Sylvain BONNOT

MAT 360
Geometric Structures

Spring 2008

We will meet on TuTh : 12:50pm to 2:10pm in Library N3063.

First day of class: Jan 29th, 2008.
Final exam : TBA.

Office hours:
every Thursday from 3:30 pm to 6:00 pm in my office, 5D-148 in the Math Tower.
My office is in the I.M.S (Institute for Math. Sciences), located on floor 5 and a half.

How to contact me?
the best way is to email me there: `bonnot at math dot sunysb dot edu`

Our textbook:

Textbook: *Euclidean and non Euclidean Geometry : Development and History*, (Fourth Edition) by Marvin Jay Greenberg, Editor: W.H. Freeman and Company. (I will use the latest edition, i.e the fourth edition, so if you have another edition just make sure that you do the required exercises.)

Link to Current Homework: The Homework is an important part of this class. Click [here]() to go to the homework page.

Course notes and announcements:

- And here are [the solutions for the practice final](). On tuesday afternoon I'll be available for office hours, so don't hesitate to find me...

- The solutions for the last 2 HW have been posted, and on Monday I will post the solutions for the practice exam. If you need to find me, I'll be in my office as usual, so stop by, or email me if you have any questions.

- Please try this practice final and don't forget to prepare questions for the review session on Thursday!
- I decided that you won't have a new HW due next week: so the last one was HW8 (that you can return on Tuesday May 6th).
- There is a new HW for next week, but it is a short one...
- Solutions of Midterm II Please have a look at these solutions for the midterm II. It seems that most of you found it quite hard, so very likely the average should be a bit lower than last time...
- Solutions Practice Midterm II Please read these solutions for the practice midterm II.
- For next week : Don't forget that midterm II is this on Thursday April 17th, same room, same time... Also don't forget to bring with you the list of axioms! On the HW page you have now the solutions of the last two HW. There won't be any HW for next week (the week of the exam). On Monday I'll post the solutions for the practice exam. Also prepare your questions for Tuesday: at least half the time will be spent reviewing for the test.
- Practice exam for next week Please try this practice midterm II and very soon I'll post the solutions together with the solutions for the last HW...
- Splitting of letter grades: Here is what I propose for the letter grades, concerning our last midterm: A: anything above 85//A-: between 80 and 85//B: between 65 and 80//B-: between 59 and 65//C: between 59 and 49// C-: between 49 and 30//D: below 30.No F for this midterm...
- Solutions for Midterm I: Here are the solutions of the midterm in one big file. Your exam will be graded soon (grade available on request, in few days...) Have a nice break! Also, there will be a new HW available soon, but you will have lots of time for it...
- Solutions for the practice exam: first half, second half. Tomorrow is the review session, so bring your questions with you...
- No HW is due for the week of the exam! But I suggest that you try the practice exam and prepare some questions for me on Tuesday. This tuesday we will have a review session, as I told you.
- Practice Midterm I is available! Please, do try this at home: PracticeMidterm I.
- Midterm is next week! Don't forget that Midterm I is on Thursday March 13th, usual room, usual time. It will cover the first 3 chapters, and will contain 4 or 5 problems (similar to the HW problems).
- Very important! For next week's test you need to print this Official-Prof.Bonnot-approved-cheat-sheet that is gathering the axioms we studied. Again you need to print this and to bring it to the midterm. "Unapproved cheat sheets" are not welcome, and will be destroyed on location (as well as their owners)...

- Hmmm...all right, since my explanations were confused last time, here is what I should have said the other day: a clearer explanation
- Please notice that I made a little change in HW3 (exercise 2): I had forgotten to define the lines in the model, now it is ok...
- HW3 is now available on the HW page, it is shorter than the previous one...
- HW2 is available on the HW page.
- There was a small mistake in the first problem, it is corrected now (it's 2AM.CB instead of just AM.CB).
- HW1 is available on the HW page... Also if you want to witness how a circular motion can be transformed into a linear one by an "inversor", you should have a look at that page... Then click on movie, but you can also click on the little animation to the right and drag one point: you will see the other one moving on a vertical line.
- First day of class is on Tuesday January 29th.

Quick intro: The goal for this class is to really build plane geometry from scratch (I mean, from a small number of axioms). Also we will show how different sets of axioms can lead to more "exotic" geometries. But my personal goal is to convince you that Geometry is "everywhere" and that it is at the root of mathematics, both historically and conceptually.

Link to Current Homework: Regularly you will have to consult this homework page to know what has been assigned.

Syllabus (T.B.A) :

Week	**Sections Covered**
1	Intro to euclidean, and hyperbolic geometry, Euclid's axioms
2	Classical ruler and compass constructions, altitudes, medians, Pythagorean theorem, geometry of the triangle, circle inversion
3	Various proofs about circle inversions, basics of logic, incidence axioms, affine plane, intro to projective plane
4	(Prevision) End of construction of the projective plane, Hilbert axioms of betweenness

---

Exams:

Midterm 1	Th. March 13th	Usual room
Midterm 2	Th. April 17th	Usual room
Final	Thursday, May 15, 11:00 AM- 1:30	Usual room

	PM	

Homework and grading policy: Here is how your final grade will be computed. of the following:

Exam I	25%
Exam II	25%
Final Exam	35%
Homework	15%

Late homework will not be accepted.

DSS advisory:

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

MAT 360 Homework Assignments

Spring 2008

Link to main page for MAT 360.

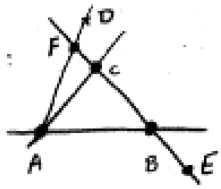
Mathematics department

#	Problems	Due Date
HW1	pdf file for HW1 Solutions for HW1	Th Feb. 14th
HW2	pdf file for HW2 Solutions for HW2, first page Solutions for HW2, second page	Th Feb. 21st
HW3	pdf file for HW3 Solutions for HW3, first page Solutions for HW3, second page Solutions for HW2, 3rd page	Th Feb. 28th
HW4	file for HW4 Solutions for HW4, first page Solutions for HW4, second page	Th March 6th
HW	No HW is due for March 13th ! (Yes, I changed my mind) Instead, try the practice exam for yourself! Solutions for the practice exam to be posted on Monday	Th March 13th
HW5	pdf file for HW5 Solutions for HW5, first page (poor resolution, sorry...) Solutions for HW5, second page	Th April 3rd
HW6	pdf file for HW6 Solutions for HW6, first page Solutions for HW6, second page	Th April 10th
HW7	pdf file for HW7 Solutions for HW7	Th April 24th
HW8	pdf file for HW8 Solutions for HW8, first page Solutions for HW8, second page	Th May 1st extended to May 6th
HW8 was the last HW !		

Solutions - Practice Final

#1 D is outside ΔABC so D must be opposite to at least one vertex, say D and A are on opposite sides of $\overleftrightarrow{BC}$. Thus segment DA must cut $\overleftrightarrow{BC}$ at a pt F.

Three cases: ① $B * C * F$:



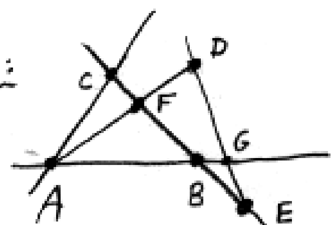
Pick any E on $\overleftrightarrow{BC}$ such that $C * B * E$. Let's show that $\overleftrightarrow{DE}$ is outside ΔABC.

Since we have $\begin{cases} D * F * A \\ B * C * F \end{cases}$ we know that D, F, C are together w.r. to $\overleftrightarrow{AB}$.

Now the entire ray $\overrightarrow{ED}$ is on the opposite side of A (w.r.t $\overleftrightarrow{BC}$) so it's outside ΔABC.

Finally the opposite ray to $\overrightarrow{ED}$ is on the opposite side of C w.r to $\overleftrightarrow{AB}$ (because D and E are on opposite sides of $\overleftrightarrow{AB}$).

② $C * F * B$:



Again pick any E on $\overleftrightarrow{BC}$ s.t $C * B * E$ and show that $\overleftrightarrow{DE}$ is outside ΔABC.

Similarly, the entire ray $\overrightarrow{ED}$ is on the opposite side of A w.r. to $\overleftrightarrow{BC}$, hence is outside ΔABC.

Now the opposite ray to $\overrightarrow{ED}$ is on the opposite side of C w.r to $\overleftrightarrow{AB}$ (to see this, notice that there is a pt $G \in$ segment DE and on $\overleftrightarrow{AB}$, and ray $\overrightarrow{GE}$ is on the opposite side of C).

③ $C * B * F$: same as ① (just permute B and C).

#2 See HW #4.

#3 It's an immediate consequence of the axioms:

Pick any line. By I2 there are at least 2 pts B, D on it.

Now by B2 there exist pts A, C, E on $\overleftrightarrow{BD}$ s.t $A * B * D$, $B * C * D$, $B * D * E$.

And we are done!

(#4) Let $\mathcal{M}$ be a projective plane, therefore it satisfies: all the incidence axioms + any two lines meet + every line has at least 3 distinct pts on it.

Let's check that $\mathcal{M}'$ is a projective plane: $\left\{ \begin{array}{l} \mathcal{M}'\text{-pts. correspond to } \mathcal{M}\text{-lines} \\ \mathcal{M}'\text{-lines} \quad " \quad " \quad \mathcal{M}\text{-points} \end{array} \right\}$

$\boxed{I\ 1:}$

Since any two distinct $\mathcal{M}$-lines intersect at a unique $\mathcal{M}$-pt, $I1$ is true for $\mathcal{M}'$!

$\boxed{I\ 2:}$

We want to show the following: for every $\mathcal{M}$-pt there exist at least two $\mathcal{M}$-lines incident with it.

Indeed we know that in $\mathcal{M}$ there are at least 3 pts such that no line is incident with all three (axiom $I3$ for $\mathcal{M}$). Thus given any pt P, consider the lines joining P to these pts: at least 2 of them are distinct so we are done.

$\boxed{I\ 3}$: We want to show the following:

There exist 3 distinct $\mathcal{M}$-lines such that no $\mathcal{M}$-point is incident with all three.

But this is a consequence of $I3$ for $\mathcal{M}$: take the 3 non-collinear $\mathcal{M}$-pts, A, B, C: then the 3 lines $\overleftrightarrow{AB}, \overleftrightarrow{AC}, \overleftrightarrow{BC}$ cannot be collinear.

$\boxed{\text{Any 2 } \mathcal{M}'\text{-lines meet}}$: we want to show that given any 2 $\mathcal{M}$-points there is a line incident with them. But this is just $I2$ for $\mathcal{M}$!

$\boxed{\text{Any } \mathcal{M}'\text{-line has at least 3 distinct pts on it}}$

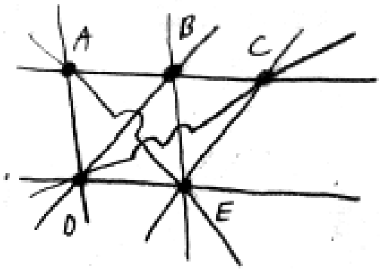
This can be rephrased as: given any $\mathcal{M}$-pt, there are at least 3 distinct lines incident with it.

Indeed by $I3$ for $\mathcal{M}$, there exist a line l not going through a given P.

Now there exist at least 3 distinct pts on that line l (because $\mathcal{M}$ is a projective plane).

Join those pts to P to get the desired lines.

#5: Just take a space with 5 pts as follows:



schematic picture of the lines

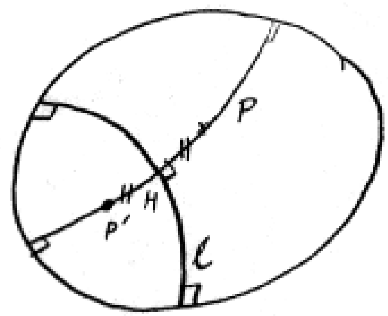
Points: A, B, C, D, E

Lines: $\{A,B,C\}, \{D,E\}, \{A,D\}, \{B,D\}, \{C,D\}$
$\{A,E\}, \{B,E\}, \{C,E\}$.

Incidence: "set membership".

Then in this model, the incidence axioms are satisfied. But $\{A,B,C\}$ has a unique parallel through D, but $\{CE\}$ has 2 parallels through D, so the model doesn't have any of the 3 properties mentioned.

#6 In the Poincaré model:

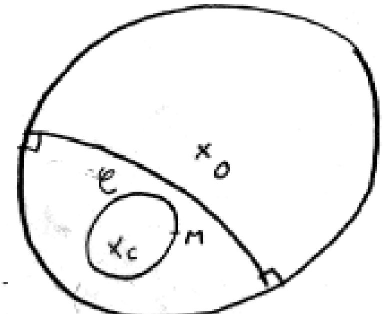


Since it is a Hilbert plane, one can drop a perpendicular through P to ℓ to get the point H and then "transport the length HP" on the other side of ℓ to get P'.

Consider now the inversion with respect to ℓ: it preserves orthogonal circles thus the "line" $\overleftrightarrow{HP}$ is globally preserved, H is preserved (because on ℓ) and the two sides of ℓ are exchanged.

We also proved that inversions preserve $\mathcal{P}$-lengths therefore $d(HP) = d(H\,i(P))$ and thus $i(P)$ coincides with P' (which is the unique pt P' on $\overleftrightarrow{HP}$ on the opposite ray to $\overrightarrow{HP}$, such that $HP \cong HP'$).

#7



Let $\mathcal{C}$ a $\mathcal{P}$-circle with center C in the radius CM Poincaré-model: so $\mathcal{C}$ is the set of pts $P \in \mathcal{P}$ such that $CP \cong CM$.

We know the existence of an inversion sending C to O.

Since such an inversion preserves the $\mathcal{P}$-length, it will send $\mathcal{C}$ to a $\mathcal{P}$-circle centered at O. But we proved that $d(OP) = d(OQ)$ iff $\overline{OP} = \overline{OQ}$ (when O is the origin), therefore a $\mathcal{P}$-circle centered at the origin is the same thing as an Euclidean circle centered at O.

Now apply the inversion one more time (since $i \circ i$ = identity we return to $\mathcal{C}$): we know that inversions send circles (not going through the origin of the inversion) to circles (Euclidean circles), thus $\mathcal{C}$ is a Euclidean circle.
(Notice that the $\mathcal{P}$-center of $\mathcal{C}$ is not the same as its Euclidean center).

Practice Final

Name:

Student I.D:

Problem 1. (15 points)

In a Hilbert plane, prove that if D is an exterior point of ΔABC, then there exists a line (DE) through D that is contained in the exterior of ΔABC.

Problem 2. (15 points) In a Hilbert plane, prove that a line cannot be contained in the interior of a triangle.

Problem 3. (10 points) Assume that you have a geometry with only the incidence and the betweenness axioms: prove that every line has at least five points on it.

Problem 4. (15 points)

Let $\mathcal{M}$ be a projective plane (see previous exams for the definition). Define a new geometry $\mathcal{M}'$ by taking as "points" of $\mathcal{M}'$ the lines of $\mathcal{M}$ and as "lines" of $\mathcal{M}'$ the points of $\mathcal{M}$, with the same incidence properties. Prove that $\mathcal{M}'$ is also a projective plane.

Problem 5. (15 points)

Construct a model of incidence geometry with a finite number of points, that has neither the elliptic, hyperbolic nor euclidean parallel properties. (Basically you want to have some points having only one parallel through them to a given line, some other points with several parallels through them, etc...)

Problem 6. (15 points)

In the Poincare model of the hyperbolic plane, prove that inversions truly "are" hyperbolic reflections.

