Remus Radu

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MAT 341: Applied Real Analysis
Spring 2017
Course Information

Synopsis

This course is an introduction to Fourier series and to their use in solving partial differential equations (PDEs). We will discuss in detail the three fundamental types of PDEs: the heat equation, the wave equation and Laplace's equation. These equations are important in many applications from various fields (mathematics, physics, engineering, economics, etc.) and illustrate important properties of PDEs in general.

Click here to download a copy of the course syllabus. Please visit the **course website on Blackboard** to see your grades and the solutions to midterms & exams.

Lectures

Tuesdays & Thursdays 2:30-3:50pm in Melville Library E4315

Instructor

Remus Radu
Office hours: Tuesday 11:30am-1:30pm in Math Tower 4-103;
Thursday 1:00-2:00 in **MLC**, or by appointment

Teaching Assistant

Qianyu Chen
Office: Math Tower S-240A
Office hours: Tuesday 4:30-6:30pm in **MLC**;
Thursday 5:00-6:00pm in Math Tower S-240A

Textbook

David Powers, *Boundary Value Problems and Partial Differential Equations*, 6th ed., Elsevier (Academic Press), 2010.

Grading Policy

Grades will be computed using the following scheme:

- Homework – 20%
- Midterm 1 – 20%
- Midterm 2 – 20%
- Final – 40%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading.

Exams

There will be two midterms and a final exam, scheduled as follows:

- Midterm 1 – Tuesday, February 28, 2:30pm-3:50pm, in Library E4315.
- Midterm 2 – Tuesday, April 11, 2:30pm-3:50pm, in Library E4315.
- Final Exam – Monday, May 15, 11:15am-1:45am, TBA.

Remus Radu

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Home Research Teaching MAT 341 (Spring 2017)

About me

From 2013 to 2017 I was a Milnor Lecturer at the **Institute for Mathematical Sciences** at Stony Brook University. I got my Ph.D. in Mathematics from **Cornell University** in 2013, under the supervision of **John H. Hubbard**.

I started my undergraduate studies at the **University of Bucharest** and after one year I transfered to **Jacobs University Bremen**, where I earned my B.S. degree in Mathematics in 2007. I got a M.S. in Computer Science from Cornell University in 2012.

Research Interests

My interests are in the areas of Dynamical Systems (in one or several complex variables), Analysis, Topology and the interplay between these fields.

My research is focused on the study of complex Hénon maps, which are a special class of polynomial automorphisms of $\mathbb{C}^2$ with chaotic behavior. I am interested in understanding the global topology of the Julia sets J, J^- and J^+ of a complex Hénon map and the dynamics of maps with partially hyperbolic behavior such as holomorphic germs of diffeomorphisms of $(\mathbb{C}^n, 0)$ with semi-neutral fixed points. Some specific topics that I work on include: relative stability of semi-parabolic Hénon maps and connectivity of the Julia set J, regularity properties of the boundary of a Siegel disk of a semi-Siegel Hénon map, local structure of non-linearizable germs of diffeomorphisms of $(\mathbb{C}^n, 0)$.

Other activities

I was organizer for the **Dynamics Seminar** at Stony Brook University.
I have also developed projects for MEC (Math Explorer's Club): **Mathematics of Web Search** and **Billiards & Puzzles**.

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MAT 341: Applied Real Analysis
Spring 2017
Schedule & Homework

Schedule

The PDF version of the schedule is available for print **here**.

Date	Topic	Section	Assignments	Due date
Jan 24	An introduction to Fourier series	1.1	**1.1:** 1abc, 2ad, 4, 7b, 8	**HW1** Due Jan 31
Jan 26	Determining Fourier coefficients; Examples	1.2	**1.2:** 1, 7c	
Jan 31	Even & odd extensions Convergence of Fourier series	1.2, 1.3	**1.2:**10b, 11b	**HW2** Due Feb 7
Feb 2	Uniform convergence of Fourier series	1.3, 1.4	**1.3:**1abd, 2ad, 6	
Feb 7	Fourier sine & cosine series Basic operations on Fourier series	1.4, 1.5	**1.4:** 1ae, 2, 3ab, 5bc **page 120:** 19, 20 [use a=3]	**HW3** Due Feb 14
Feb 9	*no class (snow storm)*			
Feb 14	Differentiation of Fourier series The heat equation	1.5, 2.1	**1.5:** 2, 5, 9 **2.1:** 2, 9	**HW4** Due Feb 23
Feb 16	The heat equation Steady-state & transient solutions	2.1, 2.2	**2.2:** 2, 6	
Feb 21	Fixed-end temperatures	2.3	**2.3:** 8 [use a=pi]	
Feb 23	Insulated bar; Examples Review	2.4	**2.3:** 6 **2.4:** 4 [use a=pi], 5, 8	**HW5** Due Mar 9
Feb 28	**Midterm 1** (2:30-3:50pm) Covers 1.1-1.5, 2.1-2.3 -- **Solutions** Practice exams: **Fall 2015** (**Solutions**) and **Spring 2015** (**Solutions**)			
Mar 2	Different boundary conditions	2.5	**2.5:** 4, 5 [use a=pi], 6	
Mar 7	Eigenvalues and eigenfunctions Convection	2.6, 2.7 **Notes**	**2.6:** 7, 9, 10	**HW6** Due Mar 23 **Problem 3c**
Mar 9	Sturm-Liouville problems	2.7	**2.7:** 1, 3abc, 7	

Mar 14	*no class (Spring break)*			
Mar 16	*no class (Spring break)*			
Mar 21	Series of eigenfunctions & examples Fourier integral & applications to PDEs	2.8, 1.9	**2.8:** 1 [use b=2] **1.9:** 1ab, 3a	**HW7** Due Mar 30
Mar 23	Semi-infinite rod The wave equation	2.10, 3.1	**2.10:** 3, 4	
Mar 28	The wave equation	3.2	**3.2:** 3, 4, 5, 7	**HW8** Due Apr 6 **Comments**
Mar 30	D'Alembert's solution; Examples	3.3, 3.4	**3.3:** 1, 2, 5	
Apr 4	The wave equation: generalizations Laplace's equation	3.4, 4.1	**page 255:** 18 **page 257:** 31	**HW9** Due Apr 20 **Comments**
Apr 6	Dirichlet's problem in a rectangle Examples & Review	4.2, 4.3	**4.1:** 2 **4.2:** 5 [use a=1, f(x)=sin(3pix)] **4.2:** 6	
Apr 11	**Midterm 2** (2:30-3:50pm) Covers 2.4-2.8, 2.10, 1.9, 3.1-3.4 -- **Solutions** Practice exams: **Fall 2015** (**Solutions**) and **Spring 2015** (**Solutions**) **Extra practice problems**			
Apr 13	Potential in a rectangle; Examples Potential in unbounded regions	4.3, 4.4	**4.3:** 2b **4.4:** 4a, 5ab	**HW10** Due Apr 27
Apr 18	Polar coordinates Potential in a disk	4.1, 4.5 **Notes**	**4.1:** 6 **4.5:** 1	
Apr 20	Dirichlet problem in a disk; Examples	4.5	**4.5:** 4	
Apr 25	Two-dimensional heat equation	5.3, 5.4 **Notes**	**5.3:** 1, 7c [use a=b=pi]	**HW11** Due May 4
Apr 27	Problems in polar coordinates Bessel's equation	5.5, 5.6	**5.4:** 5	
May 2	Temperature in a cylinder Applications: symmetric vibrations	5.6, 5.7	**5.6:** 3 [use a=1] **page 371:** 1	
May 4	Examples & Review	5.7		
May 15	**Final Exam** (11:15am-1:45pm) -- in class, **Melville Library E4315** The final is cumulative and covers: 1.1-1.5, 1.9, 2.1-2.8, 2.10, 3.1-3.4, 4.1-4.5, 5.3-5.6 Practice exams: **Fall 2015** and **Spring 2015**.			

MAT 341: APPLIED REAL ANALYSIS – SPRING 2017
GENERAL INFORMATION

Instructor. Remus Radu
Email: remus.radu@stonybrook.edu
Office: Math Tower 4-103, Phone: (631) 63**2-8266**
Office Hours: Tu 11:30am-1:30pm in Math Tower 4-103,
Th 1:00-2:00 in MLC (Math Tower S-235), or by appointment

Teaching Assistant. Qianyu Chen
Email: qianyu.chen@stonybrook.edu
Office Hours: Tu 4:30-6:30pm in MLC (Math Tower S-235)
Th 5:00-6:00pm in Math Tower S-240A

Lectures. TuTh 2:30-3:50pm in Library E4315.

Course website & Bb. Grades and announcements will be posted on Blackboard. Please login using your NetID at http://blackboard.stonybrook.edu. A detailed weekly schedule of the lectures and homework assignments will be posted on the course website:
http://www.math.stonybrook.edu/~rradu/MAT341SP17

Course Description. This course is an introduction to Fourier series and to their use in solving partial differential equations (PDEs). We will discuss in detail the three fundamental types of PDEs: the heat equation, the wave equation and Laplace's equation. These equations are important in many applications from various fields (mathematics, physics, engineering, economics, etc.) and illustrate important properties of PDEs in general.

Prerequisites. C or higher in the following: MAT 203 or 205 or 307 or AMS 261; MAT 303 or 305 or AMS 361. Advisory Prerequisite: MAT 200. It is important to be familiar with the basic techniques in ordinary differential equations.

Textbook. The following textbook is required:

David Powers, *Boundary Value Problems and Partial Differential Equations*, 6th ed., Elsevier (Academic Press), 2010.

Exams. There will be two midterms and a final exam, scheduled as follows:

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There will be no make-up exams.

Grading policy. Grades will be computed using the following scheme:

Homework	20%
Midterm 1	20%
Midterm 2	20%
Final Exam	40%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading. It is generally useful to read the corresponding section in the book before the lecture. There will be weekly homework assignments; the lowest homework score will be dropped. You may work together on your problem sets, and you are encouraged to do so. However, all solutions must be written up independently.

Extra Help. You are welcome to attend the office hours and ask questions about the lectures and about the homework assignments. In addition, math tutors are available at the MLC: http://www.math.sunysb.edu/MLC.

Information for students with disabilities. If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631) 632-6748, or at the following website http://studentaffairs.stonybrook.edu/dss/index.shtml. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential.

Academic integrity. Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty is required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary.

Critical Incident Management. Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of University Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures. Further information about most academic matters can be found in the Undergraduate Bulletin, the Undergraduate Class Schedule, and the Faculty-Employee Handbook.

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MAT 341 – Applied Real Analysis
Spring 2017

Midterm 1 – February 28, 2017
Solutions

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
TOTAL	

Problem 1: (30 points) Consider the function $f(x) = 3 - x$, $0 < x < 3$.

a) Sketch both the even and odd periodic extensions of f on the interval $[-6, 6]$.

SOLUTION.

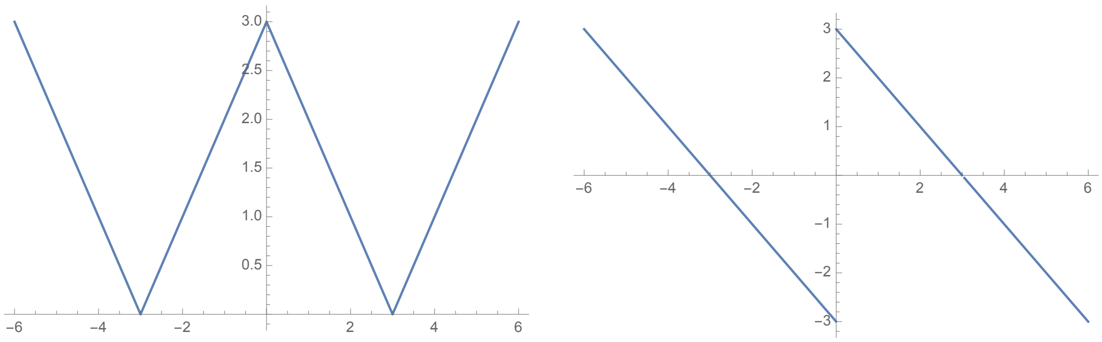

Figure 1: LEFT: even extension. RIGHT: odd extension.

□

b) Find the Fourier cosine series of f.

SOLUTION. The cosine series of f is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right),$$

where

$$a_0 = \frac{1}{3}\int_0^3 (3 - x)\,dx = \frac{1}{3}\left(3x - \frac{x^2}{2}\right)\Big|_0^3 = \frac{3}{2}$$

and

$$\begin{aligned} a_n &= \frac{2}{3}\int_0^3 (3-x)\cos\left(\frac{n\pi x}{3}\right)\,dx = 2\int_0^3 \cos\left(\frac{n\pi x}{3}\right)\,dx - \frac{2}{3}\int_0^3 x\cos\left(\frac{n\pi x}{3}\right)\,dx \\ &= -\frac{6}{(n\pi)^2}(\cos(n\pi) - 1) = \frac{6(1-(-1)^n)}{n^2\pi^2}. \end{aligned}$$

This gives

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6(1-(-1)^n)}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right).$$

□

(Problem 1 continued)

c) Find the Fourier sine series of f. The sine series of f is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right),$$

where

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 (3-x)\sin\left(\frac{n\pi x}{3}\right)\,dx = 2\int_0^3 \sin\left(\frac{n\pi x}{3}\right)\,dx - \frac{2}{3}\int_0^3 x\sin\left(\frac{n\pi x}{3}\right)\,dx \\ &= \frac{6}{n\pi}(1-\cos(n\pi)) + \frac{6}{n\pi}\cos(n\pi) = \frac{6}{n\pi}. \end{aligned}$$

This gives

$$f(x) \sim \sum_{n=1}^{\infty} \frac{6}{n\pi}\sin\left(\frac{n\pi x}{3}\right).$$

□

d) To what value does the Fourier cosine series converge at $x = 2$? At $x = 6$? To what value does the Fourier sine series converge at $x = 2$, $x = 6$? Do the series from parts b) and c) converge uniformly?

SOLUTION. At $x = 2$, the Fourier cosine series converges to $\frac{f(2-)+f(2+)}{2} = 1$. At $x = 6$ the cosine series converges to $\frac{f(0-)+f(0+)}{2} = 3$. Similarly, at $x = 2$, the Fourier sine series converges to 1. At $x = 6$, the sine series converges to 0.

The even periodic extension is continuous and piecewise smooth. Therefore the Fourier cosine series converges uniformly. The odd periodic extension is not continuous; it has a jump discontinuity at $x = 0$. Therefore the Fourier sine series does not converge uniformly (and the Gibbs phenomenon occurs). □

Problem 2: (25 points) The Fourier cosine series of $f(x) = x^2 - 2x$, $0 < x < 4$ is

$$x^2 - 2x = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n^2} \cos\left(\frac{n\pi x}{4}\right).$$

a) Does the Fourier cosine series of f converge uniformly in the interval $[0, 4]$? Explain.

