_n = \sum_{k=0}^{n} k\chi_{E_k}$ for $k \geq 1$, and let $\varphi = \sum_{k=0}^{\infty} k\chi_{E_k}$. Using the definition of the integral, compute $\int_{[0,1]\times[0,1]} \varphi_n \, dm_2$ and $\int_{[0,1]\times[0,1]} \varphi \, dm_2$. Deduce that $f \notin L^1([0,1] \times [0,1])$.

Problem 4: Consider the measure spaces $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ where $X = Y = [0,1]$, $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{B}_{[0,1]}$ is the σ-algebra of Borel subsets of $[0,1]$. Let μ be the Lebesgue measure on $\mathcal{F}_1$ and ν be the counting measure on $\mathcal{F}_2$, that is $\nu(E) =$ number of elements in E if E is finite and $\nu(E) = \infty$ otherwise. Let $D = \{(x,y) \mid x = y\}$ and consider

$$D_n = \bigcup_{k=1}^{n} \left(\left[\frac{k-1}{n}, \frac{k}{n}\right] \times \left[\frac{k-1}{n}, \frac{k}{n}\right] \right)$$

a) Show that $D = \bigcap_{n=1}^{\infty} D_n$ and that $D \in \mathcal{F}_1 \times \mathcal{F}_2$.

b) Compute $\int_0^1 \int_0^1 \chi_D(x,y) \, d\mu(x) d\nu(y)$ and $\int_0^1 \int_0^1 \chi_D(x,y) d\nu(y) d\mu(x)$ and show that they are not equal.

Recall that χ_D is the characteristic function of the set D and $\chi_D(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

Remark: This problem does not contradict Theorem 6.12 since ν is not σ-finite.

MAT324: Real Analysis – Fall 2016
Assignment 8 – Solutions

Problem 1: Suppose $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$. Prove that f also belongs to $L^3(\mathbb{R})$.

Solution. Notice that $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ implies $|f| \in L^2(\mathbb{R})$, and $f^2 \in L^2(\mathbb{R})$. By Hölder's inequality,

$$\left| \int_{\mathbb{R}} |f| f^2 dx \right| \leq \left(\int_{\mathbb{R}} |f|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f^2|^2 dx \right)^{\frac{1}{2}} < \infty$$

Hence $f \in L^3(\mathbb{R})$. □

Problem 2: Determine if the following functions belong to $L^\infty(\mathbb{R})$.

a) $f(x) = \frac{1}{x^2}\chi_{(0,n]}$ for some $n > 0$.

b) $f(x) = \frac{1}{\sqrt{x}}\chi_{[n,n^2]}$ for some $n > 0$.

Solution.

a) f is not essentially bounded. Given $M > 0$, for any $\epsilon > 0$, we have $x = \frac{1}{\sqrt{M+\epsilon}}$. Then $f(x) = M + \epsilon > M$, implying that

$$m(\{x \in \mathbb{R} | |f(x)| > M\} \geq m\left(\left(0, \frac{1}{M}\right)\right) = \frac{1}{M} > 0.$$

b) f is bounded by $\frac{1}{\sqrt{n}}$ □

Problem 3: Consider the function

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the sets $E_k = \{(x,y) \in [0,1] \times [0,1] : f(x,y) \in [k, k+1)\}$. Consider non-negative simple functions $\varphi_n = \sum_{k=0}^{n} k\chi_{E_k}$ for $k \geq 1$, and let $\varphi = \sum_{k=0}^{\infty} k\chi_{E_k}$. Using the definition of the integral, compute $\int_{[0,1]\times[0,1]} \varphi_n \, dm_2$ and $\int_{[0,1]\times[0,1]} \varphi \, dm_2$. Deduce that $f \notin L^1([0,1] \times [0,1])$.

Solution. See Example 6.15 and Exercise 6.1 in the textbook. □

Problem 4: Consider the measure spaces $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ where $X = Y = [0,1]$, $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{B}_{[0,1]}$ is the σ-algebra of Borel subsets of $[0,1]$. Let μ be the Lebesgue measure on $\mathcal{F}_1$

and ν be the counting measure on $\mathcal{F}_2$, that is $\nu(E)$ = number of elements in E if E is finite and $\nu(E) = \infty$ otherwise. Let $D = \{(x, y) \mid x = y\}$ and consider

$$D_n = \bigcup_{k=1}^{n} \left(\left[\frac{k-1}{n}, \frac{k}{n} \right] \times \left[\frac{k-1}{n}, \frac{k}{n} \right] \right)$$

a) Show that $D = \bigcap_{n=1}^{\infty} D_n$ and that $D \in \mathcal{F}_1 \times \mathcal{F}_2$.

b) Compute $\int_0^1 \int_0^1 \chi_D(x, y)\, d\mu(x) d\nu(y)$ and $\int_0^1 \int_0^1 \chi_D(x, y) d\nu(y) d\mu(x)$ and show that they are not equal.