Let's recall the definition of the hyperbolic reflection with respect to the line l: to find the reflection A' of a point A, drop a perpendicular through A to l, call H the foot of that perpendicular. Then A' is the unique point such that H is the midpoint of AA'.

Problem 7. (15 points)

In the Poincare model, you can define circles as in any Hilbert plane (they are sets of points at equal Poincare-distance to the center). Show that such "Poincare-circles" are the same as Euclidean circles.

MAT 360 04/17/2008

MIDTERM II

Name:

Student I.D:

Problem 1. (25 points) Assume you are in a Hilbert plane.

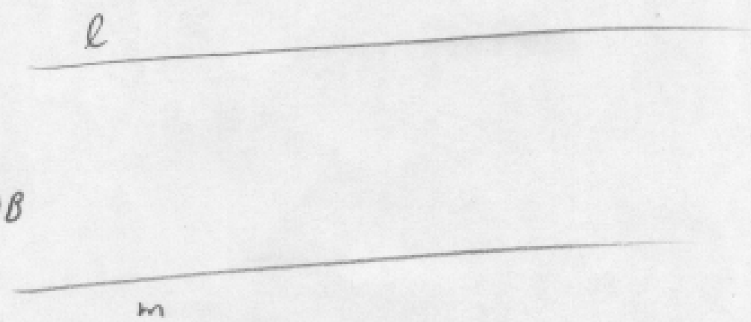
Given parallel lines l and m. Given points A and B (not on m) such that: for any point P on l, A and P are on opposite sides of m, and B and P are on opposite sides of m.

Prove, by using only the axioms and the definitions, that A and B lie on the same side of l.

Proof: by contradiction: assume A and B are on opposite sides of ℓ.

Then there would be a pt $P \in \ell$ such that $A * P * B$.

Since $P \in \ell$, there exists $\begin{cases} C \in m \text{ such that } A * C * P \\ D \in m \text{ " " } P * D * B \end{cases}$



But this implies that line m = line (CD) = line AB which is absurd.

Problem 2. (25 points) In any Hilbert plane, using the axioms, the definitions, and the relevant propositions, show that Hilbert's Parallel Postulate **implies** the following proposition:

If $k \| l$, $m \perp k$ and $n \perp l$ then either $m = n$ or $m \| n$.

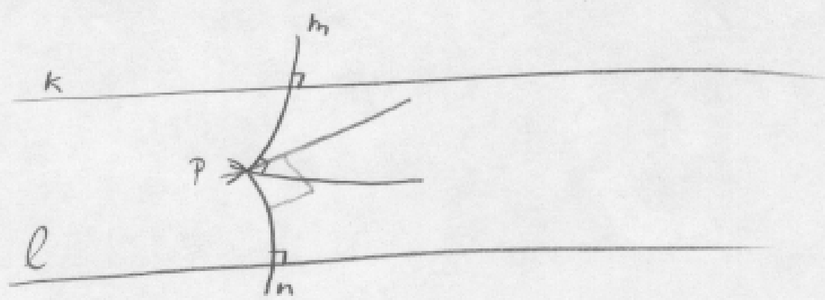
$\boxed{\text{Hilbert} \Rightarrow P:}$

Two cases: 1) <u>either</u> $m // n$ (and we are done)

2/ <u>OR</u> m intersects n: assume $m \neq n$, so there is a unique intersection point P. Since $m \neq n$, the perpendiculars through P to m and n are distinct and are parallel to n.

(use alternate interior angle thm + the fact that Hilbert $\Rightarrow$ transitivity of parallelism).

But this contradicts Hilbert, thus $m = n$.



Problem 3. (25 points) Assume you are in a Hilbert plane :

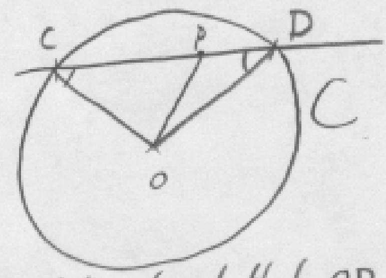
assume that the line l meets a circle $\mathcal{C}$ in two distinct points C and D. Prove the following, using axioms, definitions and any relevant propositions from the list:

a) Point P on l lies inside $\mathcal{C}$ if and only if $C * P * D$;

b) If points A and B are inside $\mathcal{C}$ and on opposite sides of l, then the point E at which segment AB meets l is between C and D.

(a)

If $C * P * D$:



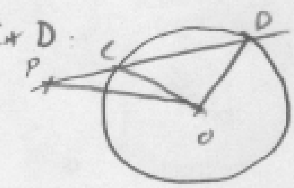
then $\underbrace{\angle OPD > \angle OCD}_{\text{Ext. Angle Thm}} \; \underbrace{\cong \angle ODC}_{\text{because } \triangle OCD \text{ isosceles}}$

Apply proposition "the greater side lies opposite the greater angle", to get that $OD > OP$

$\Rightarrow$ P is inside $\mathcal{C}$.

Let's prove that $\left.\begin{array}{c} P * C * D \\ \text{or} \\ C * D * P \end{array}\right\} \Rightarrow$ P is not inside $\mathcal{C}$

if $P * C * D$:



then $\angle ODP \cong \underbrace{\angle OCD > \angle OPC}_{\text{Ext. Angle. Thm}}$, therefore $OP > OD$.

Case $C * D * P$ is similar.

(b) Reprove that the interior of a circle is convex (cf. practice exam)

$\Rightarrow$ pt E is inside $\mathcal{C}$ and on line l

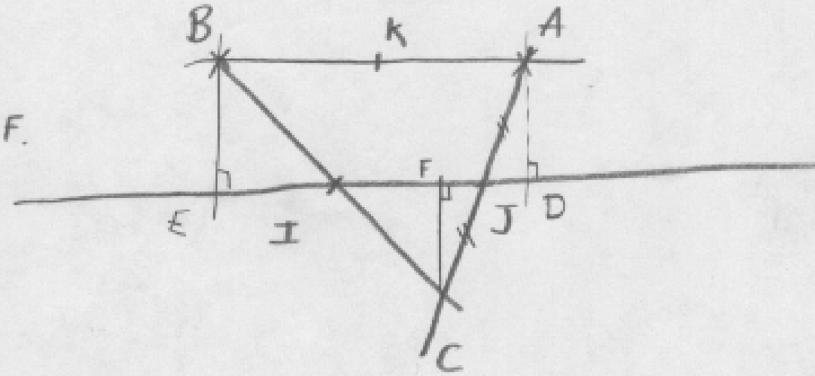
$\Rightarrow$ E is between C and D (by question (a)).

Problem 4. (25 points) Assume you are in a Hilbert plane. You can use the axioms, definitions and relevant propositions.

Consider a triangle $\triangle ABC$. Let I,J,K be the midpoints of BC, CA, AB respectively. Let D, E, F be the feet of the perpendiculars from A, B, C, respectively to line (IJ).

a) Prove that $AD \cong CF \cong BE$.

b) Prove that the perpendicular bisector of AB is also perpendicular to the line (IJ). (You can use that in a Saccheri quadrilateral the line joigning the midpoints of the summit and the base is perpendicular to both summit and base)



(a) The triangles ΔADJ and ΔCFJ are congruent by A-A-S, thus $AD \cong CF$. Similarly $\Delta BIE \cong \Delta CIF$, therefore $BE \cong CF$.

(b) By (a), BEDA is a Saccheri quadrilateral. Thus using the theorem mentioned in (b) we know that the perpendicular bisector of AB coincides with the line joigning the midpoints of the summit and the base, and therefore is $\perp$ to the base (IJ). (We used the unicity of a perpendicular to a line through a given pt.)

Solutions of Practice Midterm II

Name:

Student I.D:

Problem 1. (25 points)

1. Prove that in any Hilbert space every angle has a unique bisector. (You can use the existence and unicity of the midpoint of a segment).
2. Consider a triangle $\triangle$ABC. Using the above result, bisect the angle $\lhd A$. Explain why the bisector meets segment BC at some point D. Use this point D to reprove the triangle inequality (namely $\overline{\mathrm{AB}} + \overline{\mathrm{AC}} > \overline{\mathrm{BC}}$).

Proof.

1. Suppose you want to bisect an angle $\lhd$ BAC. You can assume, to simplify, that AB is congruent to AC (so that the triangle is isosceles). Consider D the midpoint of segment CD. By S-A-S, the two triangles ACD and ABD are congruent, therefore angle $\lhd$CAD is congruent to angle $\lhd$DAB. Now D is between B and C, therefore it is inside the angle $\lhd A$, and by angle addition we deduce that the ray $\overrightarrow{\mathrm{AD}}$ bisects the angle $\lhd A$. The bisector is unique by the axiom "transporting the angle".
2. Now we don't necessarily assume that the triangle is isosceles. But the bisector must intersect segment BC at a point D because of the crossbar theorem. Notice that now D is not necessarily the midpoint of segment BC. Now apply the Exterior Angle theorem twice to deduce that
   $(\lhd \mathrm{ADB})^o > (\lhd \mathrm{CAD})^o = (\lhd \mathrm{BAD})^o$ and
   $(\lhd \mathrm{ADC})^o > (\lhd \mathrm{BAD})^o = (\lhd \mathrm{CAD})^o$
   Apply now proposition P4.5 from the axioms list ("the greater side lies opposite the greater angle") and therefore deduce that:
   $\overline{\mathrm{AC}} > \overline{\mathrm{CD}}$ and $\overline{\mathrm{AB}} > \overline{\mathrm{BD}}$, but this implies the result by addition, using the fact that D lies between B and C.

□

Problem 2. (25 points) Prove that Hilbert's Euclidean parallel postulate is equivalent to the following proposition:

if t is a transversal to l and m, l parallel to m, and $t \perp l$ then $t \perp m$.

Proof. a) Hilbert implies P:

Assume t cuts m at P without forming a right angle, then if you consider the perpendicular through P to t you would get a line distinct from m that would b another parallel to l, a contradiction.

b) P implies Hilbert:

Consider the standard configuration: l is a line, P a point not on l; build t the perpendicular line through P to l and m the perpendicular through P to t. Assume now that there is another parallel through P to l, then it would form a different angle with respect to t, but this is a contradiction with proposition P.

$\square$

Problem 3. (25 points) Prove that in any Hilbert plane there exists one triangle that is not isosceles.

Proof. Take a line l, pick a point P not on l, drop a perpendicular through P in order to create a right angle at $Q \in l$. On one ray of the right angle, pick any length QR . On the other ray, transport twice that length to get a segment QS. Now RQS is a right triangle such that QR is not congruent to QS.

It remains to show that RS is not congruent to any other side. But the Exterior angle theorem says that each angle at R and S is acute. Therefore by proposition P4.5 ("the greater side lies opposite the greater angle"), we know that the hypotenuse is strictly greater than any other side and we are done.

$\square$

Problem 4. (25 points) In any Hilbert plane, prove that the interior of a circle is a convex set.

Proof. Let $\mathcal{C}$ be the circle with center O and with radius OM.

Given any two points A,B inside $\mathcal{C}$, we want to show that for any point P such that $A * P * B$ we have $\mathrm{OP} < \mathrm{OM}$.

One of the two angles OAB, OBA is smaller than the other (or congruent to the other): assume that $\triangleleft\,\mathrm{OAB} \leqslant \triangleleft\,\mathrm{OBA}$ for example. Then by the Exterior angle theorem, $\triangleleft\,\mathrm{OPB}$ is larger than $\triangleleft\,\mathrm{OAB}$ which is larger than $\triangleleft\,\mathrm{OBA}$.

Therefore, again by prop P4.5 we have that side OB is larger than side OP which is what we wanted.

$\square$

Practice Midterm II

Name:

Student I.D:

Problem 1. (25 points)

1. Prove that in any Hilbert space every angle has a unique bisector. (You can use the existence and unicity of the midpoint of a segment).
2. Consider a triangle $\triangle$ABC. Using the above result, bisect the angle $\lhd A$. Explain why the bisector meets segment BC at some point D. Use this point D to reprove the triangle inequality (namely $\overline{\mathrm{AB}} + \overline{\mathrm{AC}} > \overline{\mathrm{BC}}$).

Problem 2. (25 points) Prove that Hilbert's Euclidean parallel postulate is equivalent to the following proposition:
if t is a transversal to l and m, l parallel to m, and $t \perp l$ then $t \perp m$.

Problem 3. (25 points) Prove that in any Hilbert plane there exists one triangle that is not isosceles.

Problem 4. (25 points) In any Hilbert plane, prove that the interior of a circle is a convex set.

MAT 360 03/13/2008

Midterm I

Name:

Student I.D:

Problem 1. (25 points) Assume that the Incidence axioms and the Betweenness axioms are satisfied.

Given $A * B * C$ and $B * C * D$ prove that we have $A * B * D$ and $A * C * D$.

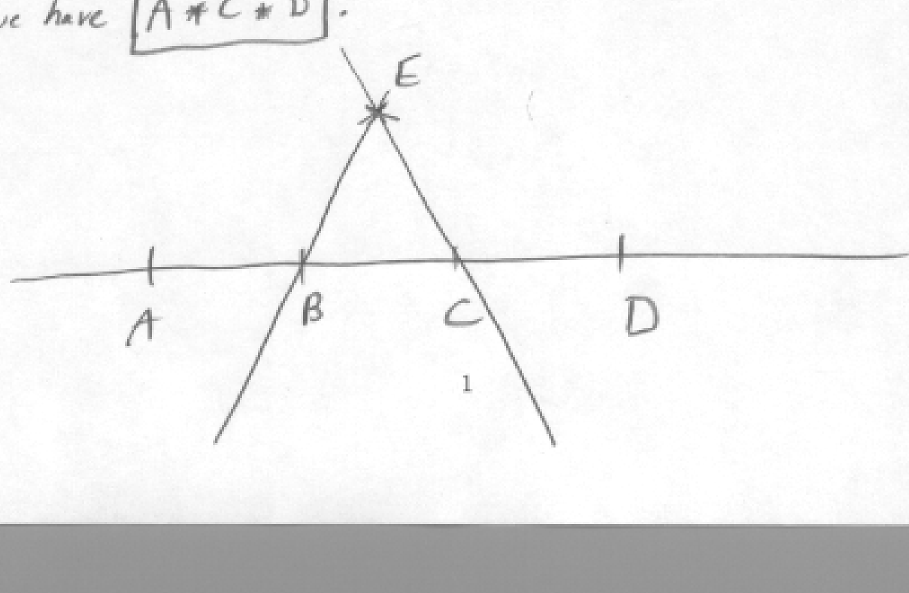
By axiom I 3 there exists a pt $E \notin \overleftrightarrow{AB}$. The line $\overleftrightarrow{EB}$ intersects $\overleftrightarrow{AB}$ only at B (because if there was another intersection pt, then I 1 would imply that line $\overleftrightarrow{AB}$ = line $\overleftrightarrow{EB}$, absurd.)

$\left.\begin{array}{l}\text{• Since } A * B * C \text{ we know that A and C are on opposite sides of } \overleftrightarrow{EB} \\ \text{• Since } B * C * D \text{, we don't have } C * B * D \text{ (axiom B-3)} \\ \quad \text{and therefore C and D are on the same side of } \overleftrightarrow{EB}\end{array}\right\} \Rightarrow$ by B4, we know that A and D are on opposite sides of $\overleftrightarrow{EB}$.

Therefore, by definition of "being on opposite sides" we know that $\boxed{A * B * D}$ is true.