SOLUTION. We notice that

$$\sum_{n=1}^{\infty} |a_n| + |b_n| \leq 4\frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

This implies that the Fourier cosine series converges uniformly. □

b) Compute the Fourier sine series of the derivative $f'(x)$ if it exists. Is the convergence uniform? If it doesn't exist, explain why it does not exist.

SOLUTION. The even periodic extension of f is continuous and piecewise differentiable. On a period interval, say $[-4, 4]$, f is not differentiable precisely when $x = -4, 0, 4$. Then the differentiated Fourier series of $f(x)$ converges to $f'(x)$ at each point where $f''(x)$ exists (see Theorem 6 from Section 1.5). Clearly, f is twice differentiable on $0 < x < 4$. It follows that the Fourier sine series of $f'(x)$ exists and

$$f'(x) = -\frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n^2} \cdot \frac{n\pi}{4} \sin\left(\frac{n\pi x}{4}\right) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n} \sin\left(\frac{n\pi x}{4}\right).$$

However, the convergence is not uniform. The even extension of f has the formula

$$f(x) = \begin{cases} x^2 + 2x & \text{if } \ -4 < x < 0, \\ x^2 - 2x & \text{if } \ \ 0 < x < 4 \end{cases}$$

Note that there is a jump discontinuity in the graph of $f'(x)$ at $x = 0$, so the Fourier sine series of f' does not converge uniformly. This could also be observed from the coefficients of the series, which grow only like $1/n$. □

(Problem 2 continued)

c) Determine the Fourier sine series of $g(x) = x^3 - 3x^2$, $0 < x < 4$.

SOLUTION. Note that $g(x) = 3\int_0^x f(t)\,dt$, so

$$\begin{aligned} g(x) &= 3\int_0^x \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1+3(-1)^n}{n^2}\cos\left(\frac{n\pi t}{4}\right)\,dt \\ &= 4x + \frac{3\cdot 16}{\pi^2}\sum_{n=1}^{\infty}\frac{1+3(-1)^n}{n^2}\frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right) \\ &= 4x + \frac{3\cdot 64}{\pi^3}\sum_{n=1}^{\infty}\frac{1+3(-1)^n}{n^3}\sin\left(\frac{n\pi x}{4}\right) \end{aligned}$$

We need to compute the Fourier sine series for x, $0 < x < 4$. Doing similar computations as in Problem 1c) we find

$$x = \sum_{n=1}^{\infty}\frac{8(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{4}\right).$$

The Fourier sine series of $g(x)$ is

$$\begin{aligned} g(x) &= 4\sum_{n=1}^{\infty}\frac{8(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{4}\right) + \frac{3\cdot 64}{\pi^3}\sum_{n=1}^{\infty}\frac{1+3(-1)^n}{n^3}\sin\left(\frac{n\pi x}{4}\right) \\ &= \sum_{n=1}^{\infty}\left(32\frac{(-1)^n}{n\pi} + 3\cdot 64\frac{1+3(-1)^n}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{4}\right). \end{aligned}$$

□

Problem 3: (20 points) Consider the partial differential equation

$$20\frac{\partial^2 u}{\partial x^2} - 10\frac{\partial u}{\partial t} + 17\,u = 0.$$

a) Let $u(x,t) = e^{\lambda t}w(x,t)$, where λ is a constant. Find the corresponding partial differential equation for w. You are not asked to solve it.

Solution. We have $u_{xx} = e^{\lambda t}w_{xx}$ and $u_t = e^{\lambda t}(\lambda w + w_t)$. Plugging back into the given equation yields

$$20u_{xx} - 10u_t + 17u = e^{\lambda t}(20w_{xx} - 10\lambda w - 10w_t + 17w) = 0.$$

Thus the PDE for w is $20w_{xx} - (10\lambda - 17)w - 10w_t = 0$. $\square$

b) Find a value for λ so that the partial differential equation for w found in part a) has no term in w. Then write the PDE as $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial w}{\partial t}$ and determine k.

Solution. We want the coefficient of w to be zero, that is $10\lambda - 17 = 0$, which gives $\lambda = \dfrac{17}{10}$. The equation for w can be written as $w_{xx} = \dfrac{10}{20}w_t$, so $k = 2$. $\square$

Problem 4: (25 points) Consider the heat problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{9}\frac{\partial u}{\partial t}, \qquad 0 < x < 3, \qquad t > 0;$$
$$u(0,t) = 20, \quad u(3,t) = 50, \qquad t > 0;$$
$$u(x,0) = 60 - 2x, \qquad 0 < x < 3.$$

a) Find the steady state solution. State the problem satisfied by the transient solution.

SOLUTION. The steady state solution verifies the equation $v''(x) = 0$, with boundary conditions $v(0) = 20$ and $v(3) = 50$. We find $v(x) = 10x + 20$. The transient solution is $w(x,t) = u(x,t) - v(x)$ and verifies the PDE:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{9}\frac{\partial w}{\partial t}, \qquad 0 < x < 3, \qquad t > 0;$$
$$w(0,t) = 0, \quad w(3,t) = 0, \qquad t > 0;$$
$$w(x,0) = 40 - 12x, \qquad 0 < x < 3.$$

□

b) Find the temperature $u(x,t)$.

SOLUTION. The solution to the homogeneous equation is

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x) e^{-k\lambda_n^2 t},$$

where $\lambda_n = \frac{n\pi}{3}$ and $k = 9$. We simplify this and get

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right) e^{-(n\pi)^2 t}.$$

We find the coefficients c_n from the initial condition $w(x,0) = 40 - 12x$:

$$\begin{aligned} c_n &= \frac{2}{3}\int_0^3 (40 - 12x)\sin\left(\frac{n\pi x}{3}\right) dx = \frac{80}{3}\int_0^3 \sin\left(\frac{n\pi x}{3}\right) dx - 8\int_0^3 x\sin\left(\frac{n\pi x}{3}\right) dx \\ &= \frac{80}{n\pi}(1 - \cos(n\pi)) + \frac{72}{n\pi}\cos(n\pi) = \frac{8}{n\pi}\left(10 - (-1)^n\right). \end{aligned}$$

Therefore, the transient solution is

$$w(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{10 - (-1)^n}{n} \sin\left(\frac{n\pi x}{3}\right) e^{-(n\pi)^2 t}$$

and $u(x,t) = w(x,t) + v(x)$.

Note that the computations are similar to Problem 1c. □

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$

MAT 341 – Applied Real Analysis
Fall 2015

Midterm 1 – October 1, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

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PROBLEM	SCORE
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TOTAL	

Problem 1: (25 points) Consider the function

$$f(x) = \begin{cases} -x & \text{if} \quad -2 \le x < 0 \\ x & \text{if} \quad 0 \le x < 2, \end{cases} \qquad f(x+4) = f(x).$$

Find the Fourier series for f. Determine whether the series converges uniformly or not. To what value does the Fourier series converge at $x = 2015$?

Problem 2: (25 points) Suppose that the Fourier series of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} e^{-341n} \cos(n\pi x)$.

a) What is the Fourier series of $1 - 2f(x)$?

b) What is the Fourier series of $F(x) = \int_0^x f(y)\, dy$?

c) Find the Fourier series of $f''(x)$ if it exists. Otherwise, explain why it does not exist.

d) What is the period of f? Can f have jump discontinuities or is it a continuous function?

Problem 3: (25 points) Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - S\frac{\partial u}{\partial x} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0$$

with boundary conditions

$$u(0,t) = T_0, \quad u(2,t) = 0, \qquad t > 0$$

and initial condition $u(x,0) = f(x)$, $0 \leq x \leq 2$. (S and T_0 are positive constants.)

a) Find the steady-state solution $v(x)$. What is the ODE that $v(x)$ satisfies?

b) State the initial value–boundary value problem satisfied by the transient solution $w(x,t)$. You are NOT asked to solve this problem.

Problem 4: (25 points) Solve the heat problem

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 1, \quad t > 0; \\
&u(0,t) = 0, \;\; u(1,t) = \beta, \quad t > 0; \\
&u(x,0) = \beta x + \sin\left(\frac{\pi x}{2}\right), \quad 0 \le x \le 1.
\end{aligned}
$$

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

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$$\sin^2(a) = \frac{1-\cos(2a)}{2} \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

MAT 341 – Applied Real Analysis
FALL 2015

Midterm 1 – October 1, 2015

SOLUTIONS

NAME: ______________________________

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PROBLEM	SCORE
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TOTAL	

Problem 1: (25 points) Consider the function

$$f(x) = \begin{cases} -x & \text{if} \quad -2 \le x < 0 \\ x & \text{if} \quad 0 \le x < 2, \end{cases} \qquad f(x+4) = f(x).$$

Find the Fourier series for f. Determine whether the series converges uniformly or not. To what value does the Fourier series converge at $x = 2015$?

Solution. Notice that f is even, so we can use the half-formulas when computing the Fourier coefficients. The Fourier series is just a cosine series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right).$$

We have $a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = 1$ and $a_n = \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)\,dx = 4\frac{(-1)^n - 1}{n^2\pi^2}$ (using the formula at the end of the booklet). The Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} 4\frac{(-1)^n - 1}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right).$$

The function is continuous, with piecewise continuous derivative, so the Fourier series converges uniformly everywhere. This can be seen also from the coefficients as

$$\sum_{n=1}^{\infty} |a_n| = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{|(-1)^n - 1|}{n^2} \le \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges. At $x = 2015$, the Fourier series converges to $f(2015) = f(2016-1) = f(-1) = 1$. We have used the fact that f is periodic of period 4. □

Problem 2: (25 points) Suppose that the Fourier series of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} e^{-341n} \cos(n\pi x)$.

a) What is the Fourier series of $1 - 2f(x)$?

SOLUTION.

$$1 - 2f(x) = 1 - 2\sum_{n=1}^{\infty} e^{-341n} \cos(n\pi x)$$

$\square$

b) What is the Fourier series of $F(x) = \int_0^x f(y)\, dy$?

SOLUTION.

$$F(x) = \int_0^x \sum_{n=1}^{\infty} e^{-341n} \cos(n\pi y)\ dy = \sum_{n=1}^{\infty} \frac{e^{-341n}}{n\pi} \sin(n\pi x)$$

$\square$

c) Find the Fourier series of $f''(x)$ if it exists. Otherwise, explain why it does not exist.

SOLUTION.

$$f''(x) = -\sum_{n=1}^{\infty} n^2\pi^2 e^{-341n} \cos(n\pi x).$$

This series converges uniformly because $\sum_{n=1}^{\infty} |n^2 a_n| = \pi^2 \sum_{n=1}^{\infty} \frac{n^2}{e^{341n}} < \infty$ (which converges by the integral test). Notice also that the denominator is a polynomial, while the nominator is an exponential, hence the series converges. $\square$

d) What is the period of f? Can f have jump discontinuities or is it a continuous function?

SOLUTION. The Fourier series is periodic of period 2, hence f is periodic of period 2. The function is continuous (by part c) we already know that f is twice differentiable, hence f is continuous). $\square$

Problem 3: (25 points) Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - S\frac{\partial u}{\partial x} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0$$

with boundary conditions

$$u(0,t) = T_0, \quad u(2,t) = 0, \qquad t > 0$$

and initial condition $u(x,0) = f(x)$, $0 \le x \le 2$. (S and T_0 are positive constants.)

a) Find the steady-state solution $v(x)$. What is the ODE that $v(x)$ satisfies?

Solution. The steady-state solution verifies the equation $v''(x) - Sv'(x) = 0$, with boundary conditions $v(0) = T_0$ and $v(2) = 0$. The characteristic equation is $r^2 - Sr = 0$ and has roots $r = S$ and $r = 0$. The solution is $v(x) = C_1 + C_2 e^{Sx}$. We find the coefficients from the boundary conditions. We have $C_1 + C_2 = T_0$ and $C_1 + C_2 e^{2S} = 0$. Hence $C_1 = \frac{T_0 e^{2S}}{e^{2S}-1}$ and $C_2 = -\frac{T_0}{e^{2S}-1}$ and

$$v(x) = \frac{T_0 e^{2S}}{e^{2S} - 1} - \frac{T_0 e^{Sx}}{e^{2S} - 1}.$$

□

b) State the initial value–boundary value problem satisfied by the transient solution $w(x,t)$. You are NOT asked to solve this problem.

Solution. By definition $w(x,t) = u(x,t) - v(x)$. Using the equations for u from the hypothesis and for v from part a) we find

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < 2, \quad t > 0;$$

$$w(0,t) = 0, \quad w(2,t) = 0, \qquad t > 0;$$

$$w(x,0) = f(x) - v(x), \quad 0 \le x \le 2.$$

where $v(x)$ is the steady-state solution from part a). □

Problem 4: (25 points) Solve the heat problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 1, \qquad t > 0;$$
$$u(0,t) = 0, \quad u(1,t) = \beta, \qquad t > 0;$$
$$u(x,0) = \beta x + \sin\left(\frac{\pi x}{2}\right), \quad 0 \le x \le 1.$$

Solution. We first find the steady-state solution $v(x) = \beta x$. As shown in the lecture, the transient solution $w(x,t)$ verifies the PDE

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{4}\frac{\partial w}{\partial t}, \qquad 0 < x < 1, \qquad t > 0;$$
$$w(0,t) = 0, \quad w(1,t) = 0, \qquad t > 0;$$
$$w(x,0) = \sin\left(\frac{\pi x}{2}\right), \quad 0 \le x \le 1.$$

and the solution of this PDE is given by

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-4n^2\pi^2 t},$$

where

$$c_n = 2\int_0^1 \sin(n\pi x)\sin\left(\frac{\pi x}{2}\right)\, dx.$$

Note that these are not orthogonal functions! These functions have different periods: $\sin(n\pi x)$ has period 2, while $\sin\left(\frac{\pi x}{2}\right)$ has period 4. We compute the integral, using the formulas at the end of the booklet and find $c_n = \frac{(-1)^n 8n}{\pi(1-4n^2)}$. The solution to the given PDE is

$$u(x,t) = \beta x + \sum_{n=1}^{\infty} \frac{(-1)^n 8n}{\pi(1-4n^2)} \sin(n\pi x) e^{-4n^2\pi^2 t}.$$

□

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

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MAT 341 – Applied Real Analysis
Spring 2015

Midterm 1 – March 10, 2015

NAME: ______________________________

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PROBLEM	**SCORE**
1	
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TOTAL	

Problem 1: (22 points) Suppose that the Fourier cosine series of a given function $f(x)$ is

$$f(x) = \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1+2(-1)^n}{2015n^3}\cos\left(\frac{n\pi x}{4}\right).$$

a) Show that $f(x) = f(x+8)$.

b) Does this Fourier cosine series converge uniformly? Explain.

c) Find the Fourier cosine series of $1 - 5f(x)$.

d) Find the Fourier cosine series of $f'(x)$ if it exists. If it does not exist, explain why it does not exist.