Recall that χ_D is the characteristic function of the set D and $\chi_D(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

Note: This problem does not contradict Theorem 6.12 since ν is not σ-finite.

SOLUTION.

a) Notice that if $(x, y) \in D_n$, then $|x - y| \leq \frac{1}{n}$. In particular, if $(x, y) \in \bigcap_{n=1}^{\infty} D_n$, then $|x - y| \leq \frac{1}{n}$, $\forall \ n \in \mathbb{N}$, hence $x = y \Rightarrow (x, y) \in D$. The other inclusion is trivial. Since D can be expressed as countable intersection of countable unions of products, it belongs to the product σ-algebra.

b)

$$\begin{aligned} \int_0^1 \int_0^1 \chi_D(x, y)\, d\mu(x) d\nu(y) &= \int_0^1 \left[\int_0^1 \chi_D(x, y) d\mu(x) \right] d\nu(y) \\ &= \int_0^1 0 \ d\nu(y) \\ &= 0 \end{aligned}$$

The second equality follows from the fact that for fixed y, $\chi_D(x, y)$ is 0 Lebesgue a.e. On the other hand,

$$\begin{aligned} \int_0^1 \int_0^1 \chi_D(x, y) d\nu(y) d\mu(x) &= \int_0^1 \left[\int_0^1 \chi_D(x, y) d\nu(y) \right] d\mu(x) \\ &= \int_0^1 1 \ d\mu(x) \\ &= 1 \end{aligned}$$

$\square$

MAT324: Real Analysis – Fall 2016
Assignment 9

Due Tuesday, **November 29**, in class.

Problem 1: Let $\mathcal{N} \subset [0,1]$ be a non-measurable set and let $\mathcal{C} \subset [0,1]$ be the Cantor middle-thirds set. Decide whether the following are true or false and explain your answer.

a) $\mathcal{N} \times \mathcal{C}$ is a Borel set;

b) $\mathcal{N} \times \mathcal{C}$ is a Lebesgue measurable set;

c) $\mathcal{N} \times \mathcal{C}$ is not measurable with respect to m_2, the Lebesgue measure on $\mathbb{R}^2$.

Problem 2: Consider the function

$$g(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_0^1 \int_0^1 g(x,y)\, dxdy = -1$ and $\int_0^1 \int_0^1 g(x,y)\, dydx = 1$. Is g an integrable function?

Problem 3: Compute

$$\int_{(0,\infty)\times(0,1)} y \sin(x) e^{-xy}\, dxdy,$$

and explain why Fubini's theorem is applicable.

MAT324: Real Analysis – Fall 2016
Assignment 9 – Solutions

Problem 1: Let $\mathcal{N} \subset [0,1]$ be a non-measurable set and let $\mathcal{C} \subset [0,1]$ be the Cantor middle-thirds set. Decide whether the following are true or false and explain your answer.

a) $\mathcal{N} \times \mathcal{C}$ is a Borel set;

b) $\mathcal{N} \times \mathcal{C}$ is a Lebesgue measurable set;

c) $\mathcal{N} \times \mathcal{C}$ is not measurable with respect to m_2, the Lebesgue measure on $\mathbb{R}^2$.

Solution. Part a) is false. Part b) is true since the set is null. Part c) is false since b) is true. □

Problem 2: Consider the function

$$g(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_0^1 \int_0^1 g(x,y)\, dxdy = -1$ and $\int_0^1 \int_0^1 g(x,y)\, dydx = 1$. Is g an integrable function?