$\left.\begin{array}{l}\text{Since } A * B * C \text{, we don't have } A * C * B \\ \quad \text{therefore A and B are on the same side of line } \overleftrightarrow{EC} \\ \text{Since } B * C * D \text{, B and D are on opposite sides of } \overleftrightarrow{EC}\end{array}\right\} \Rightarrow$ by B4, A and D are on opposite sides of $\overleftrightarrow{EC}$.

Therefore, we have $\boxed{A * C * D}$.



Problem 2. (25 points) Assume that the Incidence, Betweenness and Congruence axioms are satisfied.

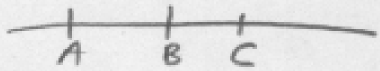
Let A,B,C be three distinct points such that AB $\cong$ BC $\cong$ AC.

Prove that A,B,C are **not** collinear.

By contradiction, assume that A, B, C are collinear.

By axiom B3, one must have one and only one of the three relations $A * B * C$, $A * C * B$, $B * A * C$.

Assume we have $A * B * C$:



Then given the segment AC, the axiom C1 says that on the ray $\overrightarrow{AC}$ there is a unique pt E (namely $E = C$ here) such that $AE \cong AC$. Since $A * B * C \Rightarrow B \in$ ray $\overrightarrow{AC}$ and that we know $AB \cong AC$ we deduce by unicity that $B = C$ (absurd).

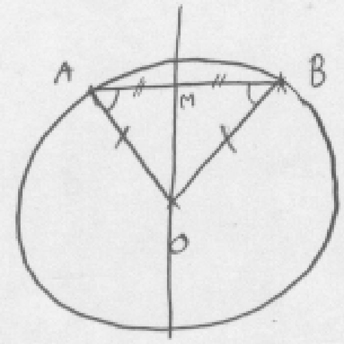
Therefore A, B, C are not collinear.

The 2 other cases ($A * C * B$, $B * A * C$) are similar (permute the letters).

Problem 3. (25 points) Assume that the Incidence, Betweenness and Congruence axioms are satisfied.

Let $\mathcal{C}$ be a circle with center O and let A and B be two distinct points on $\mathcal{C}$. The segment AB is called a chord of $\mathcal{C}$. Assume that you already constructed the midpoint M of AB.

If $O \neq M$ prove that the line (OM) is perpendicular to the line (AB).



By definition of a circle,

$$\left.\begin{array}{l} A \in \mathcal{C} \\ B \in \mathcal{C} \end{array}\right\} \Rightarrow OA \cong OB.$$

Now consider the two triangles ΔAOB and ΔBOA:

- their angles at $\measuredangle O$ are congruent
- so are the sides OA and OB

therefore by axiom C6 (SAS), the 2 triangles are congruent.

Thus $\measuredangle BAO \cong \measuredangle ABO$.

Since M is the midpoint of AB we know that $AM \cong MB$.

Now by axiom C6 (SAS) again, the 2 triangles $\left\{\begin{array}{l} \Delta MAO \\ \Delta MBO \end{array}\right.$ are congruent

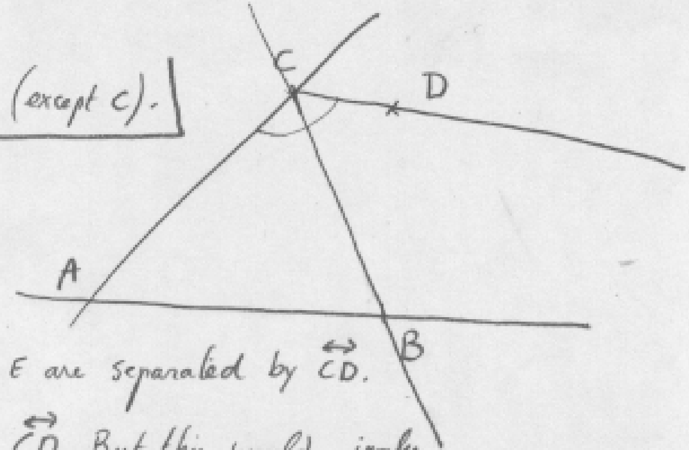
Therefore $\measuredangle AMO \cong \measuredangle BMO$. But these 2 angles are supplementary (they share one ray $\overrightarrow{MO}$, and $\overrightarrow{MA}$ is opposite to $\overrightarrow{MB}$).

Thus the angle at M is a right angle (def. of a right angle),

and therefore $\overleftrightarrow{OM}$ is perpendicular to $\overleftrightarrow{AB}$ (def. of perpendicular lines).

Problem 4. (25 points) Assume that the Incidence, Betweenness and Congruence axioms are satisfied. Consider a triangle $\triangle ABC$ and a point D such that D is in the interior of the angle $\sphericalangle CAB$ and D is outside the triangle $\triangle ABC$.

Prove that every point on ray $\overrightarrow{CB}$ (except C) is inside the angle $\sphericalangle ACD$.



Let's prove the following lemma:

$\boxed{\text{If B is inside } \sphericalangle ACD\text{, then so is every pt on } \overrightarrow{CB} \text{ (except C).}}$

Proof of lemma:

By contradiction: assume there exists $E \in \overrightarrow{CB}$ $(E \neq C)$ outside $\sphericalangle ACD$. Then B and E would be on opposite sides of one of the lines $\overleftrightarrow{CD}$ or $\overleftrightarrow{CA}$. Say that B and E are separated by $\overleftrightarrow{CD}$. Then there exists F between B and E that is on $\overleftrightarrow{CD}$. But this would imply that line $\overleftrightarrow{CB} = \overleftrightarrow{CF} = \overleftrightarrow{CD}$ which is absurd. The case where B, E are separated by $\overleftrightarrow{CA}$ is similar.

Using the lemma, we reduce the problem to the following question: $\boxed{\text{prove that B is inside } \sphericalangle ACD}$.

(a) B and D are on the same side of $\overleftrightarrow{AC}$:

indeed $D \in$ interior of $\sphericalangle CAB$ implies exactly this.

Thus it remains only to prove:

(b) B and A are on the same side of $\overleftrightarrow{CD}$:

proof by contradiction: if B and A are on opposite sides of $\overleftrightarrow{CD}$ then $\exists\, G$ between A and B s.t. $G \in \overleftrightarrow{CD}$.

Now such a G cannot be on the ray opposite to $\overrightarrow{CD}$ (because such pts are on the opposite side of B with respect to $\overleftrightarrow{AC}$, but $A * G * B$ implies that G and B are together). Therefore, if it exists, such a G must be on ray $\overrightarrow{CD}$. Thus ray $\overrightarrow{CD}$ = ray $\overrightarrow{CG}$. (indeed $H \notin \overrightarrow{CD} \Leftrightarrow H * C * D \Leftrightarrow H * C * G \Leftrightarrow H \notin \overrightarrow{CG}$).

Now $A * G * B \Rightarrow G \in$ interior of $\sphericalangle ACB$ (indeed by B3, $A * G * B$ prevents $G * A * B$ and $A * B * G$).

But our lemma implies that every pt of ray $\overrightarrow{CG}$ (except C) must be inside $\sphericalangle ACB$.

Therefore $D \in$ inside $\sphericalangle ACB$. But we already know that $D \in$ inside of $\sphericalangle CAB$, so this would imply that $D \in$ interior of triangle $\triangle ABC$. (Contradiction). So we are done.

Practice Midterm I, Solutions

#1:

- Pick a pt E not on line $\overleftrightarrow{AB}$ (exists because of I 3).
- The line $\overleftrightarrow{EC}$ intersects $\overleftrightarrow{AB}$ only at C (if there was another pt of intersection, by I 1 we would have $\overleftrightarrow{EC} = \overleftrightarrow{AB}$)
- $\begin{cases} A \text{ and } B \text{ are on the same side of } \overleftrightarrow{EC} \text{ (because } A * B * C) \\ A \text{ and } D \text{ are on opposite sides of } \overleftrightarrow{EC} \text{ (because } A * C * D) \end{cases}$

$\Longrightarrow$ B and C are on opposite sides of $\overleftrightarrow{EC}$ $\Rightarrow$ $\boxed{B * C * D}$

Now we know that $A * B * C$

and $B * C * D$,

so using line $\overleftrightarrow{EB}$ one gets: $\left.\begin{cases} A \text{ and } C \text{ are on opposite sides} \\ C \text{ and } D \text{ are on same side} \end{cases}\right\} \Rightarrow$ A and D are on opposite sides $\Rightarrow \boxed{A * B * D}$.

#2 Verification of the axioms:

Incidence: I-1: satisfied because I-1 is satisfied in the Euclidean plane.

I-2: satisfied by definition of the lines in our model.

I-3: Take $(0,0)$, $(0,1)$ and $(1,0)$. (clearly $(1,0)$ is not on the horizontal axis because the 1st coordinate is $1 \neq 0$)

Betweenness: Like in euclidean geometry we will say that $A * B * C$ iff $\vec{AB} = t\vec{AC}$ for some $t \in (0,1)$, $A \neq C$.

B-1: indeed $\vec{AB} = t\vec{AC}$ implies that line $\overleftrightarrow{AB}$ = line $\overleftrightarrow{AC}$ and that $A \neq B$, $B \neq C$.
also clearly $\vec{CB} = (1-t)\vec{CA}$ so we have $C * B * A$.

B-2: Given $B \neq D$:
- pick C the midpoint of BD, clearly $\vec{BC} = \frac{1}{2}\vec{BD}$ and C has coordinates of the form $\frac{a}{2^n}$ (because they $\frac{1}{2}\left(\frac{a}{2^n} + \frac{b}{2^m}\right)$).
- similarly, take E so that $\vec{BE} = 2\vec{BD}$, implying $B * D * E$.
then again the coordinates of E have the right form (because $2\vec{BD}$ is of the form $\frac{a}{2^n}$ and so are the coordinates of E, and adding or subtracting 2 such numbers leads to another nb of the same form: $\left\{\frac{a}{2^n}, a, n \in \mathbb{Z}\right\}$ is a subring of $\mathbb{Q}$).
- Finally pick A s.t $\vec{DA} = 2\vec{DB}$.

axiom B-3: satisfied because B-3 is already satisfied in the euclidean plane.

#3: This statement is clearly a consequence of the Angle-Side-Angle criterion.
So let's prove the A-S-A criterion (saying that $\Delta ABC \cong \Delta DEF$ if $\angle A \cong \angle D$, $\angle C \cong \angle F$ and $AC \cong DF$):

Proof of A-S-A criterion:

1/ By axiom C-1, there is a unique pt B' on ray $\overrightarrow{DE}$ s.that $DB' \cong AB$

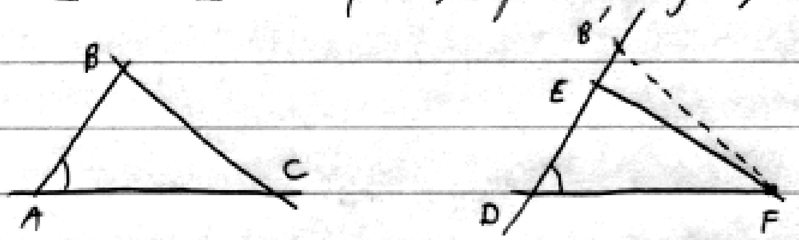
2/ Thus $\Delta ABC \cong \Delta DB'F$ (S-A-S criterion).

3/ Then $\angle DFB' \cong \angle C$ (consequence of $\Delta ABC \cong \Delta DB'F$).

4/ From C-4, we know that ray $\overrightarrow{FB'}$ = ray $\overrightarrow{FE}$ (because the angles at F are both equal to $\angle C$).

5/ Therefore $B' = E$ (unique pt of intersection of $\overleftrightarrow{DE}$ and $\overleftrightarrow{FE} = \overleftrightarrow{FB'}$).

6/ Conclusion: $\Delta ABC \cong \Delta DEF$. (in 2/ replace B' by E).



#4: We know $\angle ABC \cong \angle DEF$, we want to prove $\angle CBG \cong \angle FEH$.

1/ We can choose D, F, H so that $AB \cong DE$, $CB \cong FE$, $BG \cong EH$ (by axiom C-1).

2/ Thus $\Delta ABC \cong \Delta DEF$ (S-A-S axiom).

3/ Thus $AC \cong DF$ and $\angle A \cong \angle D$

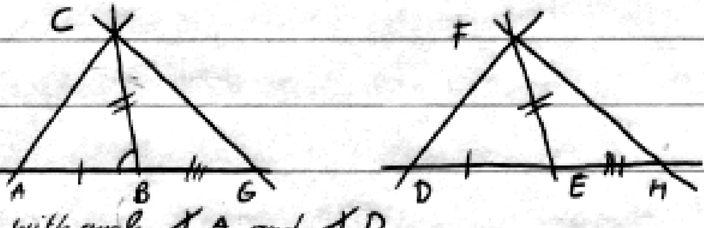
4/ Then $AG \cong DH$ (axiom C-3).

5/ So $\Delta ACG \cong \Delta DFH$ (S-A-S axiom) with angle $\angle A$ and $\angle D$

6/ Thus $CG \cong FH$, $\angle G \cong \angle H$

7/ Then $\Delta CBG \cong \Delta FEH$ (S-A-S again)

8/ Therefore $\angle CBG \cong \angle FEH$.



Practice Midterm I

Name:

Student I.D:

Problem 1. (25 points) Assume that the Incidence axioms and the Betweenness axioms are satisfied and prove the following statement:

Given $A * B * C$ and $A * C * D$, then we must have $B * C * D$ and $A * B * D$.

(Justify each step by saying which axiom you use, for example, "by I3, we know that...")

Problem 2. (25 points) Consider the following geometry:

Points: they are couples of the form $(\frac{a}{2^n}, \frac{b}{2^m})$ where a, b, n, m are integers;

Lines: they are just the standard euclidean lines joigning two such points in $\mathbb{R}^2$.

Show that the incidence axioms are satisfied and also the first three betweenness axioms.

Problem 3. (25 points) Assume that the Incidence axioms, the Betweenness axioms and the Congruence axioms are satisfied. Prove the following statement:

If in the triangle $\triangle$ABC *we have* angle $\lhd B \cong$ angle $\lhd C$ *then* AB $\cong$ AC *and the triangle* $\triangle$ABC *is isosceles.*

Problem 4. (25 points) Assume that the Incidence axioms, the Betweenness axioms and the Congruence axioms are satisfied. Prove the following statement:

Supplements of congruent angles are congruent.

(You are allowed to use these axioms, and also the following congruence propositions in the list of axioms: P3.10, P3.11, P3.12. You might want to use the Side-Angle-Side axiom, so using your angles you should build similar triangles on them...)

Definitions, Axioms, Postulates, Propositions, and Theorems from *Euclidean and Non-Euclidean Geometries* by Marvin Jay Greenberg

Undefined Terms: Point, Line, Incident, Between, Congruent.

Incidence Axioms:

IA1: For every two distinct points there exists a unique line incident on them.

IA2: For every line there exist at least two points incident on it.

IA3: There exist three distinct points such that no line is incident on all three.

Incidence Propositions:

P2.1: If l and m are distinct lines that are non-parallel, then l and m have a unique point in common.

P2.2: There exist three distinct lines such that no point lies on all three.

P2.3: For every line there is at least one point not lying on it.

P2.4: For every point there is at least one line not passing through it.

P2.5: For every point there exist at least two distinct lines that pass through it.

Betweenness Axioms:

B1: If $A * B * C$, then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

B2: Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A{*}B{*}D$, $B{*}C{*}D$, and $B{*}D{*}E$.

B3: If A, B, and C are three distinct points lying on the same line, then one and only one of them is between the other two.

