Department of Mathematics
Mat 324: Real Analysis
Fall 2012

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Classroom: Physics 129 Time: Tu,Th 10:00-11:20

Text: Measure, Integral and Probability' by Marek Capinski and Ekkehard Kopp (Springer-Verlag, Springer Undergraduate Mathematics Series, ISBN 1-85233-781-8)

Prerequisites: C or higher in MAT 203 or 205 or 307 or AMS 261; B or higher in MAT 320

Course Description: The central concepts of the course are Lebesgue measure and the Lebesgue Integral, which is a generalization of the Riemann Integral. After developing the basic theory we will give some applications to Probability.

Homework: There will be weekly assignments, though we may skip some weeks such as the week of the midterm exam.
Click on: First Assignment
Due 9/18/2012
Click on: Second Assignment
Due 10/2/2012
Click on: Third Assignment
Due 10/11/12
Click on: Fourth Assignment
Due on 10/23/12
Click on: Fifth Assignment
Due on 11/15/12
Click on: Sixth assignment
Due on 12/6/12

Examinations: There will be a midterm exam on October 25 in class and a final exam on

December 14 at 11:15AM.
See:

[Review Sheet]

[Review sheet for final]

Grading: The homework assignments will count 20%, the midterm exam 30% and the final exam 50%.

If you have a physical, psychological, medical or learning disability that my impact on your ability to carry out assigned course work, please contact the staff in the Disabled Student Service Office, Room 133, Humanities, 632-6784/TDD. DSS will review your concerns and determine with you what accommodations are necessary and appropriate. All information and documentation of disability is confidential.

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PROBLEM SET 1

1. Let X be the set of all real roots of all polynomials with integer coefficients. Is X countable or uncountable? Explain why.
2. Define $x \sim y$ if $x - y$ is rational. Prove this is an equivalence relation on the reals. How many distinct equivalence classes are there?
3. Is the function $f(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2^n x)$ Riemann integrable on $[0, 2\pi]$? Explain why or why not.
4. Is there a compact, uncountable set of real numbers which contains no rational numbers? Give an example or prove no such set exists.
5. What is the average distance between two random points in $[0, 1]$? We have not had enough theory yet to make this precise, but see if you can come up with a plausible number and explanation for it.

PROBLEM SET 2

1. Prove that the Lebesgue function F (defined on page 20 in our book) is continuous.

2. If $E \subset [0,1]$ is a null set, and $f : [0,1] \to [0,1]$ is continuous, does $f(E)$ have to be a null set as well? Prove this or find a counterexample.

3. Let $X = \{x + y : x, y \in C\}$ be the set of sums of numbers in the Cantor middle third set. What is X?

4. Prove that if $\lambda > 0$ then $m^*(\lambda E) = \lambda m^*(E)$ where $\lambda E = \{\lambda x : x \in E\}$.

5. If X is set of finite Lebesgue measure show that $m(X \cap X + t) \to 0$ as $t \to \infty$. Here $X + t = \{x + t : x \in X\}$. Does there have to be a value of t so that $m(X \cap X + t) = 0$?

PROBLEM SET 3

1. Compute the volume of a ball of radius R in n-space. You should get one type of formula for odd n and another for even n, though in both cases n will appear in the formula.

2. Suppose $\{f_n\}$ is a sequence of continuous functions and let E be the set of x's where the seqeunce converges. Show that E is Borel, and hence Lebesgue measurable.

3. Prove that every open set of real numbers is a countable union of disjoint intervals.

4. Must the continuous image of a measurable set be measureable?

5. Prove that a set E of reals is measurable if and only if for all $\epsilon > 0$, there exists an open set U such that $E \subseteq U$ and $m^*(U - E) < \epsilon$.

PROBLEM SET 4

1. Prove that every monotone function for the reals to the reals is measurable.

2. A function is called simple if it only takes on finite number of different values. If g is bounded and measurable, and $\epsilon > 0$ is given, show there is a measurable simple function f so that $\sup_x |g(x) - f(x)| \leq \epsilon$. Is this true if g is not bounded?

3. Suppose E is measurable set of real numbers and let $f(t) = m(E \cap (t-1, t+1))$. Show that f is continuous.

4. We will prove in class that if $f$is measurable then so is f^2. Is the converse also true; that is does f^2 measurable imply that f is also measurable? Prove this or give a counterexample.

5. Prove that if f is measurable and g equals f almost everywhere, the g is also measurable.

PROBLEM SET 5

1. Compute the integral of the Cantor-Lebesgue function $\int_0^1 F(x)dm$ from Chapter 2.

2. What is $\lim_{n\to\infty} \int_{-\infty}^{\infty} x^n e^{-n|x|} dm$? Find the limit and prove your answer.

3. Suppose $\{f_n\}$ is a sequence of functions that converges almost everywhere to a function f and define $F_n = \sup_{k=1,\ldots,n} |f_n|$. Show that if the integrals of F_n remain bounded as $n \to \infty$ then $\lim_n \int f_n dm = \int f dm$.