Problem 2: (30 points) Consider the function

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi \le x < 0, \\ x & \text{if} \quad 0 \le x < \pi; \end{cases} \qquad\qquad f(x + 2\pi) = f(x).$$

a) Sketch the graph of f on the interval $[0, 4\pi]$.

b) Find the Fourier series for f.

c) To what value does the Fourier series converge at:

i) $x = 0$; ii) $x = \frac{\pi}{2}$; iii) $x = 3\pi$? Explain.

d) Does the Fourier series of f converges uniformly on the interval $[0, \pi]$? Does it converge uniformly on the interval $[0, 4\pi]$? Explain.

Problem 3: (24 points) Consider the heat conduction problem in a bar that is in thermal contact with an external heat source. Then the modified heat conduction equation is

$$\frac{\partial^2 u}{\partial x^2} + s(x) = \frac{1}{k}\frac{\partial u}{\partial t}$$

where the term $s(x)$ describes the effect of the external agency; $s(x)$ is positive for a source. Suppose that the boundary conditions are

$$u(0,t) = T_0, \quad u(a,t) = T_1$$

and the initial condition is $u(x,0) = f(x)$.

a) Write $u(x,t) = w(x,t) + v(x)$, where $w(x,t)$ and $v(x)$ are the transient and steady state parts of the solution, respectively. State the boundary value problems that $v(x)$ and $w(x,t)$, respectively, satisfy.

b) Suppose $k = 1$ and $s(x) = 6x$. Find $v(x)$.

Problem 4: (24 points) Find the solution of the heat problem

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \qquad t > 0;$$

$$u(0,t) = 0, \quad u(2,t) = \pi, \qquad t > 0;$$

$$u(x,0) = \frac{\pi x}{2} - 3\sin(\pi x) + 5\sin(2\pi x), \quad 0 \le x \le 2.$$

Some useful formulas

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

MAT 341 – Applied Real Analysis
Spring 2015

Midterm 1 – March 10, 2015

Solutions

NAME: ______________________________

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PROBLEM	SCORE
1	
2	
3	
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TOTAL	

Problem 1: (22 points) Suppose that the Fourier cosine series of a given function $f(x)$ is

$$f(x) = \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{2015n^3}\cos\left(\frac{n\pi x}{4}\right).$$

a) Show that $f(x) = f(x+8)$.

SOLUTION. Clearly $\cos\left(\frac{n\pi x}{4}\right)$ is periodic of period 8 for each integer n. The sum of periodic functions of the same period (in this case 8) is again a periodic function of the same period. So $f(x)$ is periodic of period 8. □

b) Does this Fourier cosine series converge uniformly? Explain.

SOLUTION. The Fourier cosine series converges uniformly because

$$\sum_{n=1}^{\infty} |a_n| + |b_n| = \sum_{n=1}^{\infty} \frac{|1+2(-1)^n|}{2015n^3} < \sum_{n=1}^{\infty} \frac{3}{2015n^3} = \frac{3}{2015}\sum_{n=1}^{\infty}\frac{1}{n^3} < \infty.$$

We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-integral test for $p = 3 > 1$). □

c) Find the Fourier cosine series of $1 - 5f(x)$.

SOLUTION. The Fourier cosine series of 1 is just 1. Note that the function $1 - 5f(x)$ is even if f is even, so it has a Fourier cosine series which is

$$1 - 5f(x) = 1 - 5\frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{2015n^3}\cos\left(\frac{n\pi x}{4}\right) = 1 - \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{403n^3}\cos\left(\frac{n\pi x}{4}\right)$$

□

d) Find the Fourier cosine series of $f'(x)$ if it exists. If it does not exist, explain why it does not exist.

SOLUTION. We have

$$\sum_{n=1}^{\infty} |na_n| + |nb_n| = \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{|1+2(-1)^n|}{2015n^2} < \frac{3}{2015\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty,$$

since we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. This means that the differentiated Fourier series converges uniformly to the derivative $f'(x)$. Therefore the Fourier sine series of $f'(x)$ is the following

$$f'(x) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{2015n^3}\frac{n\pi}{4}\sin\left(\frac{n\pi x}{4}\right) = -\frac{1}{8060\pi}\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{n^2}\sin\left(\frac{n\pi x}{4}\right).$$

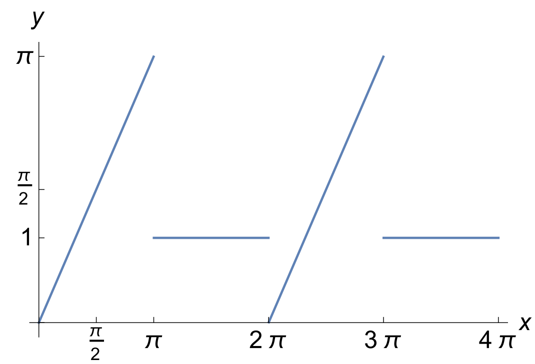
However, there are no cosine terms in this series (and the Fourier series is unique) so there is no cosine Fourier series for $f'(x)$. Alternatively, since we are given the Fourier cosine series of f, we can assume that f is even (or work with it's even extension). But $f(x) = f(-x)$ gives $f'(x) = -f'(-x)$ so the derivative f' is odd, so it has a sine series, rather than a cosine series. □

Problem 2: (30 points) Consider the function

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi \le x < 0, \\ x & \text{if} \quad 0 \le x < \pi; \end{cases} \qquad\qquad f(x+2\pi) = f(x).$$

a) Sketch the graph of f on the interval $[0, 4\pi]$.

SOLUTION.



□

b) Find the Fourier series for f.

SOLUTION. The function is periodic of period 2π so $a = \pi$. The Fourier series of f is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx \\ &= \left.\frac{\sin(nx)}{n\pi}\right|_{-\pi}^{0} + \left.\frac{\cos(nx)}{n^2\pi}\right|_{0}^{\pi} + \left.\frac{x\sin(nx)}{n\pi}\right|_{0}^{\pi} = \frac{\cos(n\pi)-1}{n^2\pi} = \frac{(-1)^n-1}{n^2\pi}. \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx \\ &= \left.\frac{-\cos(nx)}{n\pi}\right|_{-\pi}^{0} + \left.\frac{\sin(nx)}{n^2\pi}\right|_{0}^{\pi} - \left.\frac{x\cos(nx)}{n\pi}\right|_{0}^{\pi} = \frac{-1+\cos(n\pi)}{n\pi} - \frac{\pi\cos(n\pi)}{n\pi} \\ &= \frac{-1+(-1)^n}{n\pi} - \frac{(-1)^n}{n}. \end{aligned}$$

Also $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 1\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx = \frac{1}{2} + \frac{\pi}{4}$. The Fourier series is

$$f(x) = \frac{1}{2} + \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n^2\pi}\cos(nx) + \left(\frac{-1+(-1)^n}{n\pi} - \frac{(-1)^n}{n}\right)\sin(nx)\right).$$

In computing the coefficients a_n and b_n we have used the formulas provided on the last page of the exam. □

c) To what value does the Fourier series converge at:

i) $x = 0$; ii) $x = \frac{\pi}{2}$; iii) $x = 3\pi$? Explain.

Solution. Clearly the function is piecewise continuous and has a piecewise continuous derivative. The function is discontinuous (it has a jump) at $x = 0$ and $x = 3\pi$, as seen from the graph. So the Fourier series converges to $\frac{f(0+)+f(0-)}{2} = \frac{1}{2}$ at $x = 0$ and to $\frac{f(3\pi+)+f(3\pi-)}{2} = \frac{\pi+1}{2}$ at $x = 3\pi$. The function is continuous at $x = \frac{\pi}{2}$ so the Fourier series converges to $f(x) = \frac{\pi}{2}$ in this case. □

d) Does the Fourier series of f converges uniformly on the interval $[0, \pi]$? Does it converge uniformly on the interval $[0, 4\pi]$? Explain.

Solution. Both intervals $[0, \pi]$ and $[0, 4\pi]$ contain the points $x = 0$ and $x = \pi$. At $x = 0$ the Fourier series converges to $\frac{1}{2}$ as shown above. However, if we take x arbitrarily close to 0 then the function is continuous on $(0, \pi)$ and the Fourier series converges to $f(x) = x$. For example, for $x = 0.01$, the Fourier series converges to 0.01, which is far from $\frac{1}{2} = 0.5$. So in both cases the Fourier series of f does not converge uniformly. □

Problem 3: (24 points) Consider the heat conduction problem in a bar that is in thermal contact with an external heat source. Then the modified heat conduction equation is

$$\frac{\partial^2 u}{\partial x^2} + s(x) = \frac{1}{k}\frac{\partial u}{\partial t}$$

where the term $s(x)$ describes the effect of the external agency; $s(x)$ is positive for a source. Suppose that the boundary conditions are

$$u(0,t) = T_0, \quad u(a,t) = T_1$$

and the initial condition is $u(x,0) = f(x)$.

a) Write $u(x,t) = w(x,t) + v(x)$, where $w(x,t)$ and $v(x)$ are the transient and steady state parts of the solution, respectively. State the boundary value problems that $v(x)$ and $w(x,t)$, respectively, satisfy.

SOLUTION.

$$\begin{aligned} &v''(x) = -s(x) \\ &v(0) = T_0, \quad v(a) = T_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t} \\ &w(0,t) = 0, \quad w(a,t) = 0 \\ &w(x,0) = f(x) - v(x) \end{aligned}$$

□

b) Suppose $k = 1$ and $s(x) = 6x$. Find $v(x)$.

SOLUTION. We have $v''(x) = -6x$ so $v'(x) = -3x^2 + A$ and $v(x) = -x^3 + Ax + B$. From $v(0) = T_0$ we get $B = T_0$. From $v(a) = T_1$ we get $-a^3 + A \cdot a + T_0 = T_1$, which gives $A = \frac{a^3+T_1-T_0}{a} = a^2 + \frac{T_1-T_0}{a}$. Thus

$$v(x) = -x^3 + \left(a^2 + \frac{T_1 - T_0}{a}\right) x + T_0.$$

□

Problem 4: (24 points) Find the solution of the heat problem

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0;$$
$$u(0,t) = 0, \quad u(2,t) = \pi, \quad t > 0;$$
$$u(x,0) = \frac{\pi x}{2} - 3\sin(\pi x) + 5\sin(2\pi x), \quad 0 \le x \le 2.$$

Solution. We first need to find the steady-state solution $v(x)$. Note that $v''(x) = 0$ and $v(0) = 0, v(2) = \pi$. This gives $v(x) = \frac{\pi x}{2}$. We then need to solve the following homogeneous problem

$$\frac{\partial^2 w}{\partial x^2} = 4\frac{\partial w}{\partial t}, \qquad 0 < x < 2, \quad t > 0;$$
$$w(0,t) = 0, \quad w(2,t) = 0, \quad t > 0;$$
$$w(x,0) = -3\sin(\pi x) + 5\sin(2\pi x), \quad 0 \le x \le 2.$$

However, we know that the solution to this problem is given by

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 kt} \sin(\lambda_n x)$$

where $\lambda_n = \frac{n\pi}{a}$ and $k = \frac{1}{4}$, $a = 2$ in this problem. Therefore

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

The coefficients c_n can be determined from $w(x,0) = -3\sin(\pi x) + 5\sin(2\pi x)$, so $c_2 = -3$, $c_4 = 5$ and $c_n = 0$ for all other values of n. It follows that

$$w(x,t) = -3e^{-\frac{\pi^2}{4}t}\sin(\pi x) + 5e^{-\pi^2 t}\sin(2\pi x).$$

The solution to the initial problem is $u(x,t) = w(x,t) + v(x)$, so

$$u(x,t) = -3e^{-\frac{\pi^2}{4}t}\sin(\pi x) + 5e^{-\pi^2 t}\sin(2\pi x) + \frac{\pi x}{2}.$$

□

Some useful formulas

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

1

2.5. The heat equation: different boundary conditions

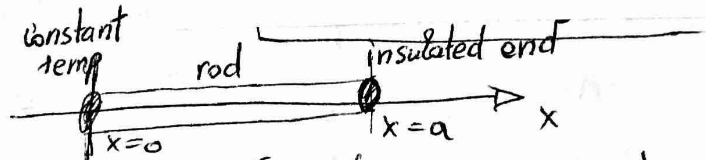
We need to solve the following PDE:

$$U_{xx} = \frac{1}{K} U_t, \quad 0<x<a, \; t>0$$

$$U(0,t) = T_0, \quad t>0$$

$$U_x(a,t) = 0, \quad t>0$$

$$U(x,0) = f(x), \quad 0<x<a$$



First, since we don't have homogeneous boundary conditions ($U(0,t) = T_0 \neq 0$) we need to find the steady state solution "substitute $u(x,t) = v(x)$". The equation for this solution is:

$v(x) \overset{def}{=} \lim_{t\to\infty} u(x,t)$

$$v''(x) = 0, \quad v(0) = T_0, \quad v'(a) = 0$$

$v(x) = Ax + B$, $B = T_0$ and $A = 0$ so $v(x) = T_0$ constant.

