MAT 362 - Differential Geometry

Announcement:

You will be provided with the following list of Christoffel Symbol formulas (.ps, .pdf) on the final exam. It is not necessary to memorize them.

Announcement:

The final exam is Wednesday, May 19th, from 11:00 am to 1:30 pm in the regular classroom.

If it were done when tis done, then twere well it were done quickly. If the assassination could trammel up the consquence and catch, with his surcease, sucess - that but this blow might be the be-all and the end-all here, but here, upon this bank and shoal of time, we'd jump the life to come. But in these cases we still have judgement here; that we but teach bloody instruction, which, being taught, return to plague its inventor. This even-handed justice commends the ingredience of our poisoned chalice to our own lips.

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General Information

Professor: Matthew Kudzin
E-mail: mkudzin@math.sunysb.edu
Office: Mathematics 2-118
Office Hours: Monday & Tuesday, 9:00 - 10:00 am , or by appointment

Classes: Monday, Wednesday, & Friday, 11:45 am - 12:40 pm in ESS room 183

Prerequisites

Students are required to have a solid understanding of multivariable calculus and linear algebra. Some knowledge of differential equations will be useful, but not necessary.

Textbook

The required textbook for this class is *Differential Geometry of Curves and Surfaces* by Manfredo doCarmo (Prentice Hall, 1976). It is an expensive book. You can compare the prices of several on-line merchants at directtextbook.com .

Homework

Homework problems will be assigned during each lecture. All of the problems assigned during the week are due at the beginning of class on the following Monday. The assignments will also be posted online, here.

Grading

The grades for this class will be based on the weekly homework and a final examination. The homework constitutes 70% of your grade. The final exam contributes the remainng 30%.

Disabled students

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.ehs.stonybrook.edu/fire/disabilities.asp.

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Math is not a spectator sport.
-- Herman Gluck

- Assignment #1 (last modified: January 30, 2004) (.ps, .pdf) - due February 2, 2004.
- Assignment #2 (last modified: February 4, 2004) (.ps, .pdf) - due February 9, 2004.

Solution to problem 1 (.ps, .pdf)

- Assignment #3 (last modified: February 12, 2004) (.ps, .pdf) - due February 16, 2004.

Solution to problem 2 (.ps, .pdf)

- Assignment #4 (last modified: February 20, 2004) (.ps, .pdf) - due February 23, 2004.
- Assignment #5 (last modified: February 27, 2004) (.ps, .pdf) - due March 1, 2004.
- Assignment #6 (last modified: March 3, 2004) (.ps, .pdf) - due March 8, 2004.
- Assignment #7 (last modified: March 12, 2004) (.ps, .pdf) - due March 15, 2004.
- Assignment #8 (last modified: March 19, 2004) (.ps, .pdf) - due March 22, 2004.
- Assignment #9 (last modified: March 24, 2004) (.ps, .pdf) - due March 29, 2004.

Solution to problem 3 (.ps, .pdf)

- Assignment #10 (last modified: March 31, 2004) (.ps, .pdf) - due April 12, 2004.
- Assignment #11 (last modified: April 14, 2004) (.ps, .pdf) - due April 19, 2004.
- Assignment #12 (last modified: April 22, 2004) (.ps, .pdf) - due April 26, 2004.
- Assignment #13 (last modified: April 28, 2004) (.ps, .pdf) - due May 3, 2004.

- Assignment #14 (last modified: May 13, 2004) (.ps, .pdf) - not for credit.

  Solution to problem 5 (.ps, .pdf)

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Books

For more about curves and surfaces in $\mathbf{R}^3$:

- O'Neill, *Elementary Differential Geometry*
- Millman & Parker, *Elements of Differential Geometry*

The same material is presented from a completely different point of view in *Differential Forms with Applications to the Physical Sciences* by Harley Flanders.

For a more advanced treatment of Differential Geometry in higher dimensions:

- doCarmo, *Riemannian Geometry*
- Gallot, Hulin, & LaFontaine, *Riemannian Geometry*
- Spivak, *A Comprehensive Introduction to Differential Geometry*

Flatland, the 1880 book about life in a two dimensional universe, is available free online . If you don't like to read books online, Dover publishes an edition for only $1.50.