Solution. Direct calculation shows that

$$\begin{aligned} \int_0^1 \left[\int_0^1 g(x,y)dx \right] dy &= \int_0^1 \left[\int_0^y \left(\frac{-1}{y}^2 \right) dx + \int_y^1 \frac{1}{x}^2 dx \right] dy \\ &= \int_0^1 (-1) dy \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \left[\int_0^1 g(x,y)dy \right] dx &= \int_0^1 \left[\int_0^x \frac{1}{x}^2 dy + \int_x^1 \left(\frac{-1}{y}^2 \right) dy \right] dx \\ &= \int_0^1 1 dx \\ &= 1 \end{aligned}$$

However, the function g is not integrable since its positive part $g^+ = f$ from Problem 3, HW8 is not integrable. □

Problem 3: Compute

$$\int_{(0,\infty)\times(0,1)} y \sin(x) e^{-xy}\, dxdy,$$

and explain why Fubini's theorem is applicable.

Solution. Notice that

$$\int_{(0,\infty)\times(0,1)} |y\sin(x)e^{-xy}|dxdy \leq \int_{(0,\infty)\times(0,1)} ye^{-xy}dxdy$$

The integrand on the right-hand side is measurable, so we can apply Tonelli's Theorem:

$$\begin{aligned}\int_{(0,\infty)\times(0,1)} ye^{-xy}dxdy &= \int_{(0,1)}\left[\int_{(0,\infty)} ye^{-xy}dx\right]dy\\ &= \int_{(0,1)}\left[\int_{(0,\infty)} \frac{d(e^{-xy})}{dx}dx\right]dy\\ &= \int_{(0,1)} 1dy = 1\end{aligned}$$

Therefore the integrand of $\int_{(0,\infty)\times(0,1)} |y\sin(x)e^{-xy}|dxdy$ is in L^1, and we can apply Fubini's theorem:

$$\begin{aligned}\int_{(0,\infty)\times(0,1)} y\sin(x)e^{-xy}dxdy &= \int_0^1 y\left[\int_0^\infty \sin(x)e^{-xy}dx\right]dy\\ &= \int_0^1 \left(\frac{y}{y^2+1}\right)dy = \frac{1}{2}\log(2)\end{aligned}$$

□

MAT324: Real Analysis – Fall 2016
ASSIGNMENT 10

Due Thursday, **December 8**, in class.

Problem 1: Let λ_1, λ_2 and μ be measures on a measurable space $(X, \mathcal{F})$. Show that if $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$ then $(\lambda_1 + \lambda_2) \ll \mu$.

Problem 2: Let $X = [0,1]$ with Lebesgue measure and consider probability measures μ and ν given by densities f and g as follows

$$\nu(E) = \int_E f\, dm \quad \text{and} \quad \mu(E) = \int_E g\, dm,$$

for every measurable subset $E \subset [0,1]$. Suppose $f(x), g(x) > 0$ for every $x \in [0,1]$. Is ν absolutely continuous with respect to μ (that is $\nu \ll \mu$)? If it is, determine the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$. Is μ absolutely continuous with respect to ν (that is $\mu \ll \nu$)?

Problem 3: Suppose μ is a σ-finite measure on $([0,1], \mathcal{F})$ and $E_1, E_2, \ldots, E_{2016}$ are measurable subsets of $[0,1]$. Define ν on $\mathcal{F}$ by $\nu(E) = \sum_{k=1}^{2016} \mu(E \cap E_k)$. Show that $\nu \ll \mu$ and find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Problem 4: Let $\lambda_1, \lambda_2, \mu$ be measures on a σ-algebra $\mathcal{F}$. Show that

a) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $(\lambda_1 + \lambda_2) \perp \mu$.

b) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$ then $\lambda_2 \perp \lambda_1$.

Problem 5: For a point x, define the Dirac measure δ_x to be

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For a fixed set B define the Lebesgue measure restricted to B by $m_B(A) = m(A \cap B)$. Let $\mu = \delta_1 + m_{[2,4]}$ and $\nu = \delta_0 + m_{(1,2)}$. Show that $\nu \perp \mu$.

MAT324: Real Analysis – Fall 2016
Assignment 10 – Solutions

Problem 1: Let λ_1, λ_2 and μ be measures on a measurable space $(X, \mathcal{F})$. Show that if $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$ then $(\lambda_1 + \lambda_2) \ll \mu$.

Solution. Suppose $E \in \mathcal{F}$ is such that $\mu(E) = 0$. Since $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, $\lambda_1(E) = \lambda_2(E) = 0$, so $(\lambda_1 + \lambda_2)(E) = 0$, thus $\lambda_1 + \lambda_2 \ll \mu$. $\square$

Problem 2: Let $X = [0, 1]$ with Lebesgue measure and consider probability measures μ and ν given by densities f and g as follows

$$\nu(E) = \int_E f \, dm \quad \text{and} \quad \mu(E) = \int_E g \, dm,$$

for every measurable subset $E \subset [0, 1]$. Suppose $f(x), g(x) > 0$ for every $x \in [0, 1]$. Is ν absolutely continuous with respect to μ (that is $\nu \ll \mu$)? If it is, determine the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$. Is μ absolutely continuous with respect to ν (that is $\mu \ll \nu$)?