The PDE that the transient solution $w(x,t) = u(x,t) - v(x)$ satisfies is the following:

$$w_{xx} = \frac{1}{K} w_t \quad 0<x<a, \; t>0$$

homogeneous boundary conditions $\begin{cases} w(0,t) = 0, \; t>0 \\ w_x(a,t) = 0, \; t>0 \end{cases}$

initial condition changes $\begin{cases} w(x,0) = f(x) - T_0 \quad 0<x<a \\ \quad\quad\quad\; = f(x) - v(x) \end{cases}$

We use separations of variables: $w(x,t) = \phi(x) T(t)$ and compute:

$w_{xx} = \frac{1}{K} w_t$ becomes $\phi''(x) T(t) = \frac{1}{K} \phi(x) T'(t)$

$$\frac{\phi''}{\phi} = \frac{1}{K} \frac{T'}{T} = \text{constant} = -\lambda^2 \quad (\lambda \text{ some real number})$$

So $\phi'' + \lambda^2 \phi = 0$

$T' + K\lambda^2 T = 0 \Rightarrow T(t) = A e^{-K\lambda^2 t}$ (we usually choose $A = 1$ here)

so $T(t) = e^{-K\lambda^2 t}$

boundary data $w(0,t) = \phi(0)T(t) = 0$ so $\phi(0) = 0$ $\left(\begin{matrix}\text{otherwise}\\ w \equiv 0\end{matrix}\right)$

$w_x(a,t) = \phi'(a)T(t) = 0$ so $\phi'(a) = 0$

we need to solve the ODE:

$$\phi'' + \lambda^2 \phi = 0, \quad \phi(0) = 0, \phi'(a) = 0$$

the characteristic equation is $r^2 + \lambda^2 = 0$ which has roots

① $r_1 = r_2 = 0$ if $\lambda = 0$

② $r_1 = i\lambda$, $r_2 = -i\lambda$, $\lambda \neq 0$

① $\underline{\lambda = 0}$: $\phi(x) = C_1 + C_2 x$, $\phi(0) = C_1 = 0$

$\phi'(a) = C_2 = 0$ so $\phi \equiv 0$ which gives $w \equiv 0$

② $\underline{\lambda \neq 0}$: $\phi(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$

$\phi(0) = C_1 = 0$

$\phi'(a) = C_2 \lambda \cos(\lambda a) = 0$ so $\cos(\lambda a) = 0$

or $\lambda a = \dfrac{(2n-1)\pi}{2}$, $n = 1, 2 \ldots$

we find $\lambda_n = \dfrac{(2n-1)\pi}{2a}$ $n = 1, 2 \ldots$

and $\phi_n(x) = \sin(\lambda_n x) = \sin\left(\dfrac{(2n-1)\pi x}{2a}\right)$ $n = 1, 2 \ldots$

(we take $C_2 = 1$ here)

The $\underline{\text{fundamental solutions}}$ of the heat equation are

$$w_n(x,t) = \phi_n(x) T_n(t) = \sin\left(\frac{(2n-1)\pi x}{2a}\right) e^{-k\left(\frac{(2n-1)\pi}{2a}\right)^2 t}$$

The general solution is

$$w(x,t) = \sum_{n=1}^{\infty} C_n w_n(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{(2n-1)\pi x}{2a}\right) e^{-k\left(\frac{(2n-1)\pi}{2a}\right)^2 t}$$

To find the coefficients C_n we have to use the initial condition $w(x,0) = f(x) - v(x) = f(x) - T_0$

$$w(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{(2n-1)\pi x}{2a}\right) = f(x) - T_0 \quad , 0<x<a$$

__Remark__: This is not the Fourier sine series of $f(x) - T_0$, because the Fourier series of $f(x) - T_0$ on the interval $0<x<a$ would involve terms of the form $\sin\left(\frac{n\pi x}{a}\right)$, not $\sin\left(\frac{(2n-1)\pi x}{2a}\right)$. One has period $2a$, while the other has period $4a$.

We can use the orthogonality of the functions $\sin\left(\frac{(2n-1)\pi x}{2a}\right)$, $n=1,2,\dots$ on $0<x<a$

$$\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{a}{2} & \text{if } m = n \end{cases}$$ (← See pg 5 *)

$$\lambda_n = \frac{(2n-1)\pi}{a}, \quad \lambda_m = \frac{(2m-1)\pi}{a}.$$

$$f(x) - T_0 = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x) \quad \Big/ \cdot \sin(\lambda_m x)$$

$$\int_0^a (f(x) - T_0)\sin(\lambda_m x)\,dx = \sum_{n=1}^{\infty} C_n \int_0^a \sin(\lambda_n x)\sin(\lambda_m x)\,dx$$

$$= C_m \cdot \frac{a}{2}$$

So $C_m = \frac{2}{a}\int_0^a (f(x) - T_0)\sin(\lambda_m x)\,dx$ for all $m = 1,2,\dots$

The solution to the original PDE is

$$u(x,t) = T_0 + \sum_{n=1}^{\infty} C_n \sin(\lambda_n x)\, e^{-k\lambda_n^2 t}, \quad \lambda_n = \frac{(2n-1)\pi}{a}.$$

Example: Solve the PDE

$$u_{xx} = \frac{1}{4} u_t \quad , 0<x<\pi \ , \ t>0$$
$$u(0,t) = 0, \ u_x(\pi,t) = 0 \ , \ t>0$$
$$u(x,0) = \sin\left(\frac{x}{2}\right) - 17 \sin\left(\frac{3x}{2}\right) \quad , \ 0<x<\pi$$

We have already shown that

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x)\, e^{-k\lambda_n^2 t} \quad , \ \lambda_n = \frac{(2n-1)\pi}{2a}$$

here $k=4$, $a=\pi$ so $\lambda_n = \frac{(2n-1)\pi}{2\pi} = \frac{2n-1}{2}$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)x}{2}\right) = \sin\left(\frac{x}{2}\right) - 17\sin\left(\frac{3x}{2}\right)$$

so $c_1 = 1$, $c_2 = -17$, $c_n = 0$ for $n \geq 3$. There is no need to actually compute them!

The solution is then:

$$u(x,t) = \sin\left(\frac{x}{2}\right) e^{-4\cdot\frac{1}{4}t} - 17\sin\left(\frac{3x}{2}\right) e^{-4\cdot\frac{9}{4}t}$$
$$u(x,t) = \sin\left(\frac{x}{2}\right) e^{-t} - 17\sin\left(\frac{3x}{2}\right) e^{-9t}$$

Other types of PDEs:

Example:
$$u_{xx} = \frac{1}{k} u_t + \gamma^2 u \quad , \ 0<x<a, \ t>0$$
$$u(0,t) = 0, \ u_x(a,t) = 0 \ , \ 0<t$$
$$u(x,0) = f(x) \quad , \ 0<x<a$$

use separation of variables $u(x,t) = \phi(x)T(t)$ and write down ODEs for ϕ and for T. Do not solve!

$$\phi'' T = \frac{1}{k}\phi T' + \gamma^2 \phi T = \left(\frac{1}{k}T' + \gamma^2 T\right)\phi$$

$$\frac{\phi''}{\phi} = \frac{\frac{1}{K}T' + \gamma^2 T}{T} = \frac{1}{K}\frac{T'}{T} + \gamma^2 = \lambda$$

Remark here we don't know that $\frac{\phi''}{\phi} \leq 0$ always as in the heat equation so we have to consider all cases, not just $\frac{\phi''}{\phi} = -\lambda^2$.

We get $\phi'' - \lambda\phi = 0$

$$T' + K(\gamma^2 - \lambda)T = 0$$

boundary data: $u(0,t) = 0$, $u_x(a,t) = 0$ so

$$\phi(0) = 0, \quad \phi'(a) = 0.$$

---

* Fact:

$$\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)\,dx = \begin{cases} 0, & m \neq n \\ \frac{a}{2}, & m = n \end{cases}, \quad \lambda_n = \frac{(2n-1)\pi}{2a}$$

if $m \neq n$: by trig identities we get

$$\frac{\sin[(\lambda_m - \lambda_n)x]}{2(\lambda_m - \lambda_n)} - \frac{\sin[(\lambda_m + \lambda_n)x]}{2(\lambda_m + \lambda_n)}\Bigg|_0^a =$$

$$= \frac{\sin[(\lambda_m - \lambda_n)a]}{2(\lambda_m - \lambda_n)} - \frac{\sin[(\lambda_m + \lambda_n)a]}{2(\lambda_m + \lambda_n)} = 0 \text{ if } m \neq n$$

which is equivalent to $(\lambda_m + \lambda_n)\sin((\lambda_m - \lambda_n)a) = (\lambda_m - \lambda_n)\sin((\lambda_m + \lambda_n)a)$

$$\frac{(2m-1)\pi + (2n-1)\pi}{2a}\sin\left(\frac{2(m-n)\pi}{2}\right) = \frac{(2m-1)\pi - (2n-1)\pi}{2a}\sin\left(\frac{2(m+n-1)\pi}{2}\right)$$

$$(m+n-1)\pi\,\underbrace{\sin((m-n)\pi)}_{=0} = (m-n)\underbrace{\sin((m+n-1)\pi)}_{=0}$$

$\sin(m\pi)\cos(n\pi) - \cos(m\pi)\sin(n\pi)$

if $m = n$ then just integrate $\int_0^a \sin^2(\lambda_n x)\,dx = \frac{a}{2}$ using $1 - \cos(2x) = \sin^2 x$.

$$\int_0^a \cos 2\lambda_n x\,dx = \frac{\sin 2x\lambda_n}{2\lambda_n}\Bigg|_0^a = \sin(2n-1)\pi = 0$$

Consider the problem:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi'(a) = 0$$

Suppose that ϕ_n, ϕ_m are ~~eigenvalues~~ eigenfunctions corresponding to eigenvalues λ_n, λ_m such that $\lambda_n \neq \lambda_m$.

Then ϕ_n, ϕ_m are <u>orthogonal</u>: $\int_0^a \phi_m(x)\phi_n(x)\,dx = 0$.

There is no need to solve the problem to observe this, so we do an indirect computation:

$$\phi_m'' + \lambda_m \phi_m = 0 \;\Big|\cdot \phi_n \quad \Rightarrow \quad \phi_m''\phi_n + \lambda_m \phi_n\phi_m = 0$$

$$\phi_n'' + \lambda_n \phi_n = 0 \;\Big|\cdot \phi_m \qquad\quad \phi_n''\phi_m + \lambda_n \phi_n\phi_m = 0$$

so $\int_0^a \phi_m''\phi_n - \phi_n''\phi_m + \int_0^a (\lambda_m - \lambda_n)\phi_m\phi_n = 0$ so since $\lambda_m - \lambda_n \neq 0$ we just need to show that

$$\int_0^a \underbrace{\phi_m''\phi_n - \phi_n''\phi_m}_{\text{by parts}}\,dx = 0.$$

$$\underbrace{\left(\phi_m'\phi_n - \phi_n'\phi_m\right)\Big|_0^a}_{0 \text{ by } \phi(0)=0,\ \phi'(a)=0} - \int_0^a \big(\underbrace{\phi_m'\phi_n' - \phi_n'\phi_m'}_{0}\big)\,dx$$

So $\int_0^a \phi_m\phi_n\,dx = 0$.

we can do a similar proof to show that:

<u>Thm:</u> The (simplified) Sturm-Liouville problem:

$$\phi'' + \lambda^2 \phi = 0, \quad 0 < x < a$$

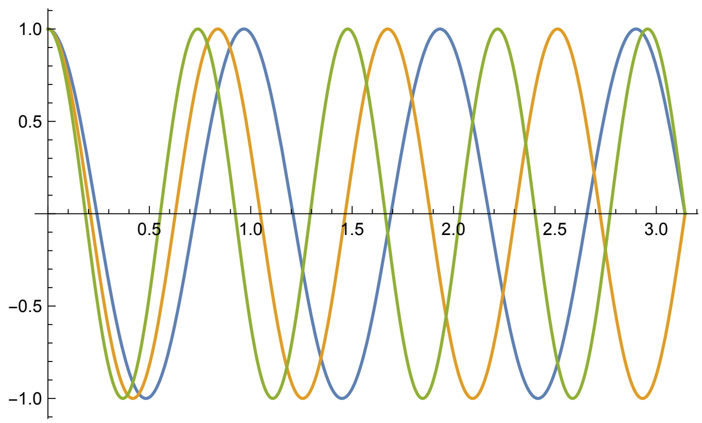
$$\alpha_1 \phi(0) - \alpha_2 \phi'(0) = 0$$

$$\beta_1 \phi(a) + \beta_2 \phi'(a) = 0$$

has an infinite number of eigenfunctions $\phi_1, \phi_2, \ldots$ each corresponding to a different eigenvalue $\lambda_1, \lambda_2, \ldots$ If $m \neq n$ then ϕ_n and ϕ_m are orthogonal and $\int_0^a \phi_m \phi_n \, dx = 0$.

```
(* Problem 2.7:3c *)
(* we can take any value for a,
say a=π. We have plotted the graphs of the 7th, 8th, and 9th eigenfunction.*)
a = π;
n = 7;
Plot[{Cos[(2 * n - 1) π * x/(2 * a)],
  Cos[(2 * (n + 1) - 1) π * x/(2 * a)], Cos[(2 * (n + 2) - 1) π * x/(2 * a)]}, {x, 0, a}]
```

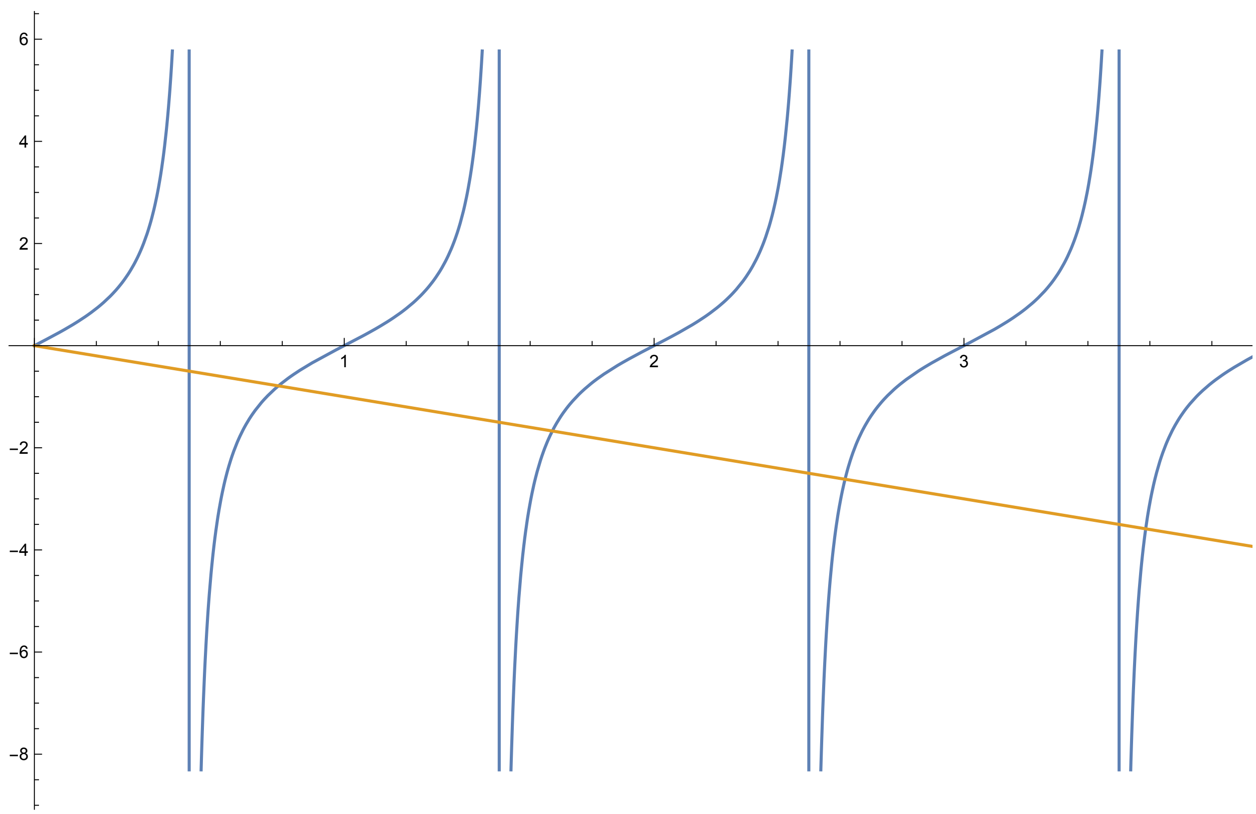
Out[57]=




```
(* Problem 2.7:3c *)
```