B4: For every line l and for any three points A, B, and C not lying on l:

1. If A and B are on the same side of l, and B and C are on the same side of l, then A and C are on the same side of l.
2. If A and B are on opposite sides of l, and B and C are on opposite sides of l, then A and C are on the same side of l.

Corollary If A and B are on opposite sides of l, and B and C are on the same side of l, then A and C are on opposite sides of l.

Betweenness Definitions:

Segment AB**:** Point A, point B, and all points P such that $A{*}P{*}B$.

Ray $\overrightarrow{AB}$**:** Segment AB and all points C such that $A{*}B{*}C$.

Line $\overleftrightarrow{AB}$**:** Ray $\overrightarrow{AB}$ and all points D such that $D{*}A{*}B$.

Same/Opposite Side: Let l be any line, A and B any points that do not lie on l. If $A = B$ or if segment AB contains no point lying on l, we say A and B are *on the same side of* l, whereas if $A \neq B$ and segment AB does intersect l, we say that A and B are *on opposite sides of* l. The law of excluded middle tells us that A and B are either on the same side or on opposite sides of l.

Betweenness Propositions:

P3.1: For any two points A and B:

1. $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$, and
2. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

P3.2: Every line bounds exactly two half-planes and these half-planes have no point in common.

Same Side Lemma: Given $A{*}B{*}C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point A, then B and C are on the same side of line l.

Opposite Side Lemma: Given $A{*}B{*}C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point B, then A and B are on opposite sides of line l.

P3.3: Given $A{*}B{*}C$ and $A{*}C{*}D$. Then $B{*}C{*}D$ and $A{*}B{*}D$.

P3.4: If $C{*}A{*}B$ and l is the line through A, B, and C, then for every point P lying on l, P either lies on ray $\overrightarrow{AB}$ or on the opposite ray $\overrightarrow{AC}$.

P3.5: Given $A{*}B{*}C$. Then $AC = AB \cup BC$ and B is the only point common to segments AB and BC.

P3.6: Given $A{*}B{*}C$. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.

Pasch's Theorem: If A, B, and C are distinct points and l is any line intersecting AB in a point between A and B, then l also intersects either AC, or BC. If C does not lie on l, then l does not intersect both AC and BC.

Angle Definitions:

Interior: Given an angle $\sphericalangle CAB$, define a point D to be in the *interior* of $\sphericalangle CAB$ if D is on the same side of $\overleftrightarrow{AC}$ as B and if D is also on the same side of $\overleftrightarrow{AB}$ as C. Thus, the interior of an angle is the intersection of two half-planes. (Note: the interior does not include the angle itself, and points not on the angle and not in the interior are on the exterior).

Ray Betweenness: Ray $\overrightarrow{AD}$ is *between* rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ provided $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to $\sphericalangle CAB$.

Interior of a Triangle: The interior of a triangle is the intersection of the interiors of its thee angles. Define a point to be *exterior* to the triangle if it in not in the interior and does not lie on any side of the triangle.

Triangle: The union of the three segments formed by three non-collinear points.

Angle Propositions:

P3.7: Given an angle $\sphericalangle CAB$ and point D lying on line $\overleftrightarrow{BC}$. Then D is in the interior of $\sphericalangle CAB$ iff $B{*}D{*}C$.

"Problem 9": Given a line l, a point A on l and a point B not on l. Then every point of the ray $\overrightarrow{AB}$ (except A) is on the same side of l as B.

P3.8: If D is in the interior of $\sphericalangle CAB$, then:

1. so is every other point on ray $\overrightarrow{AD}$ except A,
2. no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\sphericalangle CAB$, and
3. if $C{*}A{*}E$, then B is in the interior of $\sphericalangle DAE$.

P3.9:

1. If a ray r emanating from an exterior point of $\triangle ABC$ intersects side AB in a point between A and B, then r also intersects side AC or BC.
2. If a ray emanates from an interior point of $\triangle ABC$, then it intersects one of the sides, and if it does not pass through a vertex, then it intersects only one side.

Crossbar Theorem: If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC.

Congruence Axioms:

C1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.

C2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

C3: If $A{*}B{*}C$, and $A'{*}B'{*}C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

C4: Given any $\sphericalangle BAC$ (where by definition of angle, $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$ and is distinct from $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$.

C5: If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

C6 (SAS): If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

Congruence Propositions:

P3.10: If in $\triangle ABC$ we have $AB \cong AC$, then $\sphericalangle B \cong \sphericalangle C$.

P3.11: If $A{*}B{*}C$, $D{*}E{*}F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.

P3.12: Given $AC \cong DF$, then for any point B between A and C, there is a unique point E between D and F such that $AB \cong DE$.

P3.13:
1. Exactly one of the following holds: $AB < CD$, $AB \cong CD$, or $AB > CD$.
2. If $AB < CD$ and $CD \cong EF$, then $AB < EF$.
3. If $AB > CD$ and $CD \cong EF$, then $AB > EF$.
4. If $AB < CD$ and $CD < EF$, then $AB < EF$.

P3.14: Supplements of Congruent angles are congruent.

P3.15:
1. Vertical angles are congruent to each other.
2. An angle congruent to a right angle is a right angle.

P3.16: For every line l and every point P there exists a line through P perpendicular to l.

P3.17 (ASA): Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.18: In in $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

P3.19: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle GBA \cong \sphericalangle HED$. Then $\sphericalangle ABC \cong \sphericalangle DEF$.

P3.20: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle ABC \cong \sphericalangle DEF$. Then $\sphericalangle GBA \cong \sphericalangle HED$.

P3.21:
1. Exactly one of the following holds: $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle P > \sphericalangle Q$.
2. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
3. If $\sphericalangle P > \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P > \sphericalangle R$.
4. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

P3.22 (SSS): Given $\triangle ABC$ and $\triangle DEF$. If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.23: All right angles are congruent to each other.

Corollary (not numbered in text) If P lies on l then the perpendicular to l through P is unique.

Definitions:

Segment Inequality: $AB < CD$ (or $CD > AB$) means that there exists a point E between C and D such that $AB \cong CE$.

Angle Inequality: $\sphericalangle ABC < \sphericalangle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABC \cong \sphericalangle GEF$.

Right Angle: An angle $\sphericalangle ABC$ is a right angle if has a supplementary angle to which it is congruent.

Parallel: Two lines l and m are parallel if they do not intersect, i.e., if no point lies on both of them.

Perpendicular: Two lines l and m are perpendicular if they intersect at a point A and if there is a ray $\overrightarrow{AB}$ that is a part of l and a ray $\overrightarrow{AC}$ that is a part of m such that $\sphericalangle BAC$ is a right angle.

Triangle Congruence and Similarity: Two triangles are congruent if a one-to-one correspondence can be set up between their vertices so that corresponding sides are congruent and corresponding angles are congruent. Similar triangles have this one-to-one correspondence only with their angles.

Circle (with center O and radius OA): The set of all points P such that OP is congruent to OA.

Triangle: The set of three distinct segments defined by three non-collinear points.

Continuity Axioms:

Archimedes' Axiom: If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E.

Dedekind's Axiom: Suppose that the set of all points on a line l is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and visa versa. Then there is a unique point O lying on l such that $P_1 * O * P_2$ if and only if one of P_1, P_2 is in Σ_1, the other in Σ_2 and $O \neq P_1, P_2$. A pair of subsets Σ_1 and Σ_2 with the properties in this axiom is called a Dedekind cut of the line l.

Continuity Principles: Circular Continuity Principle: If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.

Elementary Continuity Principle: In one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.

Other Theorems, Propositions, and Corollaries in Neutral Geometry:

T4.1: If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.

Corollary 1: Two lines perpendicular to the same line are parallel. Hence the perpendicular dropped from a point P not on line l to l is unique.

Corollary 2: If l is any line and P is any point not on l, there exists at least one line m through P parallel to l.

T4.2 (Exterior Angle Theorem): An exterior angle of a triangle is greater than either remote interior angle.

T4.3 (see text for details): There is a unique way of assigning a degree measure to each angle, and, given a segment OI, called a unit segment, there is a unique way of assigning a length to each segment AB that satisfy our standard uses of angle and length.

Corollary 1: The sum of the degree measures of any two angles of a triangle is less than $180°$.

Corollary 2: If A, B, and C are three noncollinear points, then $\overline{AC} < \overline{AB} + \overline{BC}$.

T4.4 (Saccheri-Legendre): The sum of the degree measures of the three angles in any triangle is less than or equal to $180°$.

Corollary 1: The sum of the degree measures of two angles in a triangle is less than or equal to the degree measure of their remote exterior angle.

Corollary 2: The sum of the degree measures of the angles in any convex quadrilateral is at most $360°$ (**note: quadrilateral $\square ABCD$ is convex if it has a pair of opposite sides such that each is contained in a half-plane bounded by the other.**)

P4.1 (SAA): Given $AC \cong DF$, $\sphericalangle A \cong \sphericalangle D$, and $\sphericalangle B \cong \sphericalangle E$. Then $\triangle ABC \cong \triangle DEF$.

P4.2: Two right triangles are congruent if the hypotenuse and leg of one are congruent respectively to the hypotenuse and a leg of the other.

P4.3: Every segment has a unique midpoint.

P4.4:

1. Every angle has a unique bisector.
2. Every segment has a unique perpendicular bisector.

P4.5: In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e., $AB > BC$ if and only if $\sphericalangle C > \sphericalangle A$.

P4.6: Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\sphericalangle B < \sphericalangle B'$ if and only if $AC < A'C'$.

Note: Statements up to this point are from or form neutral geometry. Choosing Hilbert's/Euclid's Axiom (the two are logically equivalent) or the Hyperbolic Axiom will make the geometry Euclidean or Hyperbolic, respectively.

Parallelism Axioms:

Hilbert's Parallelism Axiom for Euclidean Geometry: For every line l and every point P not lying on l there is at most one line m through P such that m is parallel to l. (Note: it can be proved from the previous axioms that, assuming this axiom, there is **EXACTLY** one line m parallel to l [see T4.1 Corollary 2]).

Euclid's Fifth Postulate: If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

Hyperbolic Parallel Axiom: There exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.

Hilbert's Parallel Postulate is logically equivalent to the following:

T4.5: Euclid's Fifth Postulate.

P4.7: If a line intersects one of two parallel lines, then it also intersects the other.

P4.8: Converse to Theorem 4.1.

P4.9: If t is transversal to l and m, $l \| m$, and $t \perp l$, then $t \perp m$.

P4.10: If $k \| l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \| n$.

P4.11: The angle sum of every triangle is 180°.

Wallis: Given any triangle $\triangle ABC$ and given any segment DE. There exists a triangle $\triangle DEF$ (having DE as one of its sides) that is similar to $\triangle ABC$ (denoted $\triangle DEF \sim \triangle ABC$).

Theorems 4.6 and 4.7 (see text) are used to prove P4.11. They define the *defect* of a triangle to be the 180° minus the angle sum, then show that if one defective triangle exists, then all triangles are defective. Or, in contrapositive form, if one triangle has angle sum 180°, then so do all others. They do not assume a parallel postulate.

Theorems Using the Parallel Axiom

Parallel Projection Theorem: Given three parallel lines l, m, and n. Let t and t' be transversals to these parallels, cutting them in points A, B, and C and in points A', B', and C', respectively. Then $\overline{AB}/\overline{BC} = \overline{A'B'}/\overline{B'C'}$.

Fundamental Theorem on Similar Triangles: Given $\triangle ABC \sim \triangle A'B'C'$. Then the corresponding sides are proportional.

HYPERBOLIC GEOMETRY

L6.1: There exists a triangle whose angle sum is less than $180°$.

Universal Hyperbolic Theorem: In hyperbolic geometry, from every line l and every point P not on l there pass through P at least two distinct parallels to l.

T6.1: Rectangles do not exist and all triangles have angle sum less than $180°$.

Corollary: In hyperbolic geometry, all convex quadrilaterals have angle sum less than $360°$.

T6.2: If two triangles are similar, they are congruent.

T6.3: If l and l' are any distinct parallel lines, then any set of points on l equidistant from l' has at most two points in it.

T6.4: If l and l' are parallel lines for which there exists a pair of points A and B on l equidistant from l', then l and l' have a common perpendicular segment that is also the shortest segment between l and l'.

L6.2: The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and the summit, and this segment is shorter than the sides.

T6.5: If lines l and l' have a common perpendicular MM', then they are parallel and MM' is unique. Moreover, if A and B are points on l such that M is the midpoint of segment AB, then A and B are equidistant from l'.

T6.6: For every line l and every point P not on l, let Q be the foot of the perpendicular from P to l. Then there are two unique rays $\overrightarrow{PX}$ and $\overrightarrow{PX'}$ on opposite sides of $\overleftrightarrow{PQ}$ that do not meet l and have the property that a ray emanating from P meets l if and only if it is between $\overrightarrow{PX}$ and $\overrightarrow{PX'}$. Moreover, these limiting rays are situated symmetrically about $\overleftrightarrow{PQ}$ in the sense that $\measuredangle XPQ \cong \measuredangle X'PQ$.

T6.7: Given m parallel to l such that m does not contain a limiting parallel ray to l in either direction. Then there exists a common perpendicular to m and l, which is unique.

Results from chapter 7 (Contextual definitions not included):

P7.1
1. $P = P'$ if and only if P lies on the circle of inversion γ.
2. If P is inside γ then P' is outside γ, and if P is outside γ, then P' is inside γ.
3. $(P')' = P$.

P7.2 Suppose P is inside γ. Let TU be the chord of γ which is perpendicular to $\overleftrightarrow{OP}$. Then the inverse P' of P is the pole of chord TU, i.e., the point of intersection of the tangents to γ at T and U.

P7.3 If P is outside γ, let Q be the midpoint of segment OP. Let σ be the circle with center Q and radius $\overline{OQ} = \overline{QP}$. Then σ cuts γ in two points T and U, $\overleftrightarrow{PT}$ and $\overleftrightarrow{PU}$ are tangent to γ, and the inverse P' of P is the intersection of TU and OP.

P7.4 Let T and U be points on γ that are not diametrically opposite and let P be the pole of TU. Then $PT \cong PU$, $\measuredangle PTU \cong \measuredangle PUT$, $\overleftrightarrow{OP} \perp \overleftrightarrow{TU}$, and the circle δ with center P and radius $\overline{PT} = \overline{PU}$ cuts γ orthogonally at T and U.

L7.1 Given that point O does not lie on circle δ.

1. If two lines through O intersect δ in pairs of points (P_1, P_2) and (Q_1, Q_2), respectively, then we have $(\overline{OP_1})(\overline{OP_2}) = (\overline{OQ_1})(\overline{OQ_2})$. This common product is called the *power* of O with respect to δ when O is outside of δ, and minus this number is called the power of O when O is inside δ.
2. If O is outside δ and a tangent to δ from O touches δ at point T, then $(\overline{OT})^2$ equals the power of O with respect to δ.

P7.5 Let P be any point which does not lie on circle γ and which does not coincide with the center O of γ, and let δ be a circle through P. Then δ cuts γ orthogonally if and only if δ passes through the inverse point P' of P with respect to γ.

Correction...

Ok, so it seems that there is a divine justice punishing the arrogant professor beginning a sentence like this: "Ok guys, you made many mistakes in your HW, now let me show you how things should be done..."

Now here is what I should have said the other day....

Exercise 1. Given three distinct points such that $A * B * C$, show that ray $\overrightarrow{\mathrm{AB}}$= ray $\overrightarrow{\mathrm{AC}}$.

Proof. Let's prove that $\overrightarrow{\mathrm{AB}} \subset \overrightarrow{\mathrm{AC}}$ (the other inclusion is similar).