Solution. Let E be a Lebesgue measurable set and let m denote the Lebesgue measure. Notice that since $f > 0$, $\mu(E) = \int_E f dm = 0$ iff $m(E) = 0$. In particular, if $\mu(E) = 0$, then $\nu(E) = \int_E g dm = 0$, for $m(E) = 0$, so $\nu \ll \mu$. The argument to show that $\mu \ll \nu$ is similar. Furthermore, since $\nu \ll \mu \ll m$, by proposition 7.7, (ii) in the textbook,

$$\frac{d\nu}{dm} = \frac{d\nu}{d\mu} \frac{d\mu}{dm},$$

so $\frac{d\nu}{d\mu} = \frac{f}{g}$. $\square$

Problem 3: Suppose μ is a σ-finite measure on $([0, 1], \mathcal{F})$ and $E_1, E_2, \ldots, E_{2014}$ are measurable subsets of $[0, 1]$. Define ν on $\mathcal{F}$ by $\nu(E) = \sum\limits_{k=1}^{2014} \mu(E \cap E_k)$. Show that $\nu \ll \mu$ and find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Solution. Notice that

$$\mu(E \cap E_k) = \int_E \chi_{E_k} dm,$$

So for each measure ν_k, defined by $\nu_k(E) = \mu(E \cap E_k)$, we have $\nu_k \ll m$. By problem 2, $\nu = \sum_{k=1}^{2014} \nu_k \ll m$. In addition, from $\nu_k(E) = \int_E \chi_{E_k} dm$, we know that $\frac{d\nu_k}{dm} = \chi_{E_k}$. By proposition 7.7 (i) in the textbook, $\frac{d\nu}{dm} = \sum_{k=1}^{2014} \chi_{E_k}$. $\square$

Problem 4: Let $\lambda_1, \lambda_2, \mu$ be measures on a σ-algebra $\mathcal{F}$. Show that

a) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $(\lambda_1 + \lambda_2) \perp \mu$.

b) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$ then $\lambda_2 \perp \lambda_1$.

SOLUTION.

a) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then there exist disjoint sets A_i, B_i, $i = 1, 2$, such that $A_i \cup B_i = X$, $\mu(A_i) = 0$, $\lambda_i(B_i) = 0$. Then the sets $A = A_1 \cup A_2$ and $B = B_1 \cap B_2$ satisfy $X = A \cup B$, $A \cap B = \emptyset$, and

$$\begin{aligned} \mu(A) &\le \mu(A_1) + \mu(A_2) = 0 \\ (\lambda_1 + \lambda_2)(B) &\le \lambda_1(B_1) + \lambda_2(B_2) = 0 \end{aligned}$$

This implies that $(\lambda_1 + \lambda_2) \perp \mu$.

b) Suppose $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$. Let $A, B \in \mathcal{F}$ be disjoint sets, $A \cup B = X$, and $\lambda_2(A) = 0$, $\mu(B) = 0$. Since $\lambda_1 \ll \mu$, we also have $\lambda_1(B) = 0$, so $\lambda_2 \perp \lambda_1$. $\square$

Problem 5: For a point x, define the Dirac measure δ_x to be

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For a fixed set B define the Lebesgue measure restricted to B by $m_B(A) = m(A \cap B)$. Let $\mu = \delta_1 + m_{[2,4]}$ and $\nu = \delta_0 + m_{(1,2)}$. Show that $\nu \perp \mu$.

SOLUTION. μ and ν are concentrated on disjoint sets, namely $\{1\} \cup [2, 4]$ and $\{0\} \cup (1, 2)$, respectively, so they are mutually singular. $\square$

MAT 324 – Real Analysis
Fall 2014

Final Exam – December 12, 2014

NAME: ______________________________

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Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
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TOTAL	

Problem 1: (20 points) Determine whether the following statements are true or false. No further explanation is necessary.

(1) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

The outer measure is countably subadditive.

(2) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

All Lebesgue measurable sets are also Borel sets.