Gallery of Surfaces

- f(x,y) = | |x| - |y| | - |x| - |y| is a continuous function. The partial derivatives, $f_x(0,0)$ and $f_y(0,0)$, both exist. However, f is not differentiable at (0,0). You can understand why by looking at the graph of f.
- The contour surface $(x^2+y^2+z^2)^2 - 8xyz - 10(x^2+y^2+z^2) = -20$. For more examples of interesting surfaces defined as level sets, go to The Scientific Graphics Project at MSRI.

- Mathias Weber has a Virtual Minimal Surface Museum which includes a movie of the associated family of minimal surfaces between the catenoid and the helicoid.

Mathematica programs

- I have written a Mathematica notebook to allow you to construct your own minimal surfaces by specifying the Weierstrass data. It includes several examples and a brief description of how to determine the conjugate minimal surface and the associated family.

Papers

Many recent publications are available online at the e-Print archive.

Online resources

A lot of good work is being done by the gang at Amherst. Their website includes several galleries of minimal (and other related) surfaces.

Christoffel Symbol Formulas

Many formulas involving the Christoffel Symbols are long and messy. It is not a good use of your time to memorize all of the indices. Therefore, you will be provided with the following formulas on the exam. Do not read too much into this. It does not mean that you will need all (or any) of these formulas. However, they will be available if you want them.

$$\Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2}E_1 \qquad \Gamma_{12}^1 E + \Gamma_{12}^2 F = \frac{1}{2}E_2 \qquad \Gamma_{22}^1 E + \Gamma_{22}^2 F = F_2 - \frac{1}{2}G_1$$

$$\Gamma_{11}^1 F + \Gamma_{11}^2 G = F_1 - \frac{1}{2}E_2 \qquad \Gamma_{12}^1 F + \Gamma_{12}^2 G = \frac{1}{2}G_1 \qquad \Gamma_{22}^1 F + \Gamma_{22}^2 G = \frac{1}{2}G_2$$

If $v(t) = a(u_1(t), u_2(t))\mathbf{x}_1 + b(u_1(t), u_2(t))\mathbf{x}_2$, then

$$\frac{Dv}{dt} = (a' + \Gamma_{11}^1 a u_1' + \Gamma_{12}^1 a u_2' + \Gamma_{21}^1 b u_1' + \Gamma_{22}^1 b u_2')\mathbf{x}_1 + (b' + \Gamma_{11}^2 a u_1' + \Gamma_{12}^2 a u_2' + \Gamma_{21}^2 b u_1' + \Gamma_{22}^2 b u_2')\mathbf{x}_2$$

Gauss Equation:

$$(\Gamma_{12}^2)_1 - (\Gamma_{11}^2)_2 + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2 = -EK$$

Codazzi Equations:

$$e_2 - f_1 = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2$$

$$f_2 - g_1 = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{21}^1) - g\Gamma_{21}^2$$

Problem Set #1

due Monday, February 2, 2004

1. Draw a straight line. Prove that it is straight.
2. Prove that the curve $x^3 = y^2$ does not have a C^1 regular parameterization.
3. Consider the function

$$f(x) = \begin{cases} x \sin(\frac{\pi}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Is the graph of $f(x)$ between $x = 0$ and $x = 1$ rectifiable? If so, compute its length.

Problem Set #2

due Monday, February 9, 2004

1. Prove the following version of the Mean Value Theorem for vector valued functions:

   Let $\alpha : [a,b] \to \mathbb{R}^3$ be a regular, smooth curve. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|c-d| < \delta$ then there exists a $\tau \in [c,d]$ for which

$$\big|\alpha(c) - \alpha(d) - \alpha'(\tau)|c-d|\big| < \epsilon\big|c-d\big|$$

2. Let $\alpha(s)$ be a regular curve parameterized by arclength and let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. If $\beta(s) = R \circ \alpha(s)$, then
   (a) Compute $\beta'(s)$ is terms of $\alpha'(s)$.
   (b) Prove that $\beta(s)$ is a regular curve, parameterized by arclength, and that $\kappa_\beta(s) = \kappa_\alpha(s)$.