Consider a point $D \in \overrightarrow{\mathrm{AB}}$:

according to the definition of a ray, there are 2 cases to study:

1. ***D* is a point of the segment AB**:

   either a) D is an endpoint of AB ($D = A$ or $D = B$): we are done because these points are already on ray $\overrightarrow{\mathrm{AC}}$.

   or b) we have $A * D * B$: then it is enough to show that in this case we have $A * D * C$ or $A * C * D$ (because both mean that D is on the ray $\overrightarrow{\mathrm{AC}}$).
   By axiom B3, we just need to rule out one single case, namely $C * A * D$
   Ruling out $\boldsymbol{C * A * D}$: take a line l intersecting line AB at A (exists because of I3). Now B and C are on the same side of l (otherwise l would intersect the segment BC at A between B and C, impossible because we know that $A * B * C$), and D and B are on the same side of l (same reason, we know that $A * D * B$ is true), thus D and C are on the same side of l and this says that $C * A * D$ is impossible.

2. **D satisfies $\boldsymbol{A * B * D}$**:

   Consider the points A,C, D: by axiom B3 there are only 3 possibilities:

   $A * C * D$ (but in this case we are done, D is on ray $\overrightarrow{\mathrm{AC}}$), $A * D * C$ (but in this case we are done, D is on ray $\overrightarrow{\mathrm{AC}}$), and $D * A * C$.

   Ruling out $\boldsymbol{D * A * C}$: pick a line m intersecting line AB only at A (exists because I3). Then $A * B * C \Longrightarrow$ B and C are on the same side of m, and $A * B * D \Longrightarrow$ B and D are on the same side. Thus we know that C and D must be on the same side (axiom B4), but this rules out exactly $D * A * C$.

Remark:

when I made a picture of Case 1), it appeared that case CAD was ruled out, but also I tried desperately to rule out ACD (but I didn't need to). Now is it true that ACD must be ruled out, or is it just the picture that is misleading us?

□

HW 1

Exercise 1. Let (A, B, C) be a triangle in the plane, and let M be the midpoint of the segment BC. Show the following equality:
$AB^2 - AC^2 = 2\overrightarrow{AM}.\overrightarrow{CB}$

Exercise 2. I give you a segment of length 1. Now, using ruler and compass, explain how to construct a segment of length $\frac{\sqrt{3}}{5}$.

Exercise 3. Let A and B be two points in the plane. What is the set $\mathcal{S}$ of all points M in the plane satisfying the following equality: $\overrightarrow{AM}.\overrightarrow{AB} = AB^2$? (I want you to describe it in geometric terms, for example, I don't know, "it's the circle with center blabla and radius blibli", etc)

Exercise 4. Let A and B be two points in the plane. What is the set $\mathcal{T}$ of all points M in the plane such that $\overrightarrow{MA}.\overrightarrow{MB} = 0$?

Exercise 5. Squaring linkage. In this problem I propose to describe a linkage made of rods, articulations that automatically returns the square of a length. (the same object can construct square roots too).

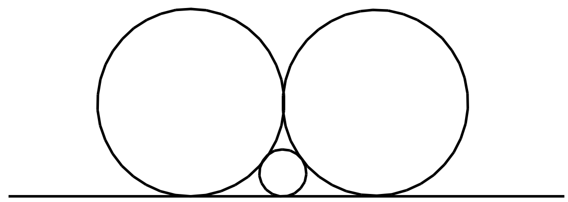
a) **Bisector**. Describe a linkage, made of rods and articulations that allows you to double, or divide by 2 any length. (we did that in class, now you have to remember it!)

b) **Adding linkage**. Come up with a linkage that creates a segment of length $a + b$, given a length a and a length b. (you can use rods, articulations, cursors sliding along a line, etc...)

c) **Inversor**. Draw the linkage we saw in class (the one that takes a length r and returns $1/r$) .

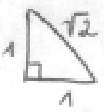
d) **Combination of these linkages**. Given a length r explain how to combine the previous mechanisms in order to return a segment with length r^2.
(Giant hint: $\frac{1}{x-1} - \frac{1}{x+1} = ?$)

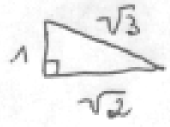
Exercise 6. What is the radius r of the little disk stuck between his two big brothers (having the same radius R) ? All the three disks are tangent to each other (even if they don't look like that...) and tangent to the horizontal line also.



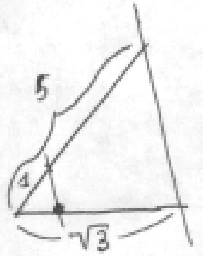
HW 1 Solutions

#1: $AB^2 - AC^2 = (\vec{AB}+\vec{AC})\cdot(\vec{AB}-\vec{AC}) = 2\vec{AM}\cdot\vec{CB}$.

#2: 

and then 

. Finish with "Thales theorem" (similitude of triangles):



#3: $\vec{AM}\cdot\vec{AB} = AB^2 \Leftrightarrow (\vec{AB}+\vec{BM})\cdot\vec{AB} = AB^2 \Leftrightarrow AB^2 + \vec{BM}\cdot\vec{AB} = AB^2 \Leftrightarrow \vec{BM}\cdot\vec{AB} = 0$

So $\{M \,/\, \vec{AM}\cdot\vec{AB} = AB^2\} = \{M \,/\, \vec{BM}$ perpendicular to $\vec{AB}\}$ = the line passing through B and perpendicular to AB.

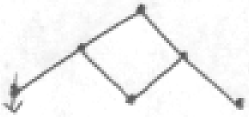
#4: $\vec{MA}\cdot\vec{MB} = 0 \Leftrightarrow (\vec{MI}+\vec{IA})\cdot(\vec{MI}+\vec{IB}) = 0$ (with I midpoint of AB)

$\Leftrightarrow MI^2 + \vec{MI}\cdot\underbrace{(\vec{IA}+\vec{IB})}_{=\vec{0}} - IA^2 = 0$ (because $\vec{IA} = -\vec{IB}$)

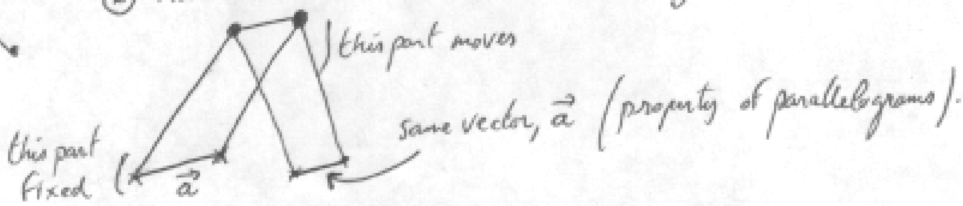
$\Leftrightarrow MI = IA$

So the set τ is the circle with diameter AB.

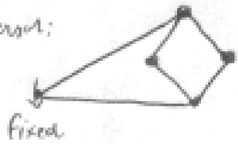
#5: (a) "pantograph"!



(b) How to add a fixed vector $\vec{a}$ to any vector $\vec{OM}$:



(c) inversor:



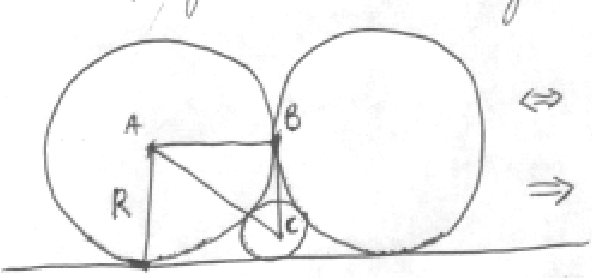
(d) We know how to do $x \mapsto x \pm 1$ and $x \mapsto \frac{1}{x}$,

so compose them: $x \mapsto x-1 \mapsto \frac{1}{x-1}$

Now $\frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$ so use (a) to get $\frac{1}{x^2-1}$, apply (c) to get x^2-1

apply (b) to get x^2.

#6: Pythagorean thm in ABC yields:



$(R+r)^2 = R^2 + (R-r)^2$

$\Leftrightarrow \cancel{R^2} + 2rR + \cancel{r^2} = \cancel{R^2} + R^2 - 2rR + \cancel{r^2}$

$\Rightarrow 4rR = R^2$

$\Rightarrow \boxed{r = \frac{R}{4}}$

MAT 360

HW 2

Exercise 1. Are the two following propositions equivalent?

1. $(A \Rightarrow B) \Rightarrow (B \text{ or } C)$
2. $(A \Rightarrow (B \text{ or } C)) \Rightarrow C$

Exercise 2. The following is taken from LSAT and GRE tests:

People should be held accountable for their own behavior, and if holding people accountable for their own behavior entails capital punishment, then so be it. However, no person should be held accountable for behavior over which he or she had no control.

Which of the following is the most logical conclusion of the argument above?

1. People should not be held accountable for the behavior of other people.
2. People have control over their own behavior.
3. People cannot control the behavior of other people.
4. Behavior that cannot be controlled should not be punished.
5. People have control over behavior that is subject to capital punishment.

Exercise 3. Suppose that in a given model for incidence geometry, every line has at least three distinct points lying on it. What is the least number of points and the least number of line such a model can have? Suppose further that the model has the Euclidean parallel property, i.e is an affine plane. Show then that 9 is now the least number of points and 12 the least number of lines such a model can have.

Exercise 4. Fix a circle in the Euclidean plane. Interpret "point" to mean a Euclidean point inside the circle, interpret "line" to mean a chord of the circle, and let "incidence" mean that the point lies on the chord. (A chord of a circle is a segment whose endpoints lie on the circle).

For this model, are the Incidence axioms satisfied? What about the 5th postulate of Euclid for this model?

Exercise 5. Let S be the following statement in the language of incidence geometry: if l and m are any two distinct lines, then there exists a point P that does not lie on either l or m. Show that S cannot be proved from the axioms of incidence geometry.

Exercise 6. Four distinct points, no three of which are collinear, are said to form a *quadrangle*. Let $\mathcal{P}$ be a model of incidence geometry for which every line has at least three distinct points lying on it. Show that a *quadrangle* exists in $\mathcal{P}$.

Exercise 7. Let $\mathcal{A}$ be a finite affine plane so that all lines in $\mathcal{A}$ have the same number of points lying on them; let n be this number, with $n \geqslant 2$. Show the following:

a) Each point in $\mathcal{A}$ has $n+1$ lines passing through it.

b) The total number of points in $\mathcal{A}$ is n^2.

c) The total number of lines in $\mathcal{A}$ is $n(n+1)$.

HW #2 Solutions

#1: they are not equivalent because if (A = True), (B = False), (C = True) then the first one is true, the other is false.

#2: Ans. is 2. (because if you have no control on it, then you shouldn't be accountable for it).

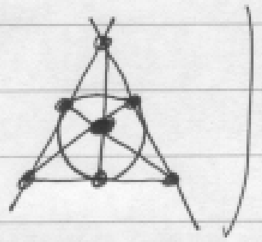
#3: Take a line ℓ and a pt P not on ℓ (this exists because of I 3).

Then ℓ has at least 3 distinct pts on it, A, B, C (by assumption).

Each line PA, PB, PC must have at least a third pt on it (by assumption) and each such pt can't be one of the pts P, A, B, C (otherwise I 1 would be contradicted). Moreover by I 1 those 3 new pts are distinct. So necessarily such a model must have at least 7 pts. But now we know the existence of the projective plane ~~ther~~ with 7 pts (which satisfies all the axioms). Thus 7 is the minimum nb of pts for such a model.

Now, using the same pts P, A, B, C we see that we already must have at least 4 lines ($(ABC), (PA), (PB), (PC)$). We saw that there must be 3 additional pts D, E, F leading to 3 more lines (necessarily distinct and distinct from the previous ones): $(DE), (DA), (EC)$. So we must have at least 7 lines.

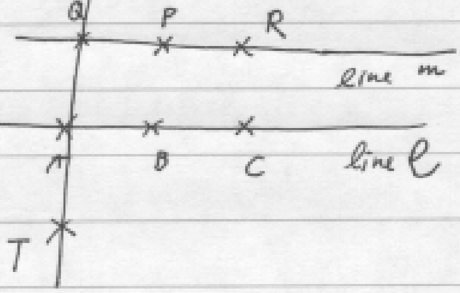
Thus 7 is the minimal nb (because it's the nb. of lines in that model:



).

Affine plane case:

Take a line ℓ (containing A, B, C) and a pt P not on ℓ. By the Euclidean parallel property, there is a line m (unique) ~~PR~~ containing P and $/\!/$ to ℓ.



The line m must contain 2 more pts Q, R.

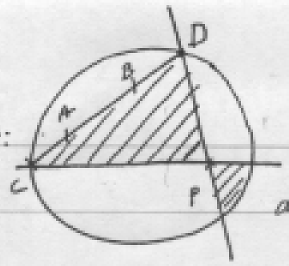
The line QA must have one more pt (at least) on it, say T.

T cannot be on ℓ nor m (that would contradict axiom I 1).

Now consider the 3 lines $(RA), (RB), (RC)$: any pair of these lines cannot have a common pt M other than R (because then the line RM would be equal to the line ℓ).

Thus each of these lines must have a 3RD pt on it, not on ℓ, not on m, and these 3 pts must be distinct.

So the minimal nb <u>is</u> 9 pts, because we know the existence of a model with 9 pts (affine plane over $\mathbb{Z}/3\mathbb{Z}$).

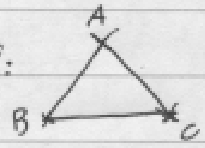
#4: 

I_1, I_2, I_3 are satisfied because they are satisfied on the euclidean plane, and the unit disk is a subset of the euclidean plane:

example of reasoning: given A, B in the disk, take the (euclidean line) joigning them and intersect with the disk to get a chord.

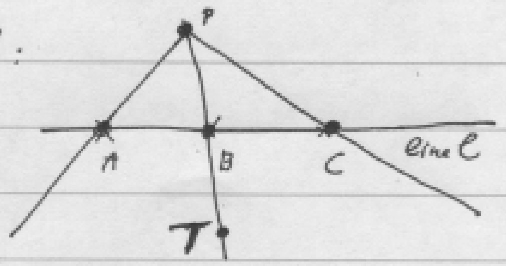
5th postulate: take P, not on the chord (AB). The chord (AB) intersects the unit circle in C, D.

Now any line through P that is outside of the sector $(\vec{PC}, \vec{PD})$ will lead to a parallel to CD: there are infinitely many such lines.

#5: The "triangle" model: 

satisfies I_1, I_2, I_3 but not S, so S is not a consequence of I_1, I_2, I_3.

#6: start with a line ℓ that has A, B, C on it, and take P not on ℓ:



The line (PB) must have a 3RD pt, T on it, and T can't be on ℓ (otherwise $I\,1$ would be contradicted).

I claim that (P, A, T, C) is a quadrangle:

indeed P, A, T are not on the same line (otherwise (line PT) would be equal to (line AB)).

similarly P, C, T ———————. And A, C, T are not on the same line (because then (line PT) would be equal to line (AB) again).

#7: (a) Take a line ℓ not going through the pt P (it exists because of $I\,3$).

Now a line going through P is either // to ℓ (but then it is unique, because we are in an affine plane), or it intersects ℓ: but ℓ has only n points on it, so there are only n lines through P that intersect ℓ (and any 2 of them are $\neq$ because otherwise $I\,1$ would be contradicted). So there are $n+1$ lines.