(3) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

For any subset $A \subset [0,1]$ the characteristic function $\mathbb{1}_A$ defined by $\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ is measurable.

(4) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

$f \in L^1(\mathbb{R})$ if and only if $|f| \in L^1(\mathbb{R})$.

(5) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

It is possible to define an inner product on the space $L^4[0,1]$ with the norm $\|\cdot\|_4$.

(6) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

There exist functions which belong to $L^4(0,1)$ but not to $L^2(0,1)$.

(7) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

If $f, g \in L^1(\mathbb{R})$ then $h \in L^1(\mathbb{R}^2)$, where $h(x,y) = f(x)g(y)$.

(8) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

Let $\mu(E) = \displaystyle\int_E e^{-\pi x^2} dx$ for any Borel subset $E \subset \mathbb{R}$. Then $\mu(\mathbb{R}) = 1$ and $\mu \ll m$.

(9) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

Let μ, ν and λ be σ-finite measures. If $\mu \ll \nu$ and $\nu \ll \lambda$ then $\mu \ll \lambda$.

(10) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

Let $\lambda_1, \lambda_2, \mu$ be σ-finite measures on a σ-field $\mathcal{F}$. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $\lambda_1 \perp \lambda_2$.

Problem 2: (12 points) Consider the set $K = \{\frac{1}{n} \mid n \geq 1\}$ and the function

$$f(x) = \begin{cases} x^2 & \text{if } x \notin K \\ 1 & \text{if } x \in K. \end{cases}$$

Explain why f is measurable and compute $\displaystyle\int_{[0,1]-K} f\, dm$.

Problem 3: (10 points) Let $E \subset [0,1]$ be a measurable set such that for any interval $(a,b) \subset [0,1]$ we have

$$m(E \cap (a,b)) \geq \frac{1}{2}(b-a).$$

Show that $m(E) = 1$.

Problem 4: (12 points) Let $f \in L^1(0,1)$. Compute the following limit if it exists or explain why it does not exist:

$$\lim_{k\to\infty} \int_0^1 k \, \ln\left(1 + \frac{|f(x)|}{k^2}\right) \, dx.$$

Problem 5: (12 points) Compute

$$\int_{(0,\pi)\times(0,\infty)} xy\sin(x)e^{-xy^2}\,dxdy$$

and explain how Fubini's theorem is used.

Problem 6: (12 points) Consider the measurable space $([0,1], \mathcal{F})$, where $\mathcal{F} = \mathcal{B}_{[0,1]}$ is the σ-algebra of Borel subsets of $[0,1]$. Let μ be a σ-finite measure on $\mathcal{F}$ with $\mu \ll m$. Let ν be the counting measure on $\mathcal{F}$, that is $\nu(E) =$ number of elements in E if E is finite and $\nu(E) = \infty$ otherwise. Let C be the Cantor middle-thirds set from $[0,1]$.

a) Explain why $D = C \times C$ belongs to the product σ-algebra $\mathcal{F} \times \mathcal{F}$.

b) Compute $\displaystyle\int_0^1 \int_0^1 \mathbb{1}_D(x,y)\, d\mu(x) d\nu(y)$. Recall that $\mathbb{1}_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D. \end{cases}$

c) Compute $\displaystyle\int_0^1 \int_0^1 \mathbb{1}_D(x,y) d\nu(y) d\mu(x)$. Does this example contradict Fubini's theorem?

Problem 7: (12 points) Suppose μ is a σ-finite measure on $([0,1],\mathcal{F})$ and E_1, E_2 are two measurable subsets of $[0,1]$. Define ν by

$$\nu(E) = \frac{1}{4}\mu(E \cap E_1) + \frac{3}{4}\mu(E \cap E_2), \text{ for all } \; E \in \mathcal{F}.$$

a) Compute $\nu(E_1 \cap E_2)$.

b) Show that $\nu \ll \mu$.

c) Find the Radon-Nikodym derivative $\dfrac{d\nu}{d\mu}$.

Problem 8: (10 points) Let $f \in L^2(0,\infty) \cap L^5(0,\infty)$. Show that $f \in L^3(0,\infty)$.

– Scratch paper –

MAT 324 – Real Analysis
Fall 2014

Final Exam (Solutions) – December 12, 2014

NAME: ______________________________

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PROBLEM	**SCORE**
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Problem 1: (20 points) Determine whether the following statements are true or false. No further explanation is necessary.

(1) $\boxed{\text{TRUE}}$

The outer measure is countably subadditive.