3. Compute the curvature of the logarithmic spiral, $\alpha(t) = (e^{-t}\cos t, e^{-t}\sin t, 0)$.

Problem Set #2

Solutions

Problem: Prove the following version of the Mean Value Theorem for vector valued functions:

Let $\alpha : [a, b] \to \mathbb{R}^3$ be a regular, smooth curve. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|c - d| < \delta$ then there exists a $\tau \in [c, d]$ for which

$$\big|\alpha(c) - \alpha(d) - \alpha'(\tau)(c - d)\big| < \epsilon\big|c - d\big|$$

Solution: Express the curve in coordinates as $\alpha(t) = (x(t), y(t), z(t))$. For any fixed interval, $[c, d]$, we can apply the standard Mean Value Theorem to the components of α to conclude that there exist real numbers $\tau_x, \tau_y, \tau_z \in [c, d]$ such that

$$\begin{aligned} x(c) - x(d) &= x'(\tau_x)(c - d) \\ y(c) - y(d) &= y'(\tau_y)(c - d) \\ z(c) - z(d) &= z'(\tau_z)(c - d) \end{aligned}$$

or, equivalently, in vetor notation

$$\alpha(c) - \alpha(d) = (x'(\tau_x), y'(\tau_y), z'(\tau_z))(c - d) \tag{1}$$

If we let $\tau = \tau_x$, we no longer have an equality, but by the triangle inequality, we have

$$\begin{aligned} \big|\alpha(c) - \alpha(d) - \alpha'(\tau_x)(c - d)\big| \leq {} & \big|\alpha(c) - \alpha(d) - (x'(\tau_x), y'(\tau_y), z'(\tau_z))(c - d)\big| + \\ & \big|(x'(\tau_x), y'(\tau_y), z'(\tau_z))(c - d) - \alpha'(\tau_x)(c - d)\big| \end{aligned} \tag{2}$$

The first term on the right hand side is zero, by equation (1). Using the fact that $\alpha'(\tau_x) = (x'(\tau_x), y'(\tau_x), z'(\tau_x))$, we can subtract the two vectors in the second term and get

$$\begin{aligned} \big|\alpha(c) - \alpha(d) - \alpha'(\tau_x)(c - d)\big| &\leq \big|(0, y'(\tau_y) - y'(\tau_x), z'(\tau_z) - z'(\tau_x))(c - d)\big| \\ &= \sqrt{(y'(\tau_y) - y'(\tau_x))^2 + (z'(\tau_z) - z'(\tau_x))^2}\,\big|c - d\big| \end{aligned}$$

In order to prove that the term under the radical can be made arbitrarily small, we use the fact that y' and z' are continuous functions, so that $\lim_{\tau_x \to \tau_y} y'(\tau_x) = y'(\tau_y)$. In particular, for every $\epsilon > 0$, there exists a $\delta_y > 0$ such that if $|\tau_y - \tau_x| < \delta_y$, then $|y'(\tau_y) - y'(\tau_x)| < \epsilon/2$. Similarly, using the continuity of z', we can construct a number δ_z sufficiently small to guarantee that $|z'(\tau_z) - z'(\tau_x)| < \epsilon/2$.

Finally, let $\delta = \min\{\delta_y, \delta_z\}$. Since τ_x, τ_y, and τ_z are all contained in the interval $[c, d]$, if $|c - d| < \delta$, then $|\tau_y - \tau_x| < \delta \leq \delta_y$ and $|\tau_z - \tau_x| < \delta \leq \delta_z$. With this, the previous estimate becomes

$$\big|\alpha(c) - \alpha(d) - \alpha'(\tau_x)(c - d)\big| < \sqrt{(\epsilon/2)^2 + (\epsilon/2)^2}\,\big|c - d\big| < \epsilon\big|c - d\big|$$

as desired.

Problem Set #3

due Monday, February 16, 2004

1. doCarmo, section 1.5, # 2,9,14
2. Let $\alpha(s)$ be a regular curve, parameterized by arclength, such that $\kappa(s) \neq 0$ and $\tau(s) \neq 0$ for all s.
   (a) Prove that if α lies on the sphere of radius r, centered at p, then
   $$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$$
   (b) Prove that the center of the sphere, p, satisfies
   $$p = \alpha(s) + \frac{1}{\kappa(s)}N(s) + \frac{\kappa'(s)}{\tau(s)\kappa^2(s)}B(s)$$
   for all s.
   (c) Prove the converse of part (a).
3. Find a minimal set of first–order, linear differential equations which are equivalent to the Frenet–Serret equations for a curve in $\mathbb{R}^3$. (Hint: You will need at least three equations.)