(b) All the pts in A can be regrouped into lines going through P (each one having $n-1$ pts $\neq$ **P** on it): so $\underbrace{(n+1)}_{\text{lines} \ni P} . \underbrace{(n-1)}_{\text{pts on each line} \ni P.} + \underbrace{1}_{\text{pt } P}$

(c) We have n^2 pts so there are $\frac{n^2(n^2-1)}{2}$ pairs of distinct pts.

Each such pair creates one line, but many pairs lead to the same line.

Actually each line contains n pts, therefore it contains $\frac{n(n-1)}{2}$ pairs of pts determining the same line.

Thus the total nb of lines is

$$\frac{\frac{n^2(n^2-1)}{2}}{\frac{n(n-1)}{2}} = n.(n+1).$$

HW 3

Exercise 1. I recall that a projective plane is a model of incidence geometry (meaning that it satisfies the incidence axioms I1, I2, I3) such that any two lines meet and that every line has at least three distinct points on it.

1. Let S be the following statement: "If l and m are any two distinct lines, then there exists a point P that does not lie on either l or m". Show that the statement S holds in any projective plane.
2. Show that any finite projective plane (meaning a projective plane as above, but with a finite number of points) has the property that all the lines have the same number of points.
   (Hint: use 1) and map the points of l onto m by projecting from the point P)

Exercise 2. Let F be the field with 2 elements $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. Let's consider the projective plane over F as follows:

Points are equivalence classes of triples $(x, y, z) \in F^3$ (with x,y,z not all equal to zero), where the equivalence is given by:

$(x, y, z) \sim (l, m, n)$ if and only if $(x, y, z) = t.(l, m, n) = (t.l, t.m, t.n)$ for some nonzero $t \in F$.

Equivalently, points can be defined as lines in F^3 going through the origin $(0, 0, 0)$.

Lines are defined as planes going through the origin (where a plane is made of all linear combinations of the form $t\vec{u} + s\vec{v}$, where t, s are in F and $\vec{u}, \vec{v}$ are linearly independent).

1. Give a list of all the points of this projective plane
2. Give a list of all the lines
3. Draw a picture of this projective plane.

Exercise 3. Some authors characterize projective planes by three axioms: Axiom I1, the elliptic parallel property (any two lines intersect), and the existence of a *quadrangle* (see previous HW).

Show that a model of those axioms is a projective plane under our definition (which is recalled in exercise 1), and conversely.

Exercise 4. In the affine plane $\mathbb{R}^2$ (the usual one you like, with vectors, etc...), let l and m be two distinct lines, x,y,z are three distinct points on l and x',y',z' are three distinct points on m, and these six points are also distinct from the intersection of l and m.

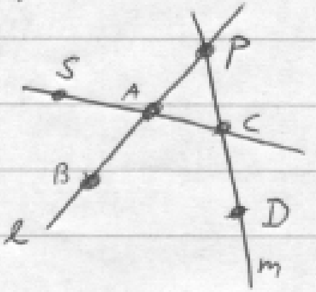
Show that the following property holds:

if the line(x,y') is parallel to the line (x',y) and the line (y,z') is parallel to the line (y',z) then the line (x,z') must be parallel to the line (x',z).

HW3 ~ Solutions

#1: ① Pick any 2 lines ℓ, m : they intersect (because we are in a projective plane) at a point P.

Now every line in a projective plane has at least 3 pts, so there exist A, B $(\neq P)$ on ℓ and C, D $(\neq P)$ on m.



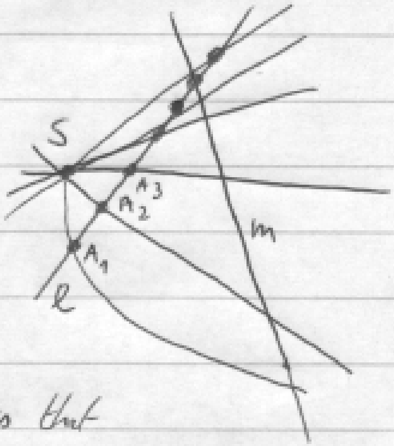
Take the line AC: it must have at least a third pt S on it. Now $(S \in \ell)$ is impossible (because I1 would imply that (line SA) = line ℓ which is absurd because $C \notin \ell$).

Similarly $(S \in m)$ is impossible.

② Pick any 2 lines ℓ, m. By ① there exists $S \notin (\ell \cup m)$

For any pt $A_1, \dots, A_n$ on ℓ, consider the line SA_i.

~~By a~~ Since any 2 lines meet in a proj. plane, SA_i meets m in ~~a pt B_i~~ a point B_i. Now any 2 points B_i, B_j must be distinct (otherwise axiom I1 is contradicted), and thus line m has at least n pts. Doing the same construction with m shows that (nb of pts on m) $\leqslant$ (nb. of pts on ℓ) and thus the 2 nb are equal.



#2: ① Pts are triples (x,y,z) where not all components are zero, so we get 7 pts:

$(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$.

② Given any two pts (a,b,c), (x,y,z), the plane containing these and the origin is given by the following linear combinations $0.(a,b,c) + 1.(x,y,z) = (x,y,z)$. So each plane has 3 pts on it.

$$1.(a,b,c) + 0(x,y,z) = (a,b,c)$$

$$1.(a,b,c) + 1(x,y,z) = 3\text{rd pt}$$

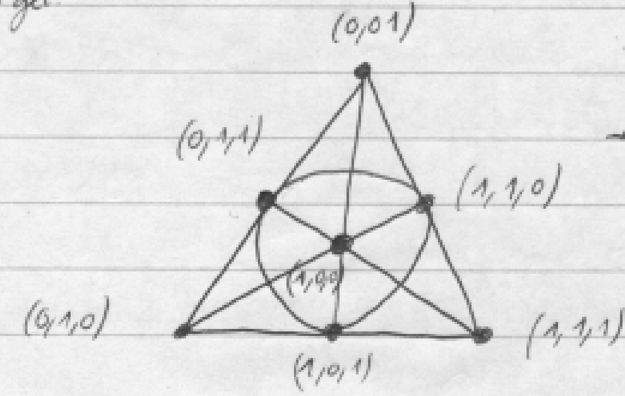
Example: plane generated by $(0,0,1)$ and $(0,1,1)$ is $\{(0,0,1), (0,1,1), (0,1,0)\}$ because $(0,0,1)+(0,1,1) = (0,1,0)$ in F_2.

So we get 7 such "lines": $\{(0,0,1),(0,1,0),(0,1,1)\}$; $\{(0,0,1),(1,0,0),(1,0,1)\}$; $\{(0,0,1),(1,1,0),(1,1,1)\}$

$\{(0,1,0),(1,0,0),(1,1,0)\}$; $\{(0,1,0),(1,0,1),(1,1,1)\}$;

$\{(0,1,1),(1,0,0),(1,1,1)\}$; $\{(0,1,1),(1,0,1),(1,1,0)\}$.

③ Picture: draw dots for pts and "connect the dots" (if they are on the same line).
You will get:



It looks familiar, no?

#3: We need to prove that "the 3 Axioms" ⟹ "the axioms for projective plane" and vice versa.

so: ⓐ "3 Axioms" ⟹ "Proj. plane":

indeed I 1 is already satisfied, as well as the property that "any 2 lines intersect".

Now the existence of a quadrangle ⟹ there exists 3 pts not collinear (this is I 3).

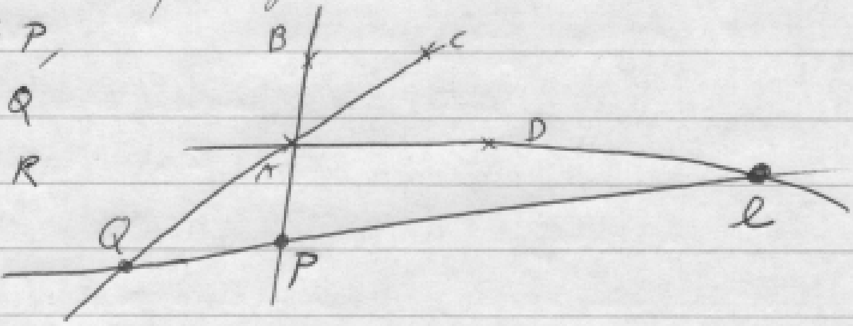
So it remains to show that every line has at least 3 pts on it:

pick any line l: ~~...~~ we know that there exists a quadrangle somewhere ABCD.

The line AB intersects l ("any 2 lines meet") at P,

—— AC ———————— Q

—— AD ———————— R

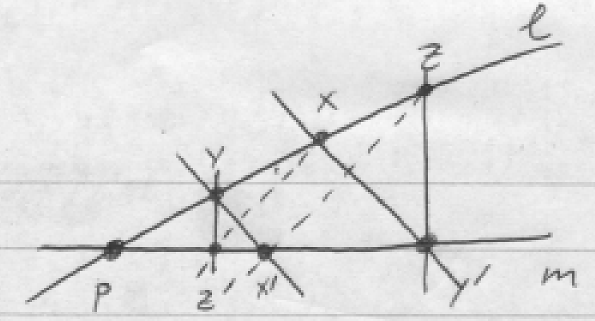


Now these 3 pts must be distinct (otherwise axiom I 1 would be contradicted). So we are done.

ⓑ "Proj. plane" ⟹ "The 3 axioms":

We already proved in last HW that a proj. plane must have a quadrangle (because every line has at least 3 pts on it).

#4.



First case: ℓ and m intersect (at P):

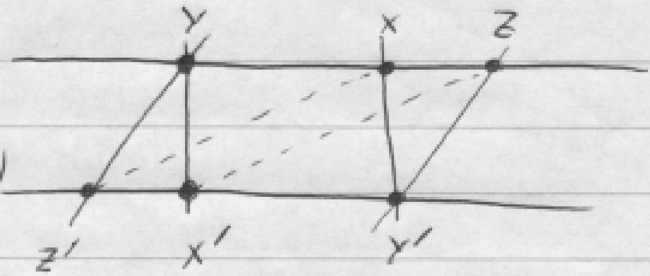
Goal: prove that $\begin{cases}\overrightarrow{PZ} = k\,\overrightarrow{PX} \\ \overrightarrow{PX'} = k\,\overrightarrow{PZ'}\end{cases}$ (same k)

But we know two things: $\begin{cases}\overrightarrow{PZ} = \ell\,\overrightarrow{PY} \\ \overrightarrow{PY'} = \ell\,\overrightarrow{PZ'}\end{cases}$ and $\begin{cases}\overrightarrow{PX} = m\,\overrightarrow{PY} \\ \overrightarrow{PY'} = m\,\overrightarrow{PX'}\end{cases}$ (similar triangles).

so we deduce $\overrightarrow{PZ} = \ell\overrightarrow{PY} = \frac{\ell}{m}\overrightarrow{PX}$

and $\overrightarrow{PX'} = \frac{1}{m}\overrightarrow{PY'} = \frac{\ell}{m}\overrightarrow{PZ'}$ and we are done.

Second case: $\ell \parallel m$



just notice that: $\overrightarrow{XZ} = \overrightarrow{XY'} + \overrightarrow{Y'Z}$

$= \overrightarrow{YX'} + \overrightarrow{Z'Y}$ (parallelograms)

$= \overrightarrow{Z'X'}$

and we are done.

HW 4

Exercise 1. Assume that all the incidence axioms and the betweenness axioms are satisfied.

Prove the following proposition:

Given $A * B * C$, then the segment AC is the union of the segments AB and BC, and B is the only point common to segments AB and BC.

Exercise 2. Assume that all the incidence axioms and the betweenness axioms are satisfied.

Prove the following proposition:

If D is in the interior of the angle $\lhd$CAB, then:

a) so is every other point on ray $\overrightarrow{\mathrm{AD}}$ except A;

b) no point on the opposite ray to $\overrightarrow{\mathrm{AD}}$ is in the interior of angle $\lhd$CAB;

c) if $C * A * E$, then B is in the interior of angle $\lhd$DAE.

Exercise 3. Assume that all the incidence axioms and the betweenness axioms are satisfied.

Prove the following propositions:

a) If a ray r emanating from an exterior point of triangle $\triangle$ABC intersects side AB in a point between A and B, then r also intersects side AC or side BC.

b) If a ray emanates from an interior point of $\triangle$ABC, then it intersects one of the sides, and if it does not pass through a vertex, it intersects only one side.

Exercise 4. Assume that all the incidence axioms and the betweenness axioms are satisfied.

Prove the following proposition:

A line cannot be contained in the interior of a triangle.

Exercise 5. A set of points S is called convex if whenever two points A and B are in S, then the entire segment AB is contained in S. Prove that a half-plane, the interior of an angle and the interior of a triangle are all convex sets, whereas the exterior of a triangle is not convex. Is a triangle a convex set?

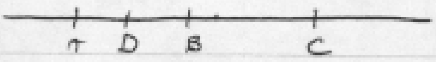
HW4 Solutions 20 pts each

total = 100 pts.

1

(a) $(AB \cup BC) \subset AC$: First let's prove $AB \subset AC$

So we know that $A * D * B$.



Assume that $D \notin AC$: then one would have $D * A * C$ or $A * C * D$, but in both cases a line ℓ_D intersecting $\overleftrightarrow{AB}$ at D would be such that A and C are on the same side of ℓ_D. We know that $A * D * B$, thus A and B are on opposite sides of $\ell_D \Rightarrow B$ and C are on opposite sides of ℓ_D, so we have $B * D * C$. But this means that C and D are on same side of a line ℓ_B crossing $\overleftrightarrow{AB}$ at B. From $A * B * C$ we know that A and C are on opposite sides of ℓ_B. Thus A and D should be on opposite sides of ℓ_B (Absurd because one has $A * D * B$). Thus D must be in AC.

Now let's prove $BC \subset AC$ = just permute the letters A and C and the above proof holds

(b) $AC \subset AB \cup BC$: pick any pt $D \in AC$, so we have $A * D * C$

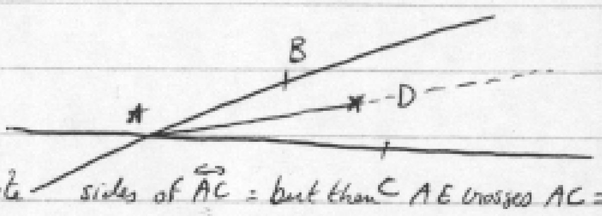
$\Rightarrow A$ and C are on opposite sides of a transversal line ℓ_D. Now either $B = D$ (and we are done) or the point B is on one side of ℓ_D: $\begin{cases} \text{if } B \text{ is with } A \text{ then one has } B * D * C \text{ (and } D \in BC). \\ \text{if } B \text{ — } C \text{ ——— } A * D * B \text{ (and then } D \in AB) \end{cases}$

(c) if a pt D is $\neq B$ and $D \in AB \cap BC$, then D is on one side of ℓ_B: $\begin{cases} \text{if it's with } A \text{ then } D * B * C \text{ true (ruling out } D \in BC) \\ \text{if it's with } C \text{, then } A * B * D \text{ true (contradicting } D \in AB). \end{cases}$

2:

(a) Pick any $E \in \overrightarrow{AD}$:

if $E \notin$ interior of $\measuredangle BAC$ then:



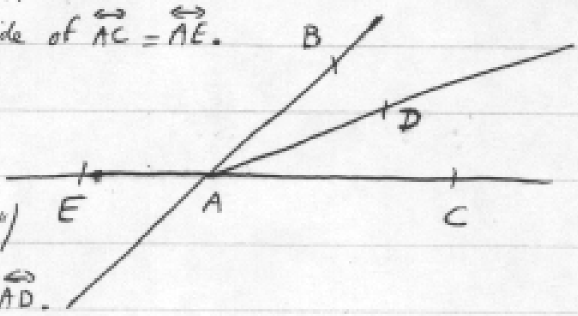
(i) either D and E are on opposite sides of $\overleftrightarrow{AC}$: but then AE crosses $AC \Rightarrow \overleftrightarrow{AE} = \overleftrightarrow{AC}$ (absurd)

(ii) or ——— $\overleftrightarrow{AB}$ ——— AE — $AB \Rightarrow \overleftrightarrow{AE} = \overleftrightarrow{AB}$ (absurd).