4. Show that $\sum_{n=1}^{\infty} \cos^n(2^n x)$ converges for a.e. x, but diverges on a dense set of x's.

5. Let m be a measure defined on Borel sets in the reals $\mathbf{R}$ by:

$$m(E) = \int_E \frac{dx}{1+x^2}.$$

   Find $m(\mathbf{R})$.

PROBLEM SET 6

1. Let $f_n(x) = \sin(nx)$. Does the sequence f_n converge in $L^1[0, 2\pi]$?

2. Prove that if X is a metric space and p and q are distinct points of X, then there exit two disjoint subsets of X, one of which contains p and the other of which contains q.

3. Give an example of a topological space X whose topology does not come from any metric. That is, there is no metric on X whose open sets coincide with the open sets of the topological space X.

4. Give an example of a continuous function f such that the improper integral $\int_{-\infty}^{\infty} f(x)dx$ exists but f is not in $L^1(\mathbf{R})$. Explain why your example cannot be a positive function.

5. Let $f_n = n1_{[0,1/n]}$. Does there exist an integrable function g such that $f_n \leq g$?

Midterm 1, MAT 324, October 28, 2007

Answer each question on the paper provided. Write neatly and give complete answers. Each question is worth 10 points.

1. Define the outer measure of a set.
2. Prove every monotone function is measurable.
3. If f is measurable, show $|f|$ is measurable. Is the converse true? Prove or find give a counterexample.
4. State the characterization of Riemann integrable functions. Given an example of Lebesgue integrable function that is not Riemann integrable.
5. If E is measurable is it true that $m(E) = m(\overline{E})$, where $\overline{E}$ denote the closure of E? (the closure of a set E is the smallest closed set containing E).
6. If $E \subset [0, 1]$ is measurable, show that for any $\epsilon > 0$ there is a an closed set K so that $K \subset E$ and $m(E \setminus K) < \epsilon$.
7. Let $\{f_n\}$ be a sequence of measurable functions. Show that the set of x where $f_n(x)$ tends to $+\infty$ is measurable.
8. Suppose $f \geq 0$ is integrable and define $h_n = \min(f, n)$. Prove that $\int |f - h_n| dm \to 0$ as $n \to \infty$.
9. State the Dominated Convergence Theorem. Give an example of a uniformly bounded sequence of integrable functions where it does not apply, i.e., $\lim_n \int f_n dx \neq \int \lim_n f_n dx$.
10. If f is integrable, show $m(\{x : |f(x)| > \lambda\}) \leq \frac{1}{\lambda} \int |f| dx$.

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Review for final

Things to know and do for the final exam:
Everything from before the midterm; e. g.:

1. Finite, countable and uncountable sets
2. The power set of a set. Prove that the number of elements in the power set of X is greater than the number of elements in X. Corollary: the set of real numbers is uncountable.
3. Prove that there are countably many rational numbers.
4. Let $f : X \to Y$. and let $S \subset Y$. Define $f^{-1}(S)$.
5. Define an equivalence relation and an equivalence class. Let x and y be real numbers and define $x \sim y$ iff $x - y$ is rational. Is this an equivalence relation?
6. Indicator or characteristic function,
7. Define open sets of the reals and also closed sets.
8. Define a sigma field and Borel sets.
9. Understand the least upper bound property of the reals
10. Balls, open sets and rectangles in $\mathbf{R}^n$.
11. Define the Riemann integral and understand the Riemann criterion which guarantees that the integral exists.
12. Define the sup-norm and L^2-norm of a function. Does sup-norm or pointwise convergence of functions imply convergence of their Riemann integrals?
13. Define outer measure of a subset of $\mathbf{R}$ and define a Lebesque measurable set.
14. Prove that the measure of an interval is the length of the interval.
15. Prove that a countable set has Lebesgue measure zero.
16. Define the Cantor set and show that it is uncountable and has measure zero.
17. Construct a non-measurable set.
18. Show that the set of Lebesgue measurable sets is a sigma field.
19. Define a probability measure and conditional probability.
20. Define independence of sets and sigma fields with respect to a probability measure.
21. Define Lebesque and Borel measurable functions.

22. Show that the sum and product of measurable functions is measurable with respect to Lebesgue or Borel measure.

Items from after the midterm

1. Simple functions and their integrals and definition and properties of Lebesque integral
2. Definition of essential supremum and essential infimum
3. Dirac measure
4. Fatou's lemma
5. Monotone and dominated convergence theorems for sequences of functions
6. The equivalence of functions defined by "almost everywhere"
7. The bell curve $f(x) = \frac{1}{\pi}e^{-x^2/2}$
8. Topological, metric and vector spaces and relations between them
9. Norms and inner products and the Schwartz inequality
10. The spaces $L^1(E)$ and $L^2(E)$
11. Beppo Levi theorem
12. Relations between the Riemann and Lebesgue integrals
13. A function is Riemann intgrable on an interval iff its discontinuities form a set of measure zero.
14. Improper Riemann integrals and their relation to the Lebesque integral.
15. $L^1(E)$ is complete