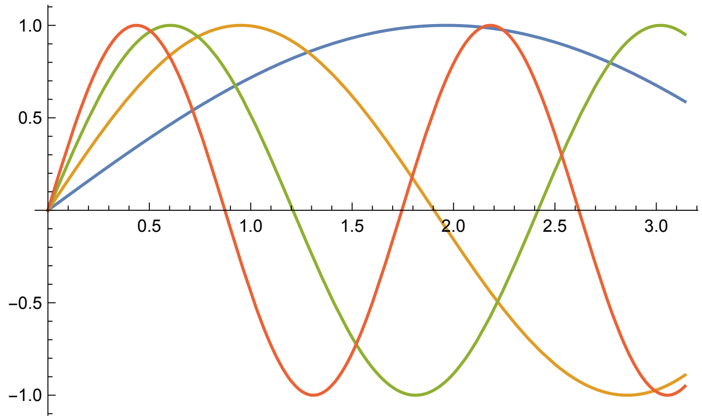
In[58]:= `Plot[{Tan[λ * a], -λ}, {λ, 0, 4}]`

Out[58]=



```
(* we observe that λ1≃.8, λ2≃.1.65, λ3≃.2.6, λ3≃.3.6,
etc. and plot the first few graphs of the eigenfunctions*)
Plot[{Sin[.8 * x], Sin[1.65 * x], Sin[2.6 * x], Sin[3.6 * x]}, {x, 0, a}]
```

Out[62]=



MAT 341: Applied Real Analysis – Spring 2017

HW8 – Comments

Sec. 3.3 – Problem 1: The problem is asking you to find some values of $u(x,t)$ such that

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \qquad t > 0;\\
&u(0,t) = 0, \quad u(a,t) = 0, \qquad t > 0;\\
&u(x,0) = f(x), \qquad t > 0;\\
&\frac{\partial u}{\partial x}(x,0) = 0, \quad 0 < x < a.
\end{aligned}$$

where $f(x)$ has the following equation:

$$f(x) = \begin{cases} \dfrac{2h}{a}x & \text{if} \quad 0 \le x \le \frac{a}{2} \\ -\dfrac{2h}{a}x + 2h & \text{if} \quad \frac{a}{2} < x \le a. \end{cases}$$

You then need to write a table with the values $u(x,t)$ at the required times, such as $u(0.25a, 0.2a/c)$. The solution $u(x,t)$ is written in Equation 13, but without the function G_e. **Note:** In the textbook, $\overline{f}_o$ means an odd periodic extension of f, while $\overline{G}_e$ means an even periodic extension of G.

Sec. 3.3 – Problem 2: You fix time $t = 0, 0.2a/c, 0.4a/c, 0.8a/c, 1.4a/c$ and you sketch 5 graphs of $u(x,t)$. For example, you need to sketch the graph of $u(x, 0.4a/c)$ as a function of x. You may assume $a = 1$ if it helps. The graphs should look like Figure 3 from Section 3.2.

Sec. 3.3 – Problem 5: The solution $u(x,t)$ verifies the PDE:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \qquad t > 0;\\
&u(0,t) = 0, \quad u(a,t) = 0, \qquad t > 0;\\
&u(x,0) = 0, \qquad 0 < x < a;\\
&\frac{\partial u}{\partial t}(x,0) = \alpha c, \quad 0 < x < a.
\end{aligned}$$

where α is just a constant, unrelated to a.

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HW9 – Comments

Sec. 4.1 – Problem 2: The sketch of the surfaces should look like the graphs below.

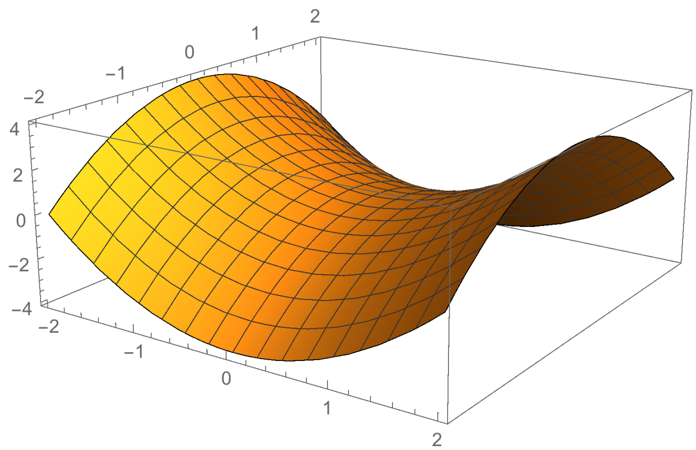

Figure 1: A sketch of the surface $z = x^2 - y^2$.

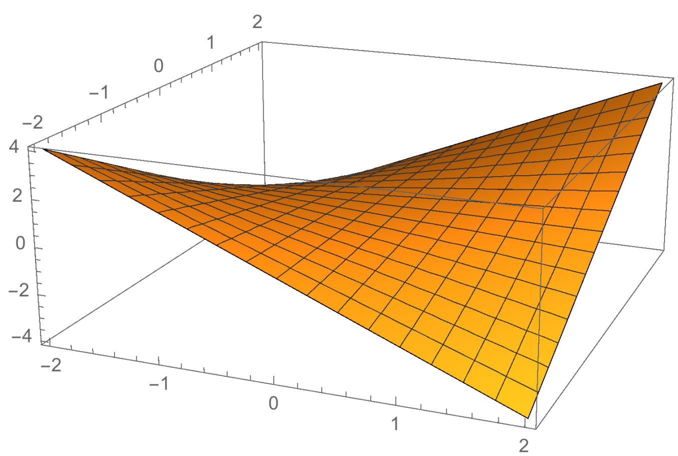

Figure 2: A sketch of the surface $z = xy$.

Regarding the boundary conditions: you have to evaluate $u(x, y)$, $\frac{\partial u}{\partial x}(x, y)$ and $\frac{\partial u}{\partial y}(x, y)$ at the given values. For example, if $u(x, y) = xy$ then $u(0, b) = 0$ and $u_x(0, b) = b$, $u_y(0, b) = 0$.

Sec. 4.2 – Problem 5: You are asked to solve the following PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 1, \qquad 0 < y < b$$

$$u(0, y) = 0, \quad u(1, y) = 0, \qquad 0 < y < b$$

$$u(x, 0) = 0, \quad u(x, b) = \sin(3\pi x), \qquad 0 < x < 1$$

You may assume that b is any constant. However, once you reach a formula for $u(x, y)$ as in Equation 9 (page 266) there is no need to compute the coefficients, simply use the fact that you already have $\sin(3\pi x)$ as a Fourier series and look for the coefficient of $n = 3$ (the rest are all zeros). To sketch the level curves, one has to do as in Figure 2, page 268 (see next page).

```
(*Problem 4.2:5 Sketch of level curve. *)
a = 1;
b = 0.5;
const = 0.1;
ContourPlot[Sinh[3 * π * y]/Sinh[3 * π * b] Sin[3 π * x] == const, {x, 0, a}, {y, 0, b}]
```

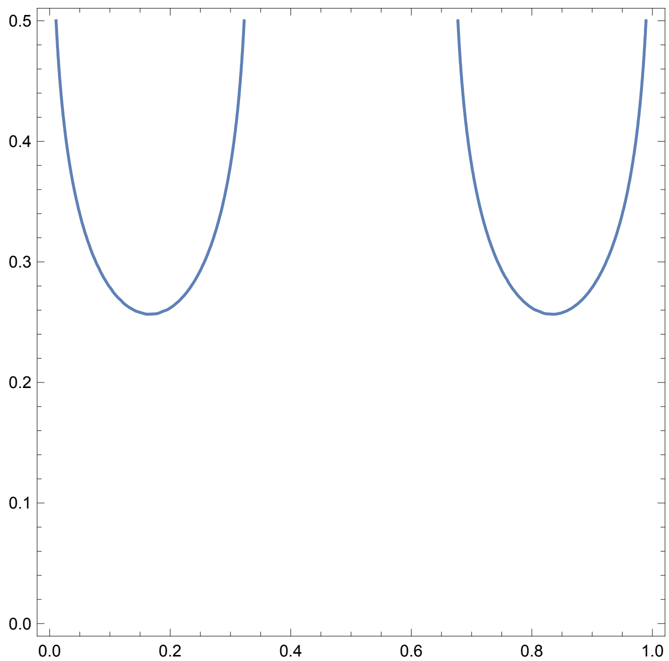

Figure 3: Level curves $u(x, y) =$ const. drawn in Mathematica.

```
Plot3D[Sinh[3*π*y]/Sinh[3*π*b] Sin[3π*x], {x, 0, a}, {y, 0, b}]
```

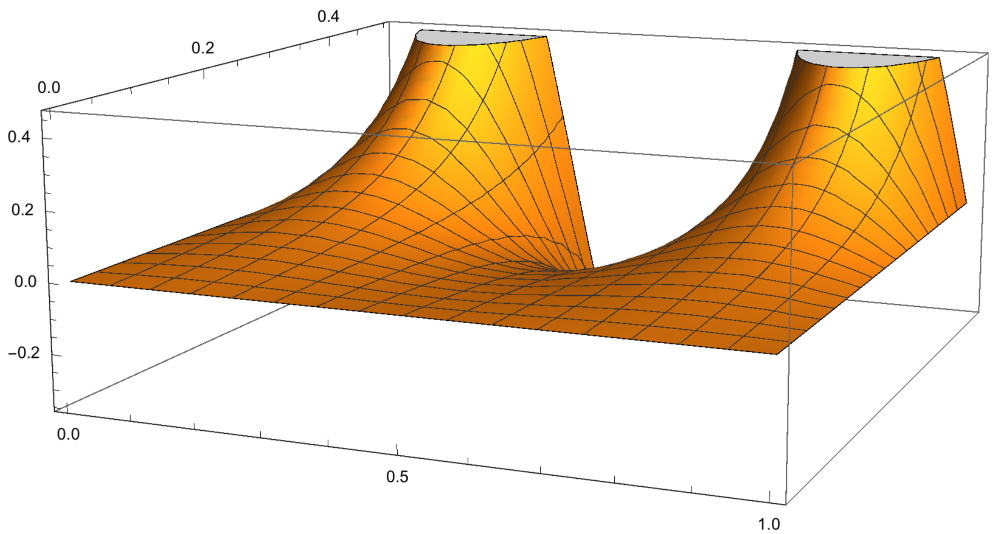

Figure 4: The surface $z = u(x, y)$. The level curves are obtained by cutting the level surface by a plane transversely.

Sec. 4.2 – Problem 6: You are asked to solve the following PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b$$

$$u(0, y) = 0, \quad u(a, y) = 1, \qquad 0 < y < b$$

$$u(x, 0) = 0, \quad u(x, b) = 0, \qquad 0 < x < a$$

MAT 341 – Applied Real Analysis
SPRING 2017

Midterm 2 – April 11, 2017

SOLUTIONS

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator. You **are** allowed to bring a note card to the exam (8.5 x 5.5in - front and back), but no other notes are allowed.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

Problem 1: (18 points) Find the Fourier integral representation of

$$f(x) = \begin{cases} \pi x & \text{if } \ 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the value of the Fourier integral at $x = 0$? At $x = 1$?

SOLUTION. We first compute the Fourier integral coefficients:

$$\begin{aligned} A(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\,dx = \frac{1}{\pi}\int_0^1 \pi x\cos(\lambda x)\,dx \\ &= \int_0^1 x\cos(\lambda x)\,dx = \left(\frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}\right)\Big|_0^1 \\ &= \frac{\sin(\lambda)}{\lambda} + \frac{\cos(\lambda)}{\lambda^2} - \frac{1}{\lambda^2} \end{aligned}$$

and

$$\begin{aligned} B(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)\,dx = \frac{1}{\pi}\int_0^1 \pi x\sin(\lambda x)\,dx \\ &= \int_0^1 x\sin(\lambda x)\,dx = \left(\frac{\sin(\lambda x)}{\lambda^2} - \frac{x\cos(\lambda x)}{\lambda}\right)\Big|_0^1 \\ &= \frac{\sin(\lambda)}{\lambda^2} - \frac{\cos(\lambda)}{\lambda} \end{aligned}$$

The Fourier integral representation is

$$\int_0^{\infty} A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\,d\lambda = \frac{f(x-) + f(x+)}{2}.$$

At $x = 0$ the integral equals 0. At $x = 1$ the integral equals $\frac{\pi}{2}$. □

Problem 2: (16 points) Consider the heat problem in a semi-infinite rod:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\pi}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0$$

$$u(x,t) \text{ bounded as } x \to \infty$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \qquad \text{where} \quad f(x) = \begin{cases} \pi x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad 1 \le x \end{cases}$$

b) Let $u(x,t) = \phi(x)T(t)$. Write down an ODE for ϕ together with the boundary and boundedness conditions.

SOLUTION. The eigenvalue problem for ϕ is

$$\phi'' + \lambda^2\phi = 0, \qquad 0 < x < \pi$$

$$\phi'(0) = 0$$

$$\phi(x) \text{ bounded as } x \to \infty$$

□

c) Find the general solution $u(x,t)$.

SOLUTION.

SOLUTION. The solution is given by

$$u(x,t) = \int_0^\infty A(\lambda)\cos(\lambda x)e^{-\pi\lambda^2 t}\,d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,dx = \frac{2}{\pi}\int_0^1 \pi x\cos(\lambda x)\,dx \\ &= 2\int_0^1 x\cos(\lambda x)\,dx = 2\frac{\lambda\sin(\lambda) + \cos(\lambda) - 1}{\lambda^2}. \end{aligned}$$

Therefore the solution is

$$u(x,t) = 2\int_0^\infty \frac{\lambda\sin(\lambda) + \cos(\lambda) - 1}{\lambda^2}\cos(\lambda x)e^{-\pi\lambda^2 t}\,d\lambda.$$

Note that the value for $A(\lambda)$ was already computed in Problem 1. □

Problem 3: (22 points) Consider the conduction of heat in a rod with insulated lateral surface whose left end is held at constant temperature and whose right end is exposed to convective heat transfer. Suppose the PDE satisfied by the temperature in the rod is:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \qquad t > 0 \\
&u(0,t) = 0, \qquad t > 0 \\
&2u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0 \\
&u(x,0) = x, \quad 0 < x < 1.
\end{aligned}$$

a) Let $u(x,t) = \phi(x)T(t)$. Write down the eigenvalue problem for ϕ (that is, and ODE satisfied by ϕ and the boundary conditions).