(b) If $F \in$ opposite ray to $\overrightarrow{AD} \Rightarrow F$ and D are on opposite sides of $\overleftrightarrow{AC} \Rightarrow F \notin$ interior of (BAC).

(c) (i) We already know that B and D are on same side of $\overleftrightarrow{AC} = \overleftrightarrow{AE}$.

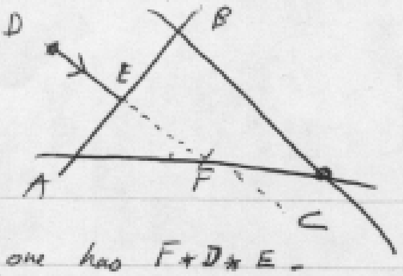
(ii) We know that $\overleftrightarrow{AD}$ separates E and C.



We need to prove that B and C are on opposite sides of $\overrightarrow{AD}$ = the proof of that is in the book ("crossbar thm")

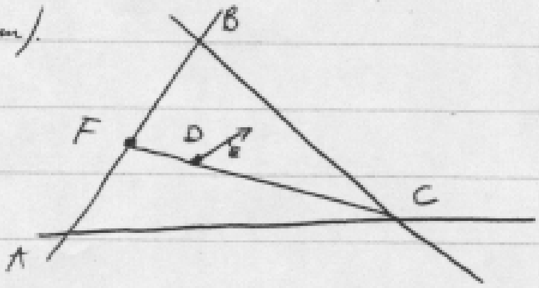
From this we deduce that E and B are on same side of $\overleftrightarrow{AD}$.

#3 (a) The line $\overleftrightarrow{DE}$ meets AB at E (between A and B), so A and B are on opposite sides of $\overleftrightarrow{DE}$. Thus C must be on an opposite side of one of these 2 pts, say A, so $\overleftrightarrow{DE}$ intersects AC at F. Now F must be on ray $\overrightarrow{DE}$: if not, then one has $F * D * E$.



Since D is outside the triangle it must be on an opposite side of one vertex to a side. Say D and C are on opposite sides of $\overleftrightarrow{AB}$: then F is on same side as C, thus D and F are opposite (this rules out $F * D * E$). The 2 other cases are similar.

(b) Consider $\overleftrightarrow{CD}$: it cuts AB at F ("crossbar" thm).



Now if ray $\overrightarrow{DE}$ doesn't intersect any side, that would mean that all pts A, B, C (and therefore F) are on the same side of $\overleftrightarrow{DE}$ (absurd: one has $F * D * C$).

#4 Take 2 pts on such a line: the ray formed must hit one side (exercise 3b). Thus the line can't be contained in the interior.

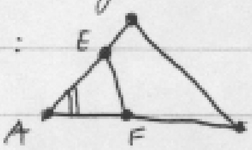
#5 Useful lemma: if S convex, T convex then $S \cap T$ is convex. (easy proof).

Now: convexity of a half-plane: if A, B are on same side of a line ℓ, assume there is a pt E between A and B that is on the other side: then AE must cross the line $\Rightarrow$ A and B are on opposite sides (absurd).

Now the interior of an angle is convex (because it's the intersection of 2 half-planes).

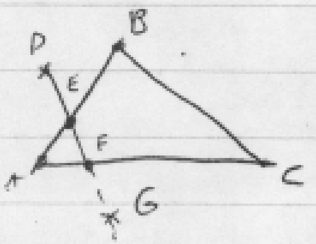
———— a triangle ——— (———— 3 interiors of angles).

. A triangle is not convex:



we proved in class that pts between E and F are in the interior of angle $\hat{A}$ (so they are not on triangle).

. Exterior is not convex:



from D, pick a ray hitting AB, it must exit somewhere (exercise 3) at F, then pick pt G so that ~~D*F*G~~ $D * F * G$.

HW 5

For all, these problems, we work in a Hilbert plane.

Exercise 1. (SAA congruence criterion). Given AC $\cong$ DF, angle $\lhd A \cong$ angle $\lhd D$ and angle $\lhd B \cong$ angle $\lhd E$. Then show that $\triangle$ABC $\cong$ $\triangle$DEF.

Exercise 2. (Hypotenuse-Leg Criterion). Two right triangles are congruent if the hypotenuse and a leg of one are congruent, respectively, to the hypotenuse and a leg of the other.

Exercise 3. Existence and unicity of the midpoint of a segment.

a) Given a segment AB, recall how to construct one midpoint of segment AB.

b) Prove that any segment has a **unique** midpoint.

Exercise 4. Prove that in any Hilbert plane, given a triangle $\triangle$ABC, AB $>$ BC if and only if angle $\lhd C >$ angle $\lhd A$ (you cannot use measures of angles, nor lengths of segments...)

Exercise 5. Prove that Hilbert's Euclidean parallel postulate is equivalent to the converse to the alternate interior angle theorem.

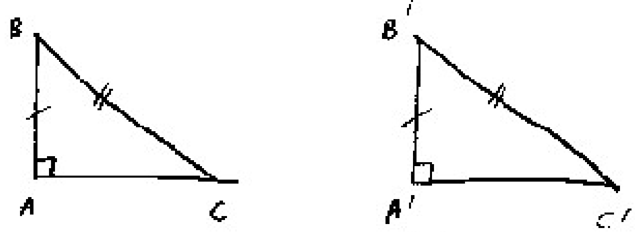
Solutions for HW5

#1 Since $AC \cong DF$, we can transport DF to AC. Equivalently we can assume $\begin{cases} D = A \\ F = C \end{cases}$

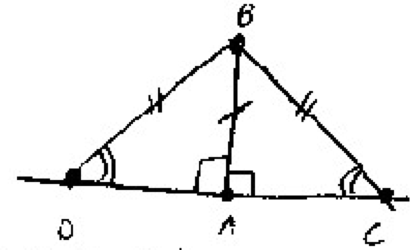
Since $\measuredangle A \cong \measuredangle D$ we can also assume that E is on ray $\overrightarrow{AB}$.

If $E \neq B$ then the angle $\measuredangle AEF$ is congruent to the angle opposite to $\measuredangle ABC$, but then the alternate interior angle then applies to the 2 lines $\overleftrightarrow{EC}$, $\overleftrightarrow{BC}$ cut by the transversal $\overleftrightarrow{AB}$ and thus $\overleftrightarrow{BC}$ and $\overleftrightarrow{EC}$ should be // which is absurd. Therefore $E = B$ and we are done.

#2 Given



Create a pt D on the ray opposite to $\overrightarrow{AC}$ so that $AD \cong A'C'$:



by S-A-S we have that $\measuredangle D \cong \measuredangle C$ (DBC is isosceles).

But then you can apply SAA (exercise #1) to get that $\Delta DAB \cong \Delta CAB$.

Since S-A-S implies that $\Delta DAB \cong \Delta C'A'B'$ we are done.

#3 (a) done in class.

(b) Unicity: assume there are 2 midpoints C_1, C_2, and assume we have $A * C_1 * C_2$.

Then we also have $A * C_1 * B$ (see midterm)

Now $A * C_1 * C_2 \Rightarrow AC_1 < AC_2$ (def. of inequality of segments), thus $AC_1 < C_2B$

Again (using midterm) we have also $C_1 * C_2 * B$ which implies $C_1B > C_2B$.

It remains to show the transitivity of the order $<$.

Namely: show that $\left.\begin{matrix} AB < CD \\ CD < EF \end{matrix}\right\} \Rightarrow AB < EF$.

Proof: "transport CD at A" to get a pt D' s.t. $AD' \cong CD$ and $A * B * D'$.

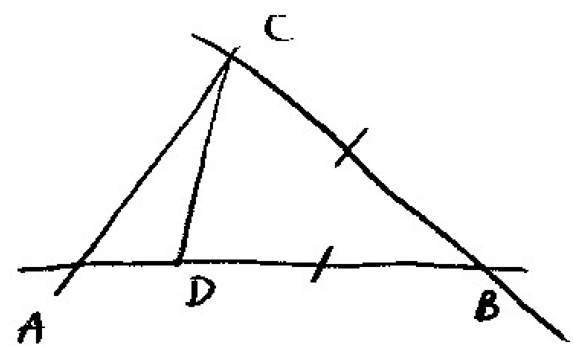
"transport EF at A" ———— F' s.t. $AF' \cong EF$ and $A * D' * F'$.

But $\left.\begin{matrix} A * B * D' \\ A * D' * F' \end{matrix}\right\} \Rightarrow A * B * F' \Rightarrow AB < AF' \cong EF$ done.

proved in midterm

#4

(a) $\boxed{\measuredangle C > \measuredangle A \Leftarrow AB > BC}$:



construct D between A and B s.t $BD \cong BC$.

We proved in class the "Exterior angle theorem"

saying that $\measuredangle BDC > \measuredangle DAC$

But ΔBDC is isosceles $\Rightarrow \measuredangle BDC \cong \measuredangle BCD$, so $\measuredangle BCD > \measuredangle DAC \cong \measuredangle B$.

Since D is inside the angles $\measuredangle ACB$ we also have $\measuredangle BCD < \measuredangle ACB$ and we are done.

(b) $\boxed{\measuredangle C > \measuredangle A \Rightarrow AB > BC}$

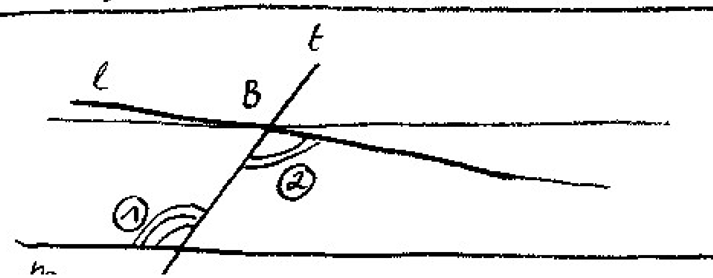
- $AB \simeq BC$ is impossible (because ΔABC would be isosceles contradicting $\measuredangle A < \measuredangle C$).
- $AB < BC$ is impossible: because part (a) would imply $\measuredangle A > \measuredangle C$.

Thus we must have $AB > BC$.

Rk: the fact that we can only have one of these cases comes from axiom B3.

#5

$\boxed{\text{Hilbert} \Rightarrow \text{converse to AIA}}$:



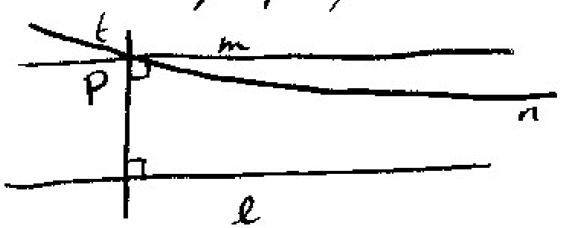
Assume $l \parallel m$ and $\measuredangle ① > \measuredangle ②$.

Then "transport the angle ① at B".

we get a line different from ℓ (because forming a different angle with line t) and $\parallel$ to m (by the A.I.A thm) which is absurd.

$\boxed{\text{converse to AIA} \Rightarrow \text{Hilbert}}$

Given ℓ, $P \notin \ell$, construct t the perpendicular to ℓ through P and m the perpendicular to t through P:



By AIA (true in any Hilbert plane) we know that $m \parallel \ell$.

Pick any ~~t~~ line $n \parallel \ell$: by the converse to AIA (which we assumed to be true) we have have n must be $\perp$ to t.

But we know ~~that~~ the unicity of the perpendicular to t through P.

So $m = n$, and we are done.

HW 6

Exercise 1. A quadrilateral ABCD is called q-convex if it has a pair of opposite sides, e.g. AB and CD such that CD is contained in one of the half-planes bounded by line(AB) and AB is contained in one of the half-planes bounded by line(CD). Define the interior of a q-convex quadrilateral to be the intersection of the interiors of its four angles. Prove that the interior of a q-convex quadrilateral is a convex set (see previous HW for the definition of a convex set) and that the point of intersection of the diagonals lies in the interior.

Exercise 2. Assume you are in a Hilbert plane and that Archimedes axiom, Dedekind axiom are both satisfied (so that you can measure segments and angles).

Given any triangle ABC, let D be the midpoint of BC, let E be the unique point on the ray opposite to $\overrightarrow{\mathrm{DA}}$ such that DE $\cong$ DA. Prove that the triangles AEC and ABC have the same angle sums, and prove that either $(\vartriangleleft \mathrm{AEC})^\circ$ or $(\vartriangleleft \mathrm{EAC})^\circ$ is $\leqslant \frac{1}{2}(\vartriangleleft \mathrm{BAC})^\circ$.

Exercise 3.
a) Prove that Hilbert's Euclidean parallel postulate is equivalent to the following statement: if a line intersects one of two parallel lines then it intersects also the other one.
b) Deduce that transitivity of parallelism is equivalent to Hilbert's Eucliden parallel postulate.

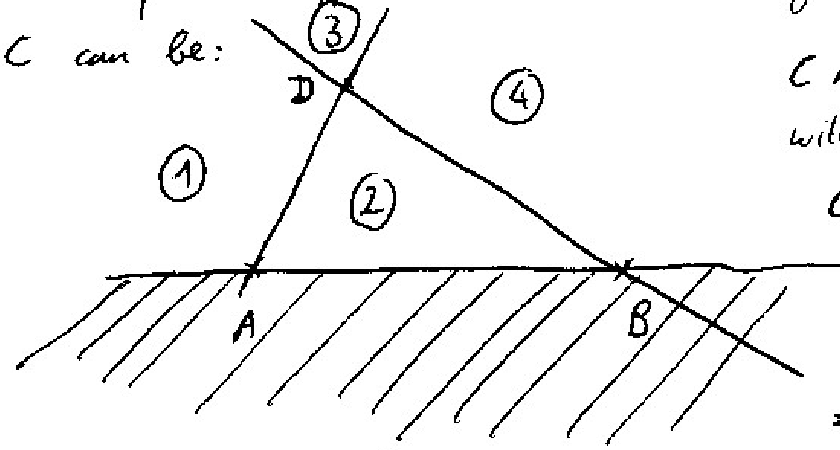
Exercise 4. Given that $A * B * C$ and that line(DC) is perpendicular to line(AC). Prove that AD>BD>CD.

Solutions of HW6

#1: The q-convexity is here just to make sure that the interior of the quadrilateral is not empty.

As in the old HW, an intersection of convex sets is convex, therefore the interior of the quadrilateral is convex.

Consider a q-convex quadrilateral ABCD: imagine that A, B, D are fixed and let's see where C can be:



C must be in the same half-plane as D with respect to $\overleftrightarrow{AB}$.

C can't be in the angles 2 nor 3 (otherwise the crossbar thm would imply that $\overleftrightarrow{DC}$ separates A and B.

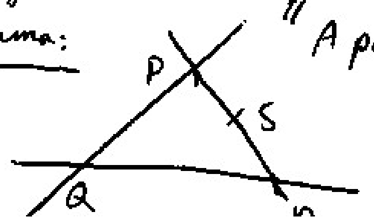
Now if $C \in$ 4, then the crossbar thm says that $\overrightarrow{AC}$ cuts DB at a point E and E is on both segments DB and AC.

