Math 362 - Differential Geometry of Curves and Surfaces (Spring 2017)

Instructor: Ben McMillan
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Locataion: TuTh 1:00pm--2:20pm in Earth & Space 181

Announcements

- The final is here.

Course Information:

The course syllabus is here. Some critical points:

- The midterm will be in class on Thursday March 23.
- My office hours this term are Tuesdays 2:30--3:30pm and Wednesdays 3--4pm in Simons 510. I will also be available in the MLC (S-235) on Thursdays 11am--12pm.

Schedule and Homework:

The following is a tentative schedule for the course. As homework is assigned it will be posted here. You are encouraged to work with others, but please make sure to write up solutions in your own words.

For each homework you turn in, please write an estimate of how long it took you for each problem.

Each week 3 problems will be graded. The ones in parenthesis are ones that I think you will benefit from, but won't be graded. You should still do them!

Week	Date	Topic(s) Covered	Reading	Homework
1	1/24	Properties of R^3	Chapter 1-2	3, 5
	1/26	Smooth maps, curves	1-3	6, 8, 10
2	1/31	Geometry of the vector product	1-4	1, 6
	2/2	Local theory of curves (invariants)	1-5	1, 2, 4, 6, 11
3	2/7	Regular surfaces	2-2	1, 8, 11, 12, 13
	2/9	Snow Day!	2-2	Read the book!
4	2/14	Regular surfaces continued	2-2	16
	2/16	Differentiable functions on surfaces	2-3	1,(2),3,13,(16)
5	2/21	Tangent planes of a regular surface	2-4	1, (2), (8), 21, (24), (25)
	2/23	Vectors, the first fundamental form	2-5	1 b & d, 3, (9, see pp 76-77)
6	2/28	Area	2-5	5, (12), (14)
	3/2	Oriented surfaces	2-6	(2), 4, 5, 7
7	3/7	The Gauss map	3-2	(4), (8)
	3/9	The Gauss map (continued)	3-2	.
8	3/14	Spring Break!	.	.
	3/16	Spring Break!	.	.
9	3/21	Review	.	.
	3/23	Midterm	.	.
10	3/28	What is Curvature	3.2 3.3	2, (4), 8, (16) 1
	3/30	Isometries	4.2	1, 2, (3)
11	4/4	Isometries	4.2	(7), (8), 9, 11
	4/6	Geodesics	4.4	3, (4), 5a
12	4/11	Connections	4.3	3, 8, (9)
	4/13	Parallel vector fields, geodesics	4.4	4, 15
13	4/18	More Parallel transport	4.4	6, 12, (13)
	4/20	The Gauss Bonnet theorem	4.5	1, (2), 4
14	4/25	Local Gauss Bonnet	4.5	Compute the Euler Characteristic of the projective plane.
	4/27	Finish proof of Gauss Bonnet	4.5	2, 3
15	5/2	So you want to know what is a category		None
	5/4	And some homotopy theory		None

Name: SID:

Math 362 Midterm

- You have 36 hours from opening this exam to complete it and write up your solutions. This is on the honor system. Turn it in to my office or to Diane Williams before 3pm on Wednesday May 17.
- Please work your answers out on scratch paper and then write your solutions neatly in the provided space. You may use the back side of each page if you need more space to work, but if you do, please make a note so that the graders will see it.
- The only sources you may consult during this exam are the textbook and any notes you have taken in class. Do not search on the internet or discuss with friends until you have turned in your exam. Again, honor system. Don't let me down :)
- Please email me ASAP if something is unclear! I will do my best to respond quickly.

Name: SID:

Problem 1. (15 points) Say that a diffeomorphism $f\colon S_1 \to S_2$ between surfaces *preserves arc length* if for all $\gamma\colon I \to S_1$,

$$\text{Length}(\gamma) = \text{Length}(f \circ \gamma).$$

Prove that f is an isometry if and only if it preserves arc length.

Problem 2. (15 points)

Consider a surface S that is diffeomerphic to $\mathbb{R}^2$ and has negative curvature everywhere. Show that any pair of distinct geodesics intersect in at most one point in S.

You may use, without proof, the following fact: given injective curves $\gamma_1\colon I_1 \to \mathbb{R}^2$ and $\gamma_2\colon I_2 \to \mathbb{R}^2$ from p to q which intersect only at at p and q, the region of $\mathbb{R}^2$ bounded by $\gamma_1(I_1) \cup \gamma_2(I_2)$ is homeomorphic to the disk.

Name: SID:

Problem 3. (10 points) Let S_1 and S_2 be regular surfaces in $\mathbb{R}^3$. Suppose that $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ which restricts to a diffeomorphism $f\colon S_1 \to S_2$. Show that f is an isometry. (Hint: Don't overthink this one! The answer should be short, using the definition of isometry.)