(2) $\boxed{\text{FALSE}}$

All Lebesgue measurable sets are also Borel sets.

(3) $\boxed{\text{FALSE}}$

For any subset $A \subset [0,1]$ the characteristic function $\mathbb{1}_A$ defined by $\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ is measurable.

(4) $\boxed{\text{TRUE}}$

$f \in L^1(\mathbb{R})$ if and only if $|f| \in L^1(\mathbb{R})$.

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(7) $\boxed{\text{TRUE}}$

If $f, g \in L^1(\mathbb{R})$ then $h \in L^1(\mathbb{R}^2)$, where $h(x,y) = f(x)g(y)$.

(8) $\boxed{\text{TRUE}}$

Let $\mu(E) = \displaystyle\int_E e^{-\pi x^2} dx$ for any Borel subset $E \subset \mathbb{R}$. Then $\mu(\mathbb{R}) = 1$ and $\mu \ll m$.

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Let μ, ν and λ be σ-finite measures. If $\mu \ll \nu$ and $\nu \ll \lambda$ then $\mu \ll \lambda$.

(10) $\boxed{\text{FALSE}}$

Let $\lambda_1, \lambda_2, \mu$ be σ-finite measures on a σ-field $\mathcal{F}$. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $\lambda_1 \perp \lambda_2$.

Problem 2: (12 points) Consider the set $K = \{\frac{1}{n} \mid n \geq 1\}$ and the function

$$f(x) = \begin{cases} x^2 & \text{if } x \notin K \\ 1 & \text{if } x \in K. \end{cases}$$

Explain why f is measurable and compute $\displaystyle\int_{[0,1]-K} f\, dm$.

Solution. The set K is countable so $m(K) = 0$. The function $f = x^2$ a.e. and $x \to x^2$ is a continuous function, hence measurable. This implies that f is measurable and $\int_{[0,1]-K} f\, dm = \int_0^1 x^2\, dx = \frac{1}{3}$. $\square$

Problem 3: (10 points) Let $E \subset [0,1]$ be a measurable set such that for any interval $(a,b) \subset [0,1]$ we have

$$m(E \cap (a,b)) \geq \frac{1}{2}(b-a).$$

Show that $m(E) = 1$.

SOLUTION. We look at the complement of E and note that

$$m(E^c \cap (a,b)) \leq \frac{1}{2}(b-a).$$

It follows (an shown in class) that $m(E^c) = 0$ so $m(E) = 1$. $\square$

Problem 4: (12 points) Let $f \in L^1(0,1)$. Compute the following limit if it exists or explain why it does not exist:

$$\lim_{k\to\infty} \int_0^1 k \,\ln\left(1 + \frac{|f(x)|}{k^2}\right)\,dx.$$

Solution. Note that

$$\left(1 + \frac{|f(x)|}{k^2}\right)^{k^2}$$

is an increasing function of k and increases to $e^{|f(x)|}$. Therefore

$$k\,\ln\left(1 + \frac{|f(x)|}{k^2}\right) = \frac{1}{k}\left(1 + \frac{|f(x)|}{k^2}\right)^{k^2} \leq \frac{|f(x)|}{k} \leq |f(x)|.$$

Since $f \in L^1(0,1)$, we can apply DCT and get that

$$\lim_{k\to\infty} \int_0^1 k\,\ln\left(1 + \frac{|f(x)|}{k^2}\right)\,dx \leq \lim_{k\to\infty} \int_0^1 \frac{|f(x)|}{k}\,dx = \lim_{k\to\infty} \frac{\|f\|_1}{k} = 0.$$

$\square$

Problem 5: (12 points) Compute

$$\int_{(0,\pi)\times(0,\infty)} xy\sin(x)e^{-xy^2}\,dxdy$$

and explain how Fubini's theorem is used.