Problem Set #3

Solutions

Problem: Let $\alpha(s)$ be a regular curve, parameterized by arclength, such that $\kappa(s) \neq 0$ and $\tau(s) \neq 0$ for all s.

(a) Prove that if α lies on the sphere of radius r, centered at p, then

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$$

(b) Prove that the center of the sphere, p, satisfies

$$p = \alpha(s) + \frac{1}{\kappa(s)}N(s) + \frac{\kappa'(s)}{\tau(s)\kappa^2(s)}B(s)$$

for all s.

(c) Prove the converse of part (a).

Solution: Assume that α lies on the surface of the sphere. That means that the distance between any point on the curve, $\alpha(s)$, and the center, p, is equal to r. Symbolically,

$$(\alpha(s) - p) \cdot (\alpha(s) - p) = r^2$$

for all s.

Notice that the equation that we are trying to derive depends on the second derivative of κ. Since curvature is, itself, computed in terms of the second derivative of α, the right hand side of the desired equation depends on the fourth derivative of α. Therefore, no matter what we do, we must compute at least four derivatives.

Differentiating once with respect to s, we get $\alpha'(s)\cdot(\alpha(s)-p)+(\alpha(s)-p)\cdot\alpha'(s) = 0$, which simplifies to

$$T \cdot (\alpha(s) - p) = 0 \tag{1}$$

Differentiating again, we get $T' \cdot (\alpha(s) - p) + T \cdot T = 0$. Using the Frenet equation $T' = \kappa N$, and the fact that T is a unit vector, this reduces to

$$N \cdot (\alpha(s) - p) = -\frac{1}{\kappa} \tag{2}$$

Differentiating a third time, we have $N' \cdot (\alpha(s) - p) + N \cdot T = (-1/\kappa)'$. The left hand can be simplified by substituting $N' = -\kappa T - \tau B$, and by using the fact that N and T are orthogonal, to get

$$(-\kappa T - \tau B) \cdot (\alpha(s) - p) = \left(-\frac{1}{\kappa}\right)' = \frac{\kappa'}{\kappa^2}$$

We can use equation (1) to further simplify this equation down to

$$B \cdot (\alpha(s) - p) = -\frac{\kappa'}{\kappa^2\tau} \tag{3}$$

Differentiating a fourth, and final, time, we get $B'\cdot(\alpha(s)-p)+B\cdot T = -(\kappa'/(\kappa^2\tau))'$. Using the Frenet equations once again, and the fact that B and T are orthogonal, this reduces to

$$\tau N \cdot (\alpha(s) - p) = -\left(\frac{\kappa'}{\kappa^2\tau}\right)'$$

Substituting equation (2) into the left hand side finishes part (a).

In order to answer part (b), notice that equations (1–3) give the projections of the vector $\alpha(s) - p$ onto the orthonormal basis $\{T, N, B\}$. We can reconstruct any vector from its projections, so

$$\alpha(s) - p = 0T - \frac{1}{\kappa}N - \frac{\kappa'}{\kappa^2\tau}B$$

which, once rearranged, is the answer. (If you are unsatisfied by this, write $\alpha(s) - p = aT + bN + cB$ and solve for the coefficients a, b, and c, as we did in the proof of the Frenet–Serrat Theorem.)

The proof of part (c) is not yet written.

Problem Set #4

due Monday, February 23, 2004

1. doCarmo, section 1.7, # 6, 13 (Hint: You might want to do problem 12(d) from section 1.5.)
2. Let α be a regular curve whose tangent indicatrix, T, is also regular. Compute the curvature and torsion of T, in terms of the curvature and torsion of α.
3. Let β be a regular closed curve of length L. Prove that if the curvature of β is bounded above by $1/R$, then $L \geq 2\pi R$.
4. Let c be a regular closed plane curve.
   (a) Prove that if c is simple, then the tangent indicatrix of c is the entire unit circle.
   (b) Give an example to demonstrate that part (a) is false without the assumption that c is simple.