SOLUTION. The eigenvalue problem for ϕ is

$$\begin{aligned}
&\phi'' + \lambda^2\phi = 0, \qquad 0 < x < 1 \\
&\phi(0) = 0 \\
&2\phi(1) + \phi'(1) = 0
\end{aligned}$$

□

b) Solve the eigenvalue problem for ϕ and determine the eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$.

SOLUTION. This is a convection problem. We know that $\lambda = 0$ is not an eigenvalue. If $\lambda^2 > 0$ then $\phi(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x)$. From $\phi(0) = 0$ we get $C_1 = 0$. From the second condition we get $2\phi(1) + \phi'(1) = 2C_2\sin(\lambda) - C_2\lambda\cos(\lambda) = 0$. This yields

$$\sin(\lambda) = -\frac{1}{2}\lambda\cos(\lambda) \quad \Rightarrow \quad \tan(\lambda) = -\frac{\lambda}{2}.$$

The eigenvalues are λ_n, $n = 1, 2, \ldots$, where λ_n is the n-th root of this equation. The eigenfunctions are $\phi_n(x) = \sin(\lambda_n x)$. □

(Problem 3 continued)

c) Find the general solution $u(x,t)$.

SOLUTION. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x) e^{-\lambda_n^2 t}$$

where

$$c_n = \frac{\displaystyle\int_0^1 x\phi_n(x)\,dx}{\displaystyle\int_0^1 \phi_n^2(x)\,dx} = \frac{\displaystyle\int_0^1 x\sin(\lambda_n x)\,dx}{\displaystyle\int_0^1 \sin^2(\lambda_n x)\,dx}$$

To finish the computation we evaluate

$$\int_0^1 x\sin(\lambda_n x)\,dx = \left.\frac{\sin(\lambda_n x)}{\lambda_n^2}\right|_0^1 - \left.\frac{x\cos(\lambda_n x)}{\lambda_n}\right|_0^1 = \frac{\sin(\lambda_n)}{\lambda_n^2} - \frac{\cos(\lambda_n)}{\lambda_n}$$

and

$$\int_0^1 \sin^2(\lambda_n x)\,dx = \int_0^1 \frac{1-\cos(2\lambda_n x)}{2}\,dx = \frac{1}{2} - \frac{\sin(2\lambda_n)}{4\lambda_n},$$

where λ_n is the n-th root of the equation $\tan(\lambda) = -\frac{\lambda}{2}$. □

Problem 4: (20 points) Find the eigenvalues and the corresponding eigenfunctions of the problem:

$$\phi'' + \lambda^2\phi = 0, \qquad 0 < x < \pi$$

$$\phi(0) - \phi(\pi) = 0, \qquad \phi'(0) = 0$$

Is this a regular Sturm-Liouville problem?

SOLUTION. If $\lambda = 0$, then $\phi(x) = Ax + B$. From $\phi'(0) = 0$ we find $A = 0$. The condition $\phi(0) - \phi(\pi) = B - B = 0$ gives no information about B. Hence $\lambda = 0$ is an eigenvalue and $\phi_0(x) = 1$ is an eigenfunction.

If $\lambda^2 > 0$, then $\phi(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$. From $\phi'(0) = C_2\lambda = 0$ we find $C_2 = 0$. Thus $\phi(x) = C_1 \cos(\lambda x)$. The first boundary condition $\phi(0) - \phi(\pi) = C_1(1 - \cos(\lambda\pi)) = 0$ yields $\cos(\lambda\pi) = 1$, so $\lambda\pi = 2n\pi$, for $n = 1, 2, \ldots$.

The eigenvalues are $\lambda_n = 2n$, for $n = 1, 2, \ldots$, and the corresponding eigenfunctions are $\phi_n(x) = \cos(2nx)$.

This is not a regular Sturm-Liouville problem. □

Problem 5: (24 points) Consider the following vibrating string problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(a,t) = 0, \qquad t > 0$$

$$u(x,0) = 0, \qquad 0 < x < a$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < a, \qquad \text{where} \quad g(x) = \begin{cases} 0 & \text{if} \quad 0 < x < \frac{a}{2} \\ 2c & \text{if} \quad \frac{a}{2} \le x < a. \end{cases}$$

a) Find $u(x,t)$ using separation of variables.

SOLUTION. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)\cos(\lambda_n ct) + b_n \sin(\lambda_n x)\sin(\lambda_n ct),$$

where $\lambda_n = \frac{n\pi}{a}$, $n = 1, 2, \ldots$. Since $u(x,0) = 0$, we get $a_n = 0$. The other initial condition gives

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{a} \sin\left(\frac{n\pi x}{a}\right) = g(x).$$

Therefore

$$\begin{aligned} b_n &= \frac{2}{n\pi c}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{n\pi c}\int_{\frac{a}{2}}^a 2c\sin\left(\frac{n\pi x}{a}\right)dx \\ &= -\frac{4}{n\pi}\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\Big|_{\frac{a}{2}}^{a} = \frac{4a}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right) \\ &= \frac{4a(-1)^{n+1}}{n^2\pi^2} \end{aligned}$$

The solution to the given PDE is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4a(-1)^{n+1}}{n^2\pi^2}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi ct}{a}\right).$$

□

(Problem 5 continued)

b) Find $u(x,t)$ using D'Alembert's solution to the wave equation.

SOLUTION. We first compute

$$G(x) = \frac{1}{c}\int_0^a g(y)\,dy = \begin{cases} 0 & \text{if} \quad 0 < x < \dfrac{a}{2} \\ 2x - a & \text{if} \quad \dfrac{a}{2} \le x < a. \end{cases}$$

Note that for $\frac{a}{2} \le x < a$ we have

$$\frac{1}{c}\int_0^a g(y)\,dy = \int_{a/2}^x 2\,dy = 2x - a.$$

The even extension G_e of G has the formula

$$G_e(x) = \begin{cases} 0 & \text{if} \quad -\dfrac{a}{2} < x < \dfrac{a}{2} \\ 2x - a & \text{if} \quad \dfrac{a}{2} \le x < a \\ -2x - a & \text{if} \quad -a < x \le -\dfrac{a}{2}. \end{cases}$$

Let $\tilde{G}_e$ be the periodic extension of G_e of period $2a$. The solution to the PDE is

$$u(x,t) = \frac{1}{2}\left(\tilde{G}_e(x+ct) - \tilde{G}_e(x-ct)\right).$$

□

c) Using the solution from part b) compute $u\left(a, \frac{a}{c}\right)$.

SOLUTION. Using part b) we find

$$u\left(a, \frac{a}{c}\right) = \frac{1}{2}\left(\tilde{G}_e(2a) - \tilde{G}_e(0)\right) = 0,$$

since $\tilde{G}_e$ is periodic of period $2a$. Note that part c) can be solved independently of part b). □

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

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$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

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MAT 341 – Applied Real Analysis
Fall 2015

Midterm 2 – November 5, 2015

NAME: ______________________________

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Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
5	
TOTAL	

Problem 1: (12 points) The *telegraph equation* governs the flow of voltage, or current, in a transmission line and has the form:

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + ku = a^2\frac{\partial^2 u}{\partial x^2} + F(x,t), \qquad 0 < x < 100, \quad t > 0.$$

The coefficients c, k, a are constants related to electrical parameters in the line. Assuming that $F(x,t) = 0$ and $u(x,t) = \phi(x)T(t)$, carry out a separation of variables and find the eigenvalue problem for ϕ. Take the boundary conditions to be

$$\frac{\partial u}{\partial x}(0,t) = 0 \;\text{ and }\; u(100,t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by $T(t)$.

Problem 2: (20 points) Solve the heat problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(2,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < 2, \qquad \text{where} \quad f(x) = \begin{cases} T_0 & \text{if} \quad 0 < x < 1 \\ T_1 & \text{if} \quad 1 \leq x < 2 \end{cases}$$

Problem 3:

a) (12 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2 \phi = 0, \quad 0 < x < 1$$

$$\phi(0) = 0, \qquad \phi'(1) - \phi(1) = 0$$

Is $\lambda = 0$ an eigenvalue?

(Problem 3 continued)

b) (5 points) Consider the function

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 < x < 0.5 \\ 1 - x & \text{if} \quad 0.5 \le x < 1. \end{cases}$$

Suppose $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is the expansion of the function $f(x)$ in terms of the eigenfunctions $\phi_n(x)$ from part *a*). Write down a formula for the coefficients c_n. You are **not** asked to compute the coefficients.

c) (7 points) To what value does the series converge at $x = 0.5$? What about at $x = 0$ and $x = 0.3$?

Problem 4: (22 points) Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0$$

$$u(x,t) \ \text{ bounded as } x \to \infty$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \quad \text{where} \quad f(x) = \begin{cases} \pi - x & \text{if } \ 0 < x < \pi \\ 0 & \text{if } \ \pi \le x \end{cases}$$

Problem 5: (22 points) If an elastic string is *free* at one end, the boundary condition to be satisfied there is that $\dfrac{\partial u}{\partial x} = 0$. On the other hand, if it is *fixed* at one end, the boundary condition to be satisfied there is that $u = 0$. Find the displacement $u(x,t)$ in an elastic string of length $a = 1$, fixed at $x = 0$ and free at $x = a$, set in motion with no initial velocity from the initial position $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$.

a) State the boundary value problem that $u(x,t)$ satisfies. Include the initial conditions.

b) Find $u(x,t)$.

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

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MAT 341 – Applied Real Analysis
Fall 2015

Midterm 2 – November 5, 2015

Solutions

NAME: ______________________________

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PROBLEM	**SCORE**
1	
2	
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TOTAL	

Problem 1: (12 points) The *telegraph equation* governs the flow of voltage, or current, in a transmission line and has the form:

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + ku = a^2\frac{\partial^2 u}{\partial x^2} + F(x,t), \qquad 0 < x < 100, \quad t > 0.$$

The coefficients c, k, a are constants related to electrical parameters in the line. Assuming that $F(x,t) = 0$ and $u(x,t) = \phi(x)T(t)$, carry out a separation of variables and find the eigenvalue problem for ϕ. Take the boundary conditions to be

$$\frac{\partial u}{\partial x}(0,t) = 0 \ \text{ and } \ u(100,t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by $T(t)$.

Solution. If we substitute $u(x,t) = \phi(x)T(t)$ we get $\phi T'' + c\phi T' + k\phi T = a^2\phi'' T$. Separation of variables gives

$$\frac{T'' + cT' + kT}{T} = a^2\frac{\phi''}{\phi} = \lambda, \qquad \text{where } \lambda \text{ is some real number.}$$

We get $a^2\phi'' - \lambda\phi = 0$ and $T'' + cT' + (k-\lambda)T = 0$, which is an ODE satisfied by T. The first boundary condition gives $\frac{\partial u}{\partial x}(0,t) = \phi'(0)T(t) = 0$ so $\phi'(0) = 0$. The second boundary condition gives $u(100,t) = \phi(100)T(t) = 0$, so $\phi(100) = 0$. □

Problem 2: (20 points) Solve the heat problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(2,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < 2, \qquad \text{where} \quad f(x) = \begin{cases} T_0 & \text{if} \quad 0 < x < 1 \\ T_1 & \text{if} \quad 1 \le x < 2 \end{cases}$$

Solution. We identify $a = 2$ and $k = 4$. The general solution to this equation is

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t}.$$

The coefficients can be found from the initial condition $u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) = f(x)$. We have $c_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{T_0+T_1}{2}$ and

$$\begin{aligned} c_n &= \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^1 T_0\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 T_1\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{2T_0}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\bigg|_0^1 + \frac{2T_1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\bigg|_1^2 \\ &= \frac{2(T_0 - T_1)}{n\pi}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

The solution is

$$u(x,t) = \frac{T_0+T_1}{2} + 2(T_0 - T_1)\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t}.$$

□

Problem 3:

a) (12 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2\phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \qquad \phi'(1) - \phi(1) = 0$$

Is $\lambda = 0$ an eigenvalue?

SOLUTION. If $\lambda = 0$ then $\phi'' = 0$ so $\phi(x) = Ax + B$. From $\phi(0) = 0$ we immediately find $B = 0$. However the relation $\phi'(1) - \phi(1) = 0$ does not give other information about A. We find $\phi(x) = Ax$ for $A \neq 0$. So $\lambda = 0$ is an eigenvalue.

If $\lambda \neq 0$ then $\phi(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x)$. The condition $\phi(0) = 0$ gives $C_1 = 0$. We can take $C_2 = 1$ at this step and write $\phi(x) = \sin(\lambda x)$. The condition $\phi'(1) - \phi(1) = 0$ gives $\lambda = \tan(\lambda)$. The eigenvalues are λ_n, the n^{th} root of the equation $\lambda = \tan(\lambda)$, for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are $\phi_n(x) = \sin(\lambda_n x)$. □

(Problem 3 continued)

b) (5 points) Consider the function

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 < x < 0.5 \\ 1 - x & \text{if} \quad 0.5 \leq x < 1. \end{cases}$$

Suppose $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is the expansion of the function $f(x)$ in terms of the eigenfunctions $\phi_n(x)$ from part a). Write down a formula for the coefficients c_n. You are **not** asked to compute the coefficients.

SOLUTION. We have

$$c_n = \frac{\displaystyle\int_0^1 f(x)\phi_n(x)\,dx}{\displaystyle\int_0^1 \phi_n^2(x)\,dx}.$$

□

c) (7 points) To what value does the series converge at $x = 0.5$? What about at $x = 0$ and $x = 0.3$?

SOLUTION. The function has a jump discontinuity at $x = 0.5$ so the series converges to $\dfrac{f(.5-) + f(.5+)}{2} = \dfrac{1.5}{2} = \dfrac{3}{4}$. The function is continuous at $x = 0.3$ so the series converges to $f(0.3) = 0.6$. At $x = 0$, we have $\phi_n(0) = 0$ from the hypothesis so the series converges to 0. □

Problem 4: (22 points) Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0$$

$$u(x,t) \text{ bounded as } x \to \infty$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \quad \text{where} \quad f(x) = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } \pi \le x \end{cases}$$

Solution. The solution is given by

$$u(x,t) = \int_0^\infty A(\lambda)\cos(\lambda x)e^{-2\lambda^2 t}\,d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(\lambda x)\,dx \\ &= 2\int_0^\pi \cos(\lambda x)\,dx - \frac{2}{\pi}\int_0^\pi x\cos(\lambda x)\,dx \\ &= \frac{2}{\lambda}\sin(\lambda x)\Big|_0^\pi - \frac{2}{\pi}\left(\frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}\right)\Big|_0^\pi \\ &= \frac{2\sin(\lambda\pi)}{\lambda} - \frac{2}{\pi}\frac{\cos(\lambda\pi)}{\lambda^2} - \frac{2\sin(\lambda\pi)}{\lambda} + \frac{2}{\pi\lambda^2} \\ &= \frac{2 - 2\cos(\lambda\pi)}{\pi\lambda^2}. \end{aligned}$$

Therefore the solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda^2}\cos(\lambda x)e^{-2\lambda^2 t}\,d\lambda.$$

□

Problem 5: (22 points) If an elastic string is *free* at one end, the boundary condition to be satisfied there is that $\dfrac{\partial u}{\partial x} = 0$. On the other hand, if it is *fixed* at one end, the boundary condition to be satisfied there is that $u = 0$. Find the displacement $u(x,t)$ in an elastic string of length $a = 1$, fixed at $x = 0$ and free at $x = a$, set in motion with no initial velocity from the initial position $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$.

a) State the boundary value problem that $u(x,t)$ satisfies. Include the initial conditions.