Solution. By Tonelli-Fubini Theorem, we get that

$$\begin{aligned}\int_{(0,\pi)\times(0,\infty)} xy\sin(x)e^{-xy^2}\,dxdy &= \int_0^\pi\int_0^\infty xy\sin(x)e^{-xy^2}\,dydx = \int_0^\pi \sin(x)\frac{1}{2}e^{-xy^2}\Big|_0^\infty\,dx\\ &= -\frac{1}{2}\cos(x)\Big|_0^\pi = 1.\end{aligned}$$

$\square$

Problem 6: (12 points) Consider the measurable space $([0,1], \mathcal{F})$, where $\mathcal{F} = \mathcal{B}_{[0,1]}$ is the σ-algebra of Borel subsets of $[0,1]$. Let μ be a σ-finite measure on $\mathcal{F}$ with $\mu \ll m$. Let ν be the counting measure on $\mathcal{F}$, that is $\nu(E) =$ number of elements in E if E is finite and $\nu(E) = \infty$ otherwise. Let C be the Cantor middle-thirds set from $[0,1]$.

a) Explain why $D = C \times C$ belongs to the product σ-algebra $\mathcal{F} \times \mathcal{F}$.

Solution. We know that $C \in \mathcal{F}$ so, by definition of $\mathcal{F} \times \mathcal{F}$, $D \in \mathcal{F} \times \mathcal{F}$. $\square$

b) Compute $\displaystyle\int_0^1 \int_0^1 \mathbb{1}_D(x,y)\, d\mu(x) d\nu(y)$. Recall that $\mathbb{1}_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D. \end{cases}$

Solution. The value of the integral is 0. $\square$

c) Compute $\displaystyle\int_0^1 \int_0^1 \mathbb{1}_D(x,y) d\nu(y) d\mu(x)$. Does this example contradict Fubini's theorem?

Solution. The value of the integral is also 0. The integrals are the same, so this does not contradict Fubini's theorem. However Fubini's theorem does not apply here because ν is not a σ-finite measure. $\square$

Problem 7: (12 points) Suppose μ is a σ-finite measure on $([0,1],\mathcal{F})$ and E_1, E_2 are two measurable subsets of $[0,1]$. Define ν by

$$\nu(E) = \frac{1}{4}\mu(E \cap E_1) + \frac{3}{4}\mu(E \cap E_2), \text{ for all } \ E \in \mathcal{F}.$$

a) Compute $\nu(E_1 \cap E_2)$.

SOLUTION. $\nu(E_1 \cap E_2) = \mu(E_1 \cap E_2)$. $\square$

b) Show that $\nu \ll \mu$.

SOLUTION. Clearly if $\mu(E) = 0$ then $\mu(E \cap E_1) = 0$ and $\mu(E \cap E_2) = 0$. So $\nu(E) = 0$ and $\nu \ll \mu$. $\square$

c) Find the Radon-Nikodym derivative $\dfrac{d\nu}{d\mu}$.

SOLUTION. As in the homework, the Radon-Nikodym derivative is $\frac{1}{4}\mathbb{1}_{E_1} + \frac{3}{4}\mathbb{1}_{E_2}$. $\square$

Problem 8: (10 points) Let $f \in L^2(0,\infty) \cap L^5(0,\infty)$. Show that $f \in L^3(0,\infty)$.

SOLUTION. Define sets $E_1 = \{x \in (0,\infty) \mid |f(x)| \leq 1\}$ and $E_2 = \{x \in (0,\infty) \mid |f(x)| > 1\}$. Then E_1 and E_2 are disjoint and $E_1 \cup E_2 = (0,\infty)$. We have

$$\begin{aligned}\int_0^\infty |f(x)|^3\,dx &= \int_{E_1} |f(x)|^3\,dx + \int_{E_2} |f(x)|^3\,dx \\ &\leq \int_{E_1} |f(x)|^2\,dx + \int_{E_2} |f(x)|^5\,dx < \infty.\end{aligned}$$

So $f \in L^3(0,\infty)$. $\square$

MAT 324: REAL ANALYSIS – FALL 2016

GENERAL INFORMATION

Instructor. Remus Radu
Email: remus.radu@stonybrook.edu
Office: Math Tower 4-103, Phone: (631) 63**2-8266**
Office Hours: Wednesday 12-2pm in Math Tower 4-103
Monday 3-4pm in MLC, or by appointment

Course Grader. El Mehdi Ainasse
Email: elmehdi.ainasse@stonybrook.edu
Office Hours: Wednesday 4-5pm in Physics D-107
Monday 4-4:30pm, Wednesday 10-11am, and Friday 12-12:30pm in MLC

Lectures. TuTh 10:00-11:20am in Library W4535.

Blackboard. Grades and some course administration will take place on Blackboard. Please login using your NetID at http://blackboard.stonybrook.edu. The course website is http://www.math.stonybrook.edu/~rradu/MAT324FA16. Please check regularly for the weekly lecture schedule updates and homework.