Problem Set #5

due Monday, March 1, 2004

Note: The definition of a regular surface that I gave in class is not the same as the one in the book. doCarmo assumes that his coordinate patches are homeomorphisms. It is a theorem (Proposition 4 in section 2.2) that this is equivalent to the coordinate patches being 1-1. You may use whichever definition is more convenient.

1. doCarmo, section 2.2, # 2, 4, 10
2. Let $c(s) = (x(s), 0, z(s))$ be a simple regular curve in the xz–plane with $x(s) > 0$ for all s. Let S be the set of points formed by rotating $c(s)$ about the z–axis.
   (a) Show that $F(s, \theta) = (x(s)\cos\theta, x(s)\sin\theta, z(s))$ is a coordinate patch for some open set in the $s\theta$–plane.
   (b) Prove that the set S is a regular surface (called a *surface of revolution*).
   (c) Prove that S is still a regular surface if $c(s)$ is a simple closed curve.

Problem Set #6

due Monday, March 8, 2004

1. doCarmo, section 2.2, # 16
2. doCarmo, section 2.3, # 1, 2, 6, 7

Problem Set #7

due Monday, March 15, 2004

1. doCarmo, section 2.4, # 2, 3, 8, 10, 15
2. Compute the inner product for the northern hemisphere in the coordinate patch $\mathbf{y}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$.

Problem Set #8

due Monday, March 22, 2004

1. doCarmo, section 2.5, # 9, 12. 14a
2. doCarmo, section 3.2, # 3, 5, 7, 8

Problem Set #9

due Monday, March 29, 2004

Warning: This is not the final draft of the assignment. More problems will be added in the course of the week.

1. doCarmo, section 3.3, # 5, 6, 16
2. Let V be a vector space with an inner product, $\langle \cdot, \cdot \rangle$, and let $\{v_1, v_2, \ldots, v_n\}$ be a (not necessarily orthonormal) basis of V. Prove that for any vector, $v = \sum c_i v_i$, the coefficients c_i can be computed by

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = (\mathfrak{g}^T)^{-1} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

   where $\mathfrak{g}$ is the matrix representing the inner product (ie. $\mathfrak{g}_{ij} = \langle v_i, v_j \rangle$) and $b_i = \langle v, v_i \rangle$.
3. Let S be a tubular surface about a curve α (see doCarmo, section 2.4, problem 10). Compute the second fundamental form, principal, Gaussian, and mean curvatures of S, in terms of the curvature and torsion of α.

Problem Set #9

Solutions

Problem: Let S be a tubular surface about a curve α (see doCarmo, section 2.4, problem 10). Compute the second fundamental form, principal, Gaussian, and mean curvatures of S, in terms of the curvature and torsion of α.

Solution: As we have discussed before, S can be parameterized as

$$\mathbf{x}(s,\theta) = \alpha(s) + r\cos\theta N(s) + r\sin\theta B(s)$$

Computing the derivatives of the parameterization, we get

$$\mathbf{x}_s = (1 - r\kappa(s)\cos\theta)T(s) + r\tau(s)\sin\theta N(s) - r\tau(s)\cos\theta B(s)$$
$$\mathbf{x}_\theta = -r\sin\theta N(s) + r\cos\theta B(s)$$

and the normal vector to the surface is

$$\nu = \cos\theta N(s) + \sin\theta B(s)$$

The second derivatives of the parameteriztion are

$$\begin{gathered}\mathbf{x}_{ss} = (-r\kappa'(s)\cos\theta - r\kappa(s)\tau(s)\sin\theta)T(s) + \\ (\kappa(s)(1 - r\kappa(s)\cos\theta) + r\tau'(s)\sin\theta - r\tau^2(s)\cos\theta)N(s) + \\ (-r\tau^2(s)\sin\theta - r\tau'(s)\cos\theta)B(s)\end{gathered}$$
$$\mathbf{x}_{s\theta} = r\kappa(s)\sin\theta T(s) + r\tau(s)\cos\theta N(s) + r\tau(s)\sin\theta B(s)$$
$$\mathbf{x}_{\theta\theta} = -r\cos\theta N(s) - r\sin\theta B(s)$$