SOLUTION. The initial value-boundary value problem is the following:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0;\\
&u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0;\\
&u(x,0) = \sin\left(\frac{3\pi x}{2}\right), \qquad 0 < x < 1;\\
&\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < 1.
\end{aligned}$$

□

b) Find $u(x,t)$.

SOLUTION. We solve the associated eigenvalue problem and find $\lambda_n = \dfrac{(2n-1)\pi}{2}$, for $n = 1, 2, \ldots$. The general solution of this PDE is therefore

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\lambda_n ct)\sin(\lambda_n x) + b_n \sin(\lambda_n ct)\sin(\lambda_n x).$$

From $\dfrac{\partial u}{\partial t}(x,0) = 0$ we find that $b_n = 0$ for all n. From the initial condition $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$ we find that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi x}{2}\right) = \sin\left(\frac{3\pi x}{2}\right)$$

The Fourier series is unique, so we just need to make the coefficients of the left-hand side equal to the coefficients of the right-hand side. This yields $a_2 = 1$ and $a_n = 0$ for all $n \neq 2$. The solution is then

$$u(x,t) = \cos\left(\frac{3\pi ct}{2}\right)\sin\left(\frac{3\pi x}{2}\right).$$

□

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

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$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$

MAT 341 – Applied Real Analysis
Spring 2015

Midterm 2 – April 16, 2015

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Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

Problem 1: Consider the heat equation

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}$$

on the interval $0 < x < 2$, with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 10, \quad u(2,t) = 100, \qquad \text{for all } t > 0.$$

a) (8 points) What is the *steady-state* temperature distribution?

b) (12 points) Find all the product solutions $w(x,t) = \phi_n(x)T_n(t)$ that satisfy the PDE and the boundary conditions for the *transient* solution. You are **NOT** asked to find the general solution!

Problem 2: (20 points) Find the Fourier integral representation of the function $f(x)$ given below:

$$f(x) = \begin{cases} \pi & \text{if} \quad 0 < x < 1 \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Problem 3: (20 points) Consider the heat conduction problem in a metal rod of semi-infinite length that is insulated on the sides:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$u(0,t) = 0, \qquad t > 0,$$

whose initial temperature distribution is $u(x,0) = f(x)$ for $0 < x < \infty$, where

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \quad \text{otherwise.} \end{cases}$$

Find the temperature $u(x,t)$ if we further assume that $u(x,t)$ remains finite as $x \to \infty$.

Problem 4:

a) (10 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2 \phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi'(1) = 0$$

b) (10 points) Find the expression of the function $f(x) = x$, $0 < x < 1$ in terms of these eigenfunctions. Does this series converge at $x = 1$?

Problem 5: (20 points) Solve the vibrating string problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0;$$

$$u(0,t) = 0, \qquad u(1,t) = 0, \quad t > 0;$$

$$u(x,0) = \sin(3\pi x), \qquad 0 < x < 1;$$

$$\frac{\partial u}{\partial t}(x,0) = \sin(5\pi x), \qquad 0 < x < 1.$$

Explain why $u(x, t+1) = u(x, t)$, which means that the solution to this problem is a function that is periodic in time of period 1.

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

MAT 341 – Applied Real Analysis
Spring 2015

Midterm 2 – Solutions – April 16, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

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PROBLEM	**SCORE**
1	
2	
3	
4	
5	
TOTAL	

Problem 1: Consider the heat equation

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}$$

on the interval $0 < x < 2$, with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 10, \quad u(2,t) = 100, \qquad \text{for all } t > 0.$$

a) (8 points) What is the *steady-state* temperature distribution?

SOLUTION. The steady-state solution $v(x)$ satisfies $v''(x) = 0$ so $v(x) = Ax + B$. From $v'(0) = 10$ and $v(2) = 100$ we find $A = 10$ and $B = 80$. So $v(x) = 10x + 80$. □

b) (12 points) Find all the product solutions $w(x,t) = \phi_n(x)T_n(t)$ that satisfy the PDE and the boundary conditions for the *transient* solution. You are **NOT** asked to find the general solution!

SOLUTION. The transient solution $w(x,t)$ satisfies $w_{xx} = 4w_t$ and $w_x(0,t) = 0$ and $w(2,t) = 0$. We write $w(x) = \phi(x)T(t)$ and get $\phi''T = 4\phi T'$. The boundary conditions are $\phi'(0) = 0$ and $\phi(2) = 0$. Separating the variables we write $\frac{\phi''}{\phi} = \frac{4T'}{T} = -\lambda^2$, so $\phi'' + \lambda^2\phi = 0$ and $T' + \frac{1}{4}\lambda^2 T = 0$. The second equation gives $T(t) = e^{-\frac{\lambda^2}{4}t}$. The first equation gives $\phi(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$. From $\phi'(0) = 0$ we find $c_2 = 0$, so $\phi(x) = c_1\cos(\lambda x)$. From $\phi(2) = 0$ we find $\cos(2\lambda) = 0$ so $\lambda = \frac{(2n-1)\pi}{4}$, for $n = 1, 2, \ldots$

The product solutions are

$$w(x,t) = \phi_n(x)T_n(t) = \cos\left(\frac{(2n-1)\pi}{4}x\right)e^{-\frac{(2n-1)^2\pi^2}{64}t},$$

for $n = 1, 2, \ldots$ □

Problem 2: (20 points) Find the Fourier integral representation of the function $f(x)$ given below:

$$f(x) = \begin{cases} \pi & \text{if} \quad 0 < x < 1 \\ 0 & \quad \text{otherwise.} \end{cases}$$

SOLUTION. The Fourier integral representation of the function $f(x)$ is

$$\int_0^\infty [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda,$$

where

$$A(\lambda) = \frac{1}{\pi}\int_0^\infty f(x)\cos(\lambda x)\, dx = \frac{1}{\pi}\int_0^1 \pi\cos(\lambda x)\, dx = \int_0^1 \cos(\lambda x)\, dx = \frac{\sin(\lambda)}{\lambda},$$

and

$$B(\lambda) = \frac{1}{\pi}\int_0^\infty f(x)\sin(\lambda x)\, dx = \frac{1}{\pi}\int_0^1 \pi\sin(\lambda x)\, dx = \int_0^1 \sin(\lambda x)\, dx = \frac{1-\cos(\lambda)}{\lambda}.$$

Putting everything together we find that

$$f(x) = \int_0^\infty \left[\frac{\sin(\lambda)}{\lambda}\cos(\lambda x) + \frac{1-\cos(\lambda)}{\lambda}\sin(\lambda x)\right] d\lambda.$$

□

Problem 3: (20 points) Consider the heat conduction problem in a metal rod of semi-infinite length that is insulated on the sides:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$
$$u(0,t) = 0, \qquad t > 0,$$

whose initial temperature distribution is $u(x,0) = f(x)$ for $0 < x < \infty$, where

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \quad \text{otherwise.} \end{cases}$$

Find the temperature $u(x,t)$ if we further assume that $u(x,t)$ remains finite as $x \to \infty$.

Solution. In this problem the constant k is 1. The general solution of this PDE is given by

$$u(x,t) = \int_0^\infty B(\lambda)\sin(\lambda x)e^{-\lambda^2 t}\,d\lambda,$$

where

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)\,dx = \frac{2}{\pi}\int_0^1 \sin(\lambda x)\,dx = \frac{2}{\pi}\frac{(-\cos(\lambda x))}{\lambda}\bigg|_0^1 = \frac{2(1-\cos(\lambda))}{\pi\lambda}.$$

Therefore the solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda)}{\lambda}\sin(\lambda x)e^{-\lambda^2 t}\,d\lambda.$$

□

Problem 4:

a) (10 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2\phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi'(1) = 0$$

Solution. If $\lambda = 0$ then $\phi(x) = Ax + B$, but $\phi'(1) = A = 0$ and $\phi(0) = B = 0$. It follows that $\lambda = 0$ is not an eigenvalue. We get that $\phi(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$ is the general solution of this ODE. From $\phi(0) = 0$ we find that $c_1 = 0$. From $\phi'(1) = c_2\lambda\cos(\lambda) = 0$ we find that $\cos(\lambda) = 0$ so $\lambda = \frac{(2n-1)\pi}{2}$ for $n = 1, 2, \ldots$ The eigenvalues are $\lambda_n = \frac{(2n-1)\pi}{2}$, while the eigenfunctions are $\phi_n(x) = \sin(\lambda_n x)$, for $n = 1, 2, \ldots$ □

b) (10 points) Find the expression of the function $f(x) = x$, $0 < x < 1$ in terms of these eigenfunctions. Does this series converge at $x = 1$?

Solution. We write $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$, where

$$c_n = \frac{\int_0^1 \phi_n(x)f(x)\,dx}{\int_0^1 \phi_n^2(x)\,dx} = \frac{\int_0^1 x\sin\left(\frac{(2n-1)\pi}{2}x\right)\,dx}{\int_0^1 \sin^2\left(\frac{(2n-1)\pi}{2}x\right)\,dx}.$$

Using the formulas at the end of the exam we compute

$$\int_0^1 x\sin\left(\frac{(2n-1)\pi}{2}x\right)\,dx = \frac{\sin\left(\frac{(2n-1)\pi}{2}\right)}{\frac{\pi^2}{4}(2n-1)^2} = \frac{4}{\pi^2}\frac{(-1)^{n+1}}{(2n-1)^2}$$

and

$$\int_0^1 \sin^2\left(\frac{(2n-1)\pi}{2}x\right)\,dx = \left.\frac{1-\cos((2n-1)\pi x)}{2}\right|_0^1 = \frac{1}{2}$$

It follows that for $0 < x < 1$ we have

$$x = \sum_{n=1}^{\infty} \frac{8}{\pi^2}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi}{2}x\right).$$

When $x = 1$ the sum becomes $\sum_{n=1}^{\infty} \frac{8}{\pi^2}\frac{1}{(2n-1)^2} < \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges. □

Problem 5: (20 points) Solve the vibrating string problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0;$$
$$u(0,t) = 0, \qquad u(1,t) = 0, \quad t > 0;$$
$$u(x,0) = \sin(3\pi x), \qquad 0 < x < 1;$$
$$\frac{\partial u}{\partial t}(x,0) = \sin(5\pi x), \qquad 0 < x < 1.$$

Explain why $u(x,t+1) = u(x,t)$, which means that the solution to this problem is a function that is periodic in time of period 1.

SOLUTION. In this problem $a = 1$ and $c = 2$. The general solution to this PDE is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(2n\pi t) + b_n \sin(2n\pi t)\right] \sin(n\pi x).$$

We check the initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = \sin(3\pi x),$$

so $a_3 = 1$ and $a_n = 0$ otherwise. From

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n 2n\pi \sin(n\pi x) = \sin(5\pi x),$$

we find $10\pi b_5 = 1$ and so $b_5 = \frac{1}{10\pi}$. The remaining b_n are all zeros. The solution to this problem is

$$u(x,t) = \cos(6\pi t)\sin(3\pi x) + \frac{1}{10\pi}\sin(10\pi t)\sin(5\pi x).$$

Clearly $u(x,t+1) = u(x,t)$ since $\cos(6\pi t)$ and $\sin(10\pi t)$ are both periodic of period 1. □

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

MAT 341: Applied Real Analysis – Spring 2017

Extra practice problems for Midterm 2

The following problems are meant for extra practice only.

Problem 1: Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad u(a,t) = T_0, \quad t > 0;$$
$$u(x,0) = T_0 + T_1 \cos\left(\frac{\pi x}{2a}\right), \quad 0 < x < a.$$

Solution: After you find the steady-state solution, this is Exercise 9 from Ch 2.5; solution at the end of the book.

Problem 2: Find the steady-state solution, the associated eigenvalue problem, and the complete solution of the following problem:

$$\frac{\partial^2 u}{\partial x^2} - \gamma^2 u = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(a,t) = 0, \quad t > 0;$$
$$u(x,0) = \frac{T_1 x}{a}, \quad 0 < x < a.$$

Solution: This is Exercise 5 from Ch 2: Miscellaneous Exercises (page 206); solution at the end of the book.

Problem 3: Find the steady-state solution, the associated eigenvalue problem, and the complete solution of the following problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \quad t > 0;$$
$$u(0,t) = T_0, \quad t > 0;$$
$$u(x,t) \text{ bounded as } x \to \infty;$$
$$u(x,0) = T_0(1 - e^{-2x}), \quad 0 < x.$$

Solution: This is Exercise 11 from Ch 2: Miscellaneous Exercises (page 207) with $\alpha = 2$; solution at the end of the book.

Problem 4: Find the eigenvalues and eigenfunctions of the problem

$$\phi'' + \lambda^2\phi = 0, \quad 0 < x < 2$$
$$\phi(0) - \phi'(0) = 0$$
$$\phi(2) + \phi'(2) = 0$$

Solution: This is Exercise 3(e) from Ch 2.7 with $a = 2$; solution at the end of the book.

Problem 5: Verify that the eigenvalues and eigenfunctions of the problem

$$(e^x\phi')' + e^x\gamma^2\phi = 0, \quad 0 < x < a$$
$$\phi(0) = 0 \quad \phi(a) = 0$$

are

$$\gamma_n^2 = \left(\frac{n\pi}{a}\right)^2 + \frac{1}{4}, \quad \phi_n(x) = e^{-\frac{x}{2}} \sin\left(\frac{n\pi x}{a}\right).$$

Is this a regular Sturm-Liouville problem? Find the coefficients for the expansion of the function $f(x) = 1$, $0 < x < a$, in terms of the ϕ_n. To what values does the series converge at $x = 0$ and $x = a$?

Solution: This is Exercise 3 from Ch 2.8; solution at the end of the book.