Course Description. In this course we will discuss Lebesgue measure, Lebesgue integration, metric spaces (including compactness, connectedness, completeness, and continuity), aspects of Fourier series, function spaces, Hilbert spaces and Banach spaces. After developing the basic theory we will also give some applications to Probability. The main results will be the Monotone Convergence Theorem, the Dominated Convergence Theorem, the Radon-Nykodyn Theorem, and the Central Limit Theorem.

Prerequisites. C or higher in MAT 203 or 205 or 307 or AMS 261; B or higher in MAT 320.

Textbook. The following textbook is required:

> Marek Capinski and Ekkehard Kopp, *Measure, Integral and Probability*, 2nd ed., Springer-Verlag, Springer Undergraduate Mathematics Series, ISBN 1-85233-781-8.

Other general references will be posted on the course webpage.

Exams. There will be a midterm and a final exam, scheduled as follows:

- Midterm – Thursday, October 25, 10:00-11:20am, in Library W4535.
- Final Exam – Friday, December 16, 11:15am-1:45pm, TBA.

There will be no make-up exams.

Grading policy. Grades will be computed using the following scheme:

Homework – 25%
Midterm – 30%
Final Exam – 45%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading. There will be (roughly) weekly homework assignments. You may work together on your problem sets, and you are encouraged to do so. However, all solutions must be written up independently.

Extra Help. You are welcome to attend the office hours and ask questions about the lectures and about the homework assignments. In addition, math tutors are available at the MLC: http://www.math.sunysb.edu/MLC.

Information for students with disabilities. If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631) 632-6748, or at the following website http://studentaffairs.stonybrook.edu/dss/index.shtml. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential.

Academic integrity. Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty is required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary.

Critical Incident Management. Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of University Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures. Further information about most academic matters can be found in the Undergraduate Bulletin, the Undergraduate Class Schedule, and the Faculty-Employee Handbook.

Oct 18, 2016

Example : Compute the limit if it exists:

$$\lim_{n\to\infty} \int_0^1 \frac{n^2 x e^{-n^2x^2}}{1+x}\, dx .$$

Solution :

by changing variables, $y = nx$, we see that

$$\int_0^1 \frac{n^2 x e^{-n^2x^2}}{1+x}\, dx = \int_0^n \frac{y e^{-y^2}}{1+\frac{y}{n}}\, dy = \int_0^\infty \frac{y e^{-y^2}}{1+\frac{y}{n}} \chi_{[0,n]}(y)\, dy$$

Let $g_n(y) = \frac{y e^{-y^2}}{1+\frac{y}{n}} \chi_{[0,n]}(y)$, $n \geq 1$. This is a sequence of measurable functions (because they are continuous) with

$$|g_n(y)| \leq g(y) \quad \text{where } g(y) = y e^{-y^2} \text{ for } y \in [0,\infty).$$

g is an integrable function as $\int_0^\infty y e^{-y^2} dy = -\frac{e^{-y^2}}{2}\Big|_0^\infty = \frac{1}{2}$.

By DCT we have

$$\lim_{n\to\infty} \int_0^\infty g_n(y)\, dy = \int_0^\infty \lim_{n\to\infty} g_n(y)\, dy = \int_0^\infty y e^{-y^2} dy = \frac{1}{2}.$$

but now

$$\int_0^1 \frac{n^2 x e^{-n^2x^2}}{1+x}\, dx = \int_0^\infty g_n(y)\, dy \quad \text{so} \quad \lim_{n\to\infty} \int_0^1 \frac{n^2 x e^{-n^2x^2}}{1+x}\, dx = \frac{1}{2}.$$

$\square$

Remarks :

① We apply DCT for the sequence $g_n(y)$, not for $f_n(x) = \frac{n^2 x e^{-n^2x^2}}{1+x}$ because we only know $|g_n(y)| \leq g(y)$ not $|f_n(x)| \leq g(y)$!

If we apply DCT for f_n we get

$$\dots = \int_0^1 \lim_{n\to\infty} \left(\frac{n^2 x e^{-n^2x^2}}{1+x}\right) dx = 0,$$ which is not the correct answer.

(2) The function $g(y) = y e^{-y^2}$ on $[0, \infty)$ has an improper Riemann integral and $g(y) \geq 0$. So the Lebesgue integral exists and is equal to the Riemann integral. So $g \in L^1(0, \infty)$ and we can apply the DCT.

Note that DCT is applicable to Lebesgue integrable functions.