Using this, we can compute the first and second fundamental forms:

$$\mathfrak{g} = \begin{pmatrix} (1 - r\kappa(s)\cos\theta)^2 + r^2\tau^2(s) & -r^2\tau(s) \\ -r^2\tau(s) & r^2 \end{pmatrix}$$
$$L_{s\theta} = \begin{pmatrix} \kappa(s)\cos\theta(1 - r\kappa(s)\cos\theta) - r\tau^2(s) & r\tau(s) \\ r\tau(s) & -r \end{pmatrix}$$

From which we mau compute various curvatures. For example, the Gaussian curvature is

$$K = \frac{\det L_{s\theta}}{\det \mathfrak{g}} = \frac{-\kappa(s)\cos\theta}{r(1 - r\kappa(s)\cos\theta)}$$

Problem Set #10

due Monday, April 12, 2004

1. doCarmo, section 3.5, # 11 – 14

Problem Set #11

due Monday, April 19, 2004

1. doCarmo, section 4.2, # 2, 3, 10, 14
2. Let $f(x)$ be a smooth function of one variable. Prove that the graph $z = f(x)$ is isometric to the xy–plane.

Problem Set #12

due Monday, April 26, 2004

1. doCarmo, section 4.3, # 2, 3

Problem Set #13

due Monday, May 3, 2004

1. doCarmo, section 4.4, # 4, 14, 18, 19
   In problem 14, skip the part about "lines of curvature".

Problem Set #14

This assignment is for your own benefit. You do not need to turn in your solutions.

1. doCarmo, section 4.5, # 1, 4, 5

Problem Set #14

Solution to section 4.5 # 5

Solution: We have seen that when you parallel transport a vector around a closed loop, the vector does not come back to itself. I tried to argue in class that the amount that the vector is rotated, the *defect angle*, is a measure of the total Gaussian curvature in the region bounded by the loop. Moreover, you can use this idea to compute the curvature at a point, p, by computing the defect angle for any infinite family of shrinking loops containing p.

This problem makes that idea explicit by asking you to perform the calculation on a sphere. We take p to be the north pole use as our family of curves the parallels, C_ϕ. For each C_ϕ, we start with a vector tangent to the curve and parallel transport it around the circle.

In order to compute the defect angle, we follow the procedure of example 1 on page 243. The crucial observation is the following: the covariant derivative of a vector field depends only on the curve, the vector field, and the tangent planes to the surface along the curve. In particular, if two surfaces are tangent to each other along the curve, then the covariant derivatives for the two surfaces will be equal. Therefore, in order to compute the defect angle, we can replace the sphere with a cone which is tangent to C_{ϕ_0}. The advantage of this is that the cone is locally isometric to the plane, where the covariant derivative is easy to compute. I will not repeat their calculation here. I will only use the result, which is that the defect angle is

$$\Delta\phi = 2\pi - \theta = 2\pi - 2\pi\sin\psi = 2\pi - 2\pi\sin(\pi/2 - \phi_0) = 2\pi(1 - \cos\phi_0)$$

(It is worth noting that the concept of parallelism is, in fact, older than the covariant derivative. It was originally defined by precisely this procedure: replace the surface with a flat one which is tangent to the orginal surface, unroll the flat surface, and compute the parallel transport in the plane.)

In order to complete the problem, we still need to compute the area, A, bounded by the parallel, C_ϕ. There are many ways to do this. For example, it can be done with elementary calculus. However, for the sake of exposition, I will show how to use the area formulas from section 2.5. We begin by parameterizing the sphere by

$$\mathbf{x}(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

so that the first fundamental form is given by

$$\mathfrak{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix}$$

We defined the area of a region, R, to be

$$A = \iint_R dA = \iint_{\mathbf{x}^{-1}(R)} \sqrt{\det\mathfrak{g}}\, d\theta d\phi$$

which, when R is the portion of the upper hemisphere bounded by C_{ϕ_0} evaluates to

$$A = \int_0^{\phi_0}\int_0^{2\pi} \sin\phi d\theta d\phi = 2\pi(1 - \cos\phi_0)$$

Finally,

$$\lim_{R \to p} \frac{\Delta\phi}{A} = \lim_{\phi_0 \to 0} \frac{2\pi(1 - \cos\phi_0)}{2\pi(1 - \cos\phi_0)} = 1$$

as expected.