Problem 6: Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \quad t > 0;$$
$$u(0,t) = 0, \quad u(a,t) = 0, \quad t > 0;$$
$$u(x,0) = 0, \quad 0 < x < a;$$
$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{a}\right), \quad 0 < x < a.$$

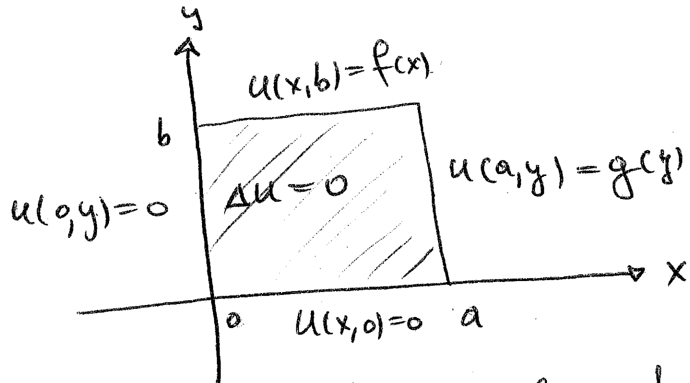
Solution: This is similar to homework Exercise 5 from Ch 3.2; solution at the end of the book. Instead of $g(x) = 1$ we use $g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{a}\right)$, which is just a Fourier series that converges uniformly to some function $g(x)$.

4.3. Potential in a rectangle

Example 1: $\Delta u = u_{xx} + u_{yy} = 0, \; 0<x<a, \; 0<y<b$

$u(0,y)=0, \; u(a,y)=g(y); \; 0<y<b$

$u(x,0)=0, \; u(x,b)=f(x), \; 0<x<a$



Remark: We need to split this problem into two problems if we want to use separation of variables.

PDE1: $\Delta u_1 = 0$

$\boxed{u_1(0,y)=0, \; u_1(a,y)=0}$ ← homogeneous vertical boundary

$u_1(x,0)=0, \; u_1(x,b)=f(x)$

PDE2: $\Delta u_2 = 0$

$u_2(0,y)=0, \; u_2(a,y)=g(y)$

$\boxed{u_2(x,0)=0, \; u_2(x,b)=0}$ ← homogeneous horizontal boundary

We then have $u(x,y)=u_1(x,y)+u_2(x,y)$. We solve PDE1 using separation of variables: $u_1(x,y)=X(x)Y(y)$ and get $X''Y+XY''=0$ so $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$

Also $X(0)=0, \; X(a)=0, \; Y(0)=0$.

From $X''+\lambda^2 X=0$ we find $X(x)=C_1\sin(\lambda x)+C_2\cos(\lambda x)$

$X(0)=0$ so $C_2=0$, $X(a)=0$ so $\sin(\lambda a)=0$ or $\lambda=\frac{n\pi}{a}$, $n=1,2\ldots$

$X_n(x)=\sin(\lambda_n x), \quad \lambda_n=\frac{n\pi}{a}$.

$Y'' - \lambda^2 Y = 0$ gives $Y(y) = C_1 \sinh(\lambda y) + C_2 \cosh(\lambda y)$

$Y(0) = 0$ so $C_2 = 0$ and $Y(y) = \sinh(\lambda y)$

$Y_n(y) = \sinh(\lambda_n y)$ and we have found

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

From $u_1(x,b) = f(x)$ we get $\displaystyle\sum_{n=1}^{\infty} \underbrace{c_n \sinh\left(\frac{n\pi b}{a}\right)}_{b_n} \sin\left(\frac{n\pi x}{a}\right) = f(x)$

Set $b_n = c_n \sinh\left(\frac{n\pi b}{a}\right)$ a constant.

Then $\displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) = f(x)$ so $\displaystyle b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

and $c_n = \dfrac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)}$. The solution of PDE1 is given by:

$$u_1(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right), \quad b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Using a similar strategy we solve <u>PDE2</u> to find $u_2(x,y)$:

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n x) \sin(\lambda_n y), \quad \lambda_n = \frac{n\pi}{b}, \; n=1,2,\ldots$$

<u>Remark</u>: We swap $x \longleftrightarrow y$ in PDE1 to get the solution for PDE2.
$a \longleftrightarrow b$
$f \longleftrightarrow g$

So one needs to pay attention on constants.

$$u_2(a,y) = g(y) = \sum_{n=1}^{\infty} \underbrace{c_n \sinh(\lambda_n a)}_{a_n} \sin(\lambda_n y)$$

$a_n = c_n \sinh(\lambda_n a)$ so $\displaystyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{b}\right) = g(y)$ gives $\displaystyle a_n = \frac{2}{b}\int_0^b g(y) \sin\left(\frac{n\pi y}{b}\right) dy$

↑ by Fourier series

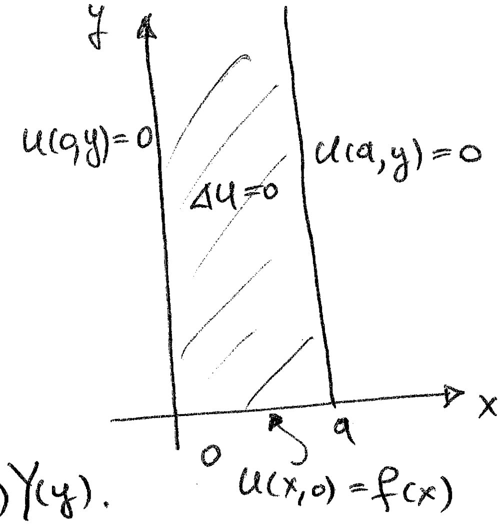
and $c_n = \dfrac{a_n}{\sinh\left(\frac{n\pi a}{b}\right)}$

The general solution of PDE2 is:

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \sin\left(\frac{n\pi y}{b}\right), \text{ where}$$

$$a_n = \frac{2}{b}\int_0^b g(y)\sin\left(\frac{n\pi y}{b}\right)dy.$$

4.4. Potential in unbounded regions



Example 2: $\Delta u = 0, \ 0<x<a, \ y>0$
$u(x,0) = f(x), \ 0<x<a$
$u(0,y) = 0, \ u(a,y) = 0, \ 0<y$
$u(x,y)$ bounded as $y \to \infty$

We set up separation of variables $u(x,y) = X(x)Y(y)$.

Then $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = -\lambda^2$ (we need to set it to $-\lambda^2$, not λ^2, because otherwise $X(x)$ will become ~~unbounded~~ zero.)

$X(0) = X(a) = 0$
$X'' + \lambda^2 X = 0$ and $Y'' - \lambda^2 Y = 0$

We find $X_n(x) = \sin(\lambda_n x)$, $\lambda_n = \frac{n\pi}{a}$, $n = 1, 2, \ldots$

and $Y(y) = C_1 e^{-\lambda y} + C_2 e^{\lambda y}$ (It is more convenient to use this notation instead of $C_1 \sinh(\lambda y) + C_2\cosh(\lambda y)$.)

$Y(y)$ bounded as $y \to \infty$ means that $C_2 = 0$ so $Y(y) = C_1 e^{-\lambda y}$. We found $Y_n(y) = e^{-\lambda_n y}$, $\lambda_n = \frac{n\pi}{a}$.

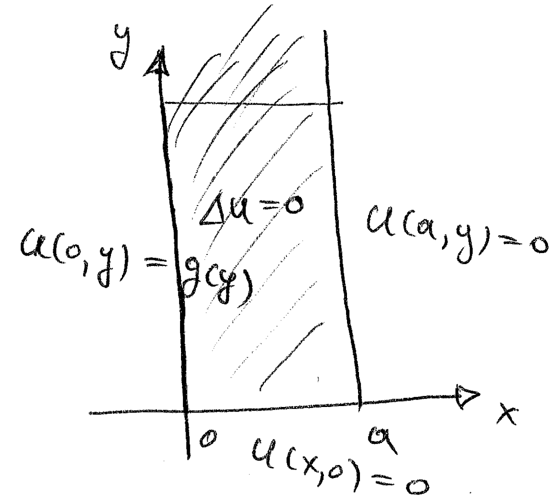
Putting all together we find:

$$u(x,y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi}{a}y}$$

From $u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)$ we get $c_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$

Example 3:

$$\Delta u = 0, \quad 0<x<a, \; 0<y$$
$$u(x,0) = 0, \quad 0<x<a$$
$$u(0,y) = g(y), \quad 0<y$$
$$u(a,y) = 0, \quad 0<y$$
$u(x,y)$ bounded as $y \to \infty$



$u(x,y) = X(x)\,Y(y)$ so $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = \lambda^2$

and:
$X(a) = 0$
$Y(0) = 0$

(here we set it equal to λ^2 because if $-\dfrac{Y''}{Y} = -\lambda^2$ then $Y'' - \lambda^2 Y = 0$ and $Y(0) = 0$ will give
$Y(y) = c_1 e^{\lambda y} + c_2 e^{-\lambda y}$, $c_2 = -c_1$
$Y(y) = c_1(e^{\lambda y} - e^{-\lambda y})$ but $Y(y)$ is bounded as $y \to \infty$ so $c_1 = 0$ and $Y \equiv 0$)

we find $\begin{cases} Y'' + \lambda^2 Y = 0 \\ X'' - \lambda^2 X = 0. \end{cases}$

So $Y(y) = c_1 \cos(\lambda y) + c_2 \sin(\lambda y)$, $Y(0) = 0$ gives $c_1 = 0$

$Y(y) = \sin(\lambda y)$ (can take $c_1 = 1$ at this step)

$X(x) = c_1 e^{-\lambda x} + c_2 e^{\lambda x}$ or

$X(x) = A \sinh(\lambda x) + B \cosh(\lambda x)$ ← this is a more convenient notation in this problem.

$X(a) = 0$ gives $A \sinh(\lambda a) + B \cosh(\lambda a) = 0$ and $A = -B \dfrac{\cosh(\lambda a)}{\sinh(\lambda a)}$

$$\text{so } X(x) = B\left(-\frac{\cosh(\lambda a)\sinh(\lambda x)}{\sinh(\lambda a)} + \cosh(\lambda x)\right)$$
$$= B\left(\frac{\cosh(\lambda x)\sinh(\lambda a) - \sinh(\lambda x)\cosh(\lambda a)}{\sinh(\lambda a)}\right)$$
$$= B\left(\frac{\sinh((a-x)\lambda)}{\sinh(\lambda a)}\right)$$

we have used the identity: $\sinh(a \pm b) = \sinh a \cosh b \pm \sinh b \cosh a$

The constant B can depend on λ.

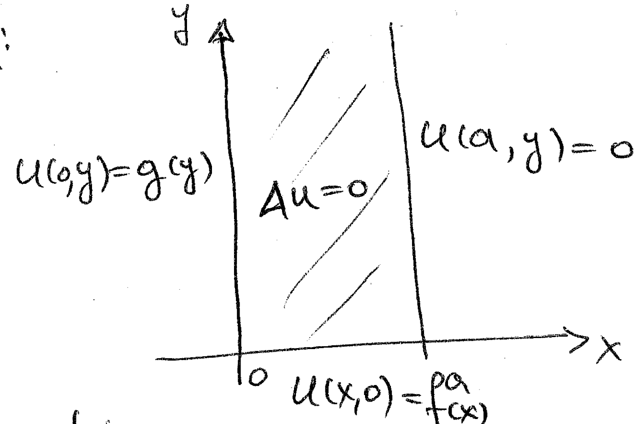
So

$$u(x,y)=\int_0^\infty B(\lambda)\,\frac{\sinh(a-x)\lambda}{\sinh(\lambda a)}\,\sin(\lambda y)\,d\lambda$$

From $u(0,y)=g(y)$ we find $u(0,y)=\int_0^\infty B(\lambda)\sin(\lambda y)\,d\lambda$

so $B(\lambda)=\frac{2}{\pi}\int_0^\infty g(y)\sin(\lambda y)\,dy$. (Review ch. 2.10 and 1.9).

Example 4:



the solution to this PDE is Example 2 + Example 3.

4.5. Potential in a disk

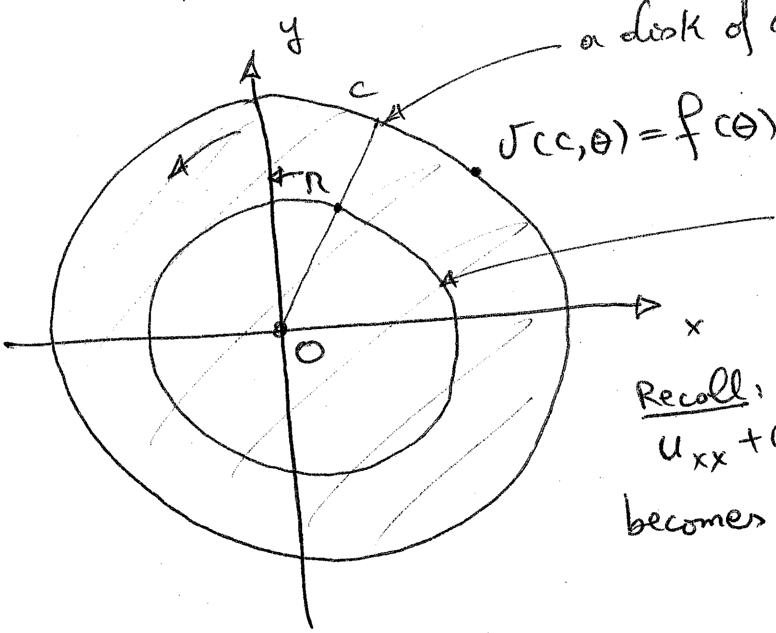
Example 5:

$$v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}=0,\quad 0\le r<c$$

$$v(c,\theta)=f(\theta),\quad -\pi<\theta<\pi$$

$$v(r,\theta+2\pi)=v(r,\theta),\quad 0<r<c$$

$v(r,\theta)$ is bounded as $r\to 0$.



Recall: $u_{xx}+u_{yy}=0$ in polar coordinates becomes $v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}=0$.

Set $v(r,\theta) = R(r)\Theta(\theta)$ so

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \text{gives}$$

$$\left(R'' + \frac{1}{r}R'\right)\Theta = -\frac{1}{r^2}R\Theta'' \quad \text{or} \quad \frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{\Theta''}{\Theta}$$

$$\text{or} \quad \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

(if we set $= -\lambda^2$ then Θ would be exponential so not periodic; we need Θ to be periodic)

$\Theta'' + \lambda^2\Theta = 0$, $\Theta(\theta + 2\pi) = \Theta(\theta)$

so $\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$ if this is periodic of period 2π then λ is an integer, so $\lambda = n$, $n = 0, 1, \ldots$

if $\lambda = 0$ we get $\Theta_0(\theta) = \text{constant}$ so we pick $\Theta_0(\theta) = 1$.

if $\lambda = n$, for $n = 1, 2, \ldots$ $\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$, $n = 1, 2, \ldots$

The equation for R is $r^2R'' + rR' - n^2R = 0$, $n = 0, 1, 2, \ldots$

There are 2 cases:

① if $n = 0$ then $r^2R'' + rR' = 0$ so $\frac{R''}{R'} = -\frac{1}{r}$ and so $R' = \frac{1}{