Problem 4. (20 points) Consider the torus given by revolving a circle around the z-axis, given as the image of

$$\begin{array}{rcl} \varphi\colon [0,2\pi]\times[0,2\pi] & \longrightarrow & S \subset \mathbb{R}^3 \\ (\theta,\psi) & \longmapsto & ((2+\cos\psi)\cos\theta, (2+\cos\psi)\sin\theta, \sin\psi) \end{array}$$

(A) Show that the curve

$$\begin{aligned} \gamma\colon [0,2\pi] &\longrightarrow S \\ t &\longmapsto (2+\cos t, 0, \sin t) \end{aligned}$$

is a geodesic.

(B) Show that the parallel transport along γ is the identity map of $T_{(3,0,0)}S$.

Problem 5. (40 points) Consider the saddle surface S, given as the image of

$$\varphi\colon \mathbb{R}^2 \to S \subset \mathbb{R}^3$$
$$(y_1, y_2) \mapsto (y_1, y_2, {y_1}^2 - {y_2}^2)$$

(A) Show that for each point $p \in S$, any neighborhood $U \subset S$ of p lies on both sides of the tangent plane T_pS. (Recall, this implies that S has negative curvature at p.)

(B) Calculate the curvature of S at $p = (0, 0, 0)$.

(C) Show that the geodesic curvature of the curve $\eta_1(t) = (t, 0, t^2)$ is zero. This means that the image of η_1 is geodesic. (Likewise, by the symmetric argument, we see that the image of $\eta_2(t) = (0, t, -t^2)$ is geodesic.)

(D) (Bonus. Read this part now because you will use it in the sequel, **but you should skip it until you have finished the rest of the exam!** This is significantly trickier than the rest of the questions.) Fix a distance $l \in \mathbb{R}^+$ and consider the geodesic α_l connecting the points $(l, 0, l^2)$ and $(0, l, -l^2)$ in S (Note: we know that α_l is unique for each l because of question 2. Also note that we don't have a formula for α_l, but we won't need one.). Prove that for any point $p = (y_1, y_2, {y_1}^2 - {y_2}^2) \in S$ with $y_1 > 0$ and $y_2 > 0$, there is L large enough that p is contained in the geodesic triangle bounded by α_L, η_1 and η_2.

(E) Now consider the geodesics

$$\alpha_l \text{ connecting } (l, 0, l^2) \text{ and } (0, l, -l^2)$$
$$\beta_l \text{ connecting } (0, l, -l^2) \text{ and } (-l, 0, l^2)$$
$$\gamma_l \text{ connecting } (-l, 0, l^2) \text{ and } (0, -l, -l^2)$$
$$\delta_l \text{ connecting } (0, -l, -l^2) \text{ and } (l, 0, l^2)$$

(Draw yourself a diagram!) For each l, these bound a geodesic quadrilateral R_l.

Prove that

$$-\iint_{R_l} K\ dA \leq 2\pi$$

for each l.

(F) Use the previous parts to show that

$$-\iint_S K dA \leq 2\pi.$$

What does this mean about the value of K for points far from $(0, 0, 0)$?

Math 362: Differential Geometry of Surfaces

Spring 2017 Stony Brook

Instructor:	Ben McMillan	**Time:**	TuTh 1:00pm–2:20am
Email:	bmcmillan@math.stonybrook.edu	**Place:**	Earth & Space 181

Course Page: You will find up to date information, homework, and announcements at

math.stonybrook.edu/~bmcmillan/math362/

Please bookmark it and check back regularly.

Office Hours: Office hours are an invaluable resource, one that you really should use!

My office hours this term are Tuesdays 2:30–3:30pm and Wednesdays 3–4pm in Simons 510. I will also be available in the MLC (S-235) on Thursdays 11am–12pm.

Textbook: The lecture will roughly follow Manfredo do Carmo's *Differential Geometry of Curves and Surfaces*, and I will generally assign homework problems from the book.

Exams: The midterm will be held in class on Thursday March 23. Please let me know if you will need DSS accomodations well in advance.

Homework: Each week I will post homework questions on the course webpage. It is due at the end of lecture on Thursday of the following week. You should expect to spend several hours on each problem set.

Grading Policy: Homework: 40%, Midterm: 25%, Final: 35%.

Americans with Disabilities Act: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631)632–6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. http://studentaffairs.stonybrook.edu/dss/index.html.

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Critical Incident Management: Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of University Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures. Further information about most academic matters can be found in the Undergraduate Bulletin, the Undergraduate Class Schedule, and the Faculty-Employee Handbook.