And if $C \in$ 1, the crossbar thm says that the diagonal $\overrightarrow{BC}$ cuts AD.

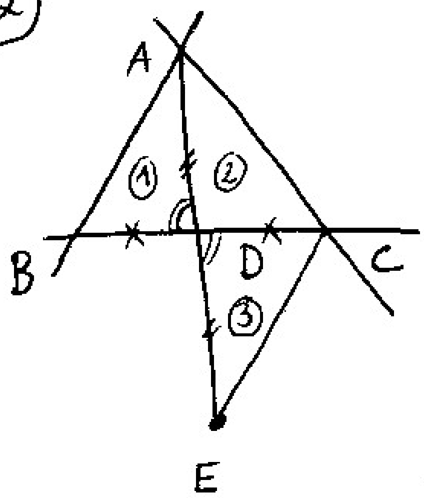
In the 2 cases, the two segments forming the diagonals intersect at a point E.

Now just use this lemma to conclude that E is in the interior:

<u>lemma:</u> "A point S on line $\overleftrightarrow{PR}$ is inside the angle $\measuredangle PQR$ if and only if S is on the segment PR."



#2



By S-A-S we have that $\Delta ABD \cong CDE$,

so $\underbrace{\text{angle sum}(1) + \text{angle sum}(2)}_{\text{angle sum}(\Delta ABC) + 180^\circ} = \underbrace{\text{angle sum}(3) + \text{angle sum}(2)}_{\text{angle sum}(\Delta AEC) + 180^\circ}$.

so this proves the first part.

Now $(\measuredangle BAC)^\circ = (\measuredangle BAD)^\circ + (\measuredangle DAC)^\circ$
$= (\measuredangle AEC)^\circ + (\measuredangle EAC)^\circ$

so $\begin{cases} \measuredangle AEC > \frac{1}{2}(\measuredangle A)^\circ \\ \text{and} \\ \measuredangle EAC > \frac{1}{2}(\measuredangle A)^\circ \end{cases}$ is impossible (since their sum is $= (\measuredangle A)^\circ$), so we are done.

#3 Let's call P the other statement, and H the Hilbert's // postulate:

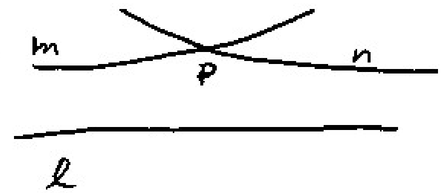
(a) $\boxed{H \Rightarrow P}$: by contradiction:

assume H true, not P. then there would exist $\ell // m$ and n intersecting ℓ but not m

$\Longrightarrow$ m would have 2 parallels going through an intersection pt (contradicts H).

$\boxed{P \Rightarrow H}$: by contradiction:

assume H not true: then we have



but then $m // \ell$ and n intersects m and not ℓ (contradicts P).

(b) $\boxed{\text{Transitivity} \Rightarrow P}$. by contrad.:

assume $\exists\ \ell // m$ and n intersecting ℓ but not m

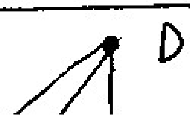
$$\Longrightarrow \begin{cases} \ell // m \\ n // m \end{cases}$$

$\Longrightarrow \ell // n$ by transitivity, but we assumed n intersects ℓ.

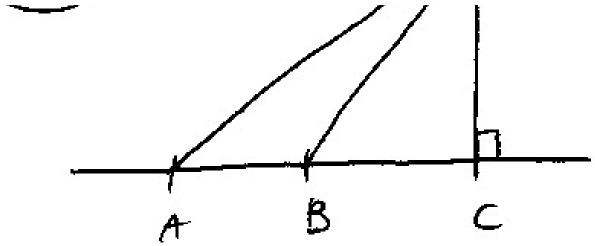
$\boxed{P \Rightarrow \text{transitivity}}$:

same method: assume $\ell // m$, $m // n$ but ℓ intersects n. This directly contradicts P (P says: if $\ell // m$, if n intersects ℓ then it must intersect m ...).

#4



We proved that in any triangle the sum of 2 angles must be



less than $180°$.

Therefore in both right triangles ADC and BDC we must have that $\measuredangle DAC$ and $\measuredangle DBC$ are acute.

Now $\measuredangle DBC$ acute $\Rightarrow \measuredangle DBA$ obtuse

$\Rightarrow \measuredangle DBA > \measuredangle DAB$ (acute)

$\Rightarrow DA > DB$ ("the greater the angle, the greater the opposite side")

Also in ΔDBC one has $\underset{\text{(acute)}}{\measuredangle DBC} < \measuredangle DCB$ (right)

$\Rightarrow DB > DC$.

HW 7

For these problems, assume that you are in a Hilbert plane satisfying the following axiom:

There exists a line l and a point P not on l such that at least two distinct lines parallel to l pass through P. (that's the negation of Hilbert's Euclidean parallel postulate)

Exercise 1. Suppose that lines l and l' have a common perpendicular MM' . Let A and B be points on l such that M is not the midpoint of segment AB. Prove that A and B are not equidistant from l' (meaning that when you drop a perpendicular to l' through each point A, B you get two segments that are not congruent).

Exercise 2. Assume that the parallel lines l and l' have a common perpendicular segment MM'. Prove that MM' is the shortest segment between any point of l and any point of l'.

Exercise 3. In a Hilbert plane where rectangles do not exist, re-explain why it is true that for every line l and every point P not on l there are infinitely many parallels to l through P.

Solutions for HW 7

2 km

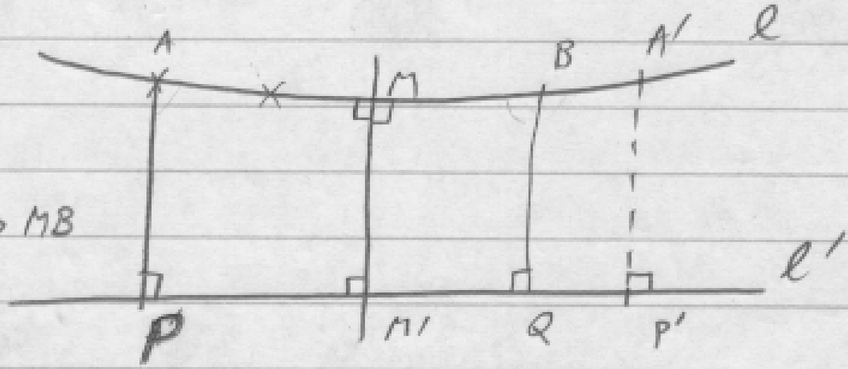
#1:

There are several cases:

① $A * M * B$:

M is not the midpoint of AB,

so assume for example that $AM > MB$



Consider the pt A' on ray $\overrightarrow{MB}$ such that $AM \simeq MA'$, and P' the orthogonal projection of A' onto line l'.

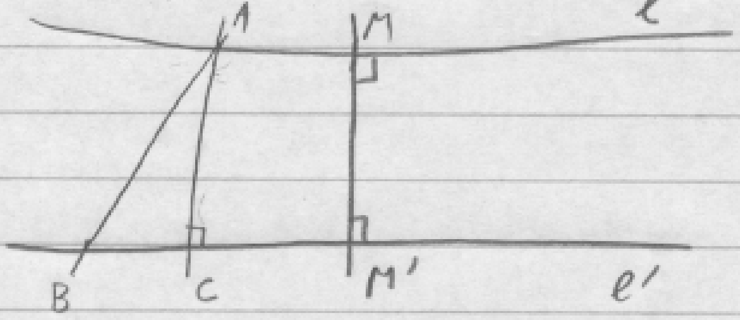
First one must have $AP \simeq A'P'$. (To see this, first show that $AM'A'$ is isosceles, deduce that the angles $\measuredangle AM'M$ and $\measuredangle A'M'M$ are congruent, and then that the angles $\measuredangle AM'P$ and $\measuredangle A'M'P'$ are congruent, and finally using AAS that $\Delta APM' \cong \Delta A'M'P'$).

Therefore we only need to treat the case

② $M * B * A'$

Since the angle $\measuredangle MBQ$ must be acute, the angle $\measuredangle A'BQ$ is obtuse, so is larger than angle $\measuredangle BA'P'$ (which is acute). Therefore, we can use the theorem saying that in any bi-right quadrilateral $BQP'A'$ if $\measuredangle B > \measuredangle A'$ then $A'P' > BQ$.

#2



Let C be the foot of the perpendicular through A to l'.

We will prove $MM' < AC < AB$.

First $AC < AB$ because $\measuredangle B$ is acute and $\measuredangle C$ is right. ("Greater angle correspond to greater opposite side").

Now angle at A must be acute so in the bi-right quadrilateral $AMM'C$ one has $\measuredangle A < \measuredangle M$ thus $MM' < AC$.

#3 See textbook (p251 for 5th edition, and p188 for 3rd edition).

HW 8

Exercise 1. Consider a sphere in $\mathbb{R}^3$ with radius 1, and centered at the point $(0,0,1)$. Find a formula for the projection from the North pole sending a point from that sphere to the horizontal plane $(z=0)$. Find a formula for the inverse also.

What is the image by this projection of a circle (on the sphere) going through the North pole?

Exercise 2. Two models of the hyperbolic plane.

In this problem we want to construct directly an isomorphism between Δ_K (the interior of the unit disk, where lines are chords: this is called Klein model) and Δ_P (the interior of the unit disk, where lines are circle arcs perpendicular to the boundary).

1. Show that the following map expressed in polar coordinates $f\colon (r,\theta) \mapsto (\frac{r}{1+\sqrt{1-r^2}}, \theta)$ is a bijection from Δ_K to Δ_P.
2. Find the expression of f in cartesian coordinates (x,y).
3. In Δ_K consider the two points P, Q with respective coordinates $(s, \sqrt{1-s^2})$, $(s, -\sqrt{1-s^2})$ where $0 < s < 1$. Find the center C and radius R of the circle γ_{PQ} going through P,Q and orthogonal to the unit circle.
4. Show that any point on the chord joigning P to Q will be mapped by f to a point on the circle with center C and radius R. (Hint: any such point has coordinates (s,y) where $-\sqrt{1-s^2} < y < \sqrt{1-s^2}$ and then use the expression of f in cartesian coordinates).
5. Conclusion: explain why chords in Δ_K are mapped to orthogonal circular arcs in Δ_P.

Solutions Hw8

#2:

① Let's find the inverse map:

we solve: $\dfrac{r}{1+\sqrt{1-r^2}} = R \iff r - R = R\cdot\sqrt{1-r^2}$

$\iff (r-R)^2 = R^2\left(1-r^2\right)$

$\iff r^2 - 2rR + \cancel{R^2} = \cancel{R^2} - r^2R^2$

$\iff r\left(r + rR^2 - 2R\right) = 0$

if $r \neq 0$ we get $r = \dfrac{2R}{1+R^2}$ so $(R,\theta) \mapsto \left(\dfrac{2R}{1+R^2}, \theta\right)$ is the inverse map.

② We use $r = \sqrt{x^2+y^2}$ and $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$.

Then $(x,y) \mapsto (r,\theta) \mapsto \underbrace{\left(\dfrac{r}{1+\sqrt{1-r^2}}, \theta\right)}_{= (R,\theta)} \mapsto (R\cos\theta, R\sin\theta) = \left(\dfrac{\cancel{\sqrt{x^2+y^2}}}{1+\sqrt{1-(x^2+y^2)}} \cdot \dfrac{x}{\cancel{\sqrt{x^2+y^2}}}, \dfrac{\cancel{\sqrt{x^2+y^2}}}{1+\sqrt{1-(x^2+y^2)}} \cdot \dfrac{y}{\cancel{\sqrt{x^2+y^2}}}\right)$

$= \left(\dfrac{x}{1+\sqrt{1-(x^2+y^2)}}, \dfrac{y}{1+\sqrt{1-(x^2+y^2)}}\right).$

③ In class we proved that the center C must be the image of the midpoint of PQ by the inversion with respect to the unit circle. Since the midpoint has coordinates $(s,0)$, its inverse C is $\left(\frac{1}{s}, 0\right)$.
And the radius is $\overline{CP} = \sqrt{\left(s-\frac{1}{s}\right)^2 + \left(\sqrt{1-s^2}\right)^2} = \sqrt{s^2 - 2 + \frac{1}{s^2} + 1 - s^2} = \sqrt{\frac{1}{s^2} - 1}$.

④ We know $(s,t) \mapsto \left(\dfrac{s}{1+\sqrt{1-(s^2+t^2)}}, \dfrac{t}{1+\sqrt{1-(s^2+t^2)}}\right) = M$. We just need to show $\overline{MC} = \overline{CP}$.

But $\overline{MC}^2 = \overline{CP}^2 \iff \left(\dfrac{s}{1+\sqrt{1-(s^2+t^2)}} - \dfrac{1}{s}\right)^2 + \dfrac{t^2}{\left(1+\sqrt{1-(s^2+t^2)}\right)^2} = \dfrac{1}{s^2} - 1$

$\iff \left(s^2 - \left(1+\sqrt{1-(s^2+t^2)}\right)\right)^2 + s^2t^2 = (1-s^2)\cdot\left(1+\sqrt{1-(s^2+t^2)}\right)^2$

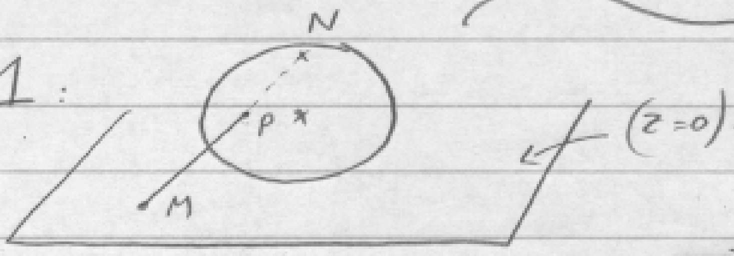
$\iff s^4 - 2s^2\left(1+\sqrt{''}\right) + \left(1+\sqrt{1-(s^2+t^2)}\right)^2 + s^2t^2 = (1-s^2)\cdot\left(1+\sqrt{1-(s^2+t^2)}\right)^2$

$\iff \left(1+\sqrt{1-(s^2+t^2)}\right)^2 - s^2\big(\underbrace{2 - t^2 - s^2 + 2\sqrt{1-(s^2+t^2)}}_{=\left(1+\sqrt{1-(s^2+t^2)}\right)^2}\big) = (1-s^2)\cdot\left(1+\sqrt{1-(s^2+t^2)}\right)^2$

Since the last equality is true, so is $\overline{MC}^2 = \overline{CP}^2$ and we are done.

(5) Any chord can be transformed by a rotation into a chord like PQ that will then be transformed into a circular arc joigning the same pts.

\# 1:



The line $\overleftrightarrow{NP}$ is the set of pts M such that $\overrightarrow{NM} = t.\overrightarrow{NP}$ for some $t \in \mathbb{R}$.

IF $P = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $M = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ one has $\begin{pmatrix} x \\ y \\ z-2 \end{pmatrix} = t.\begin{pmatrix} a \\ b \\ c-2 \end{pmatrix}$. We want $z=0$, therefore one must have $-2 = t.(c-2)$ and thus $t = \frac{2}{2-c}$. We deduce $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2}{2-c}.\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$.

Image of a circle $\ni N$:

such a circle is given by intersecting a plane through N with the sphere.
Thus the projection of such a circle
is equal to the line L (intersection of the plane through N
with the plane $(z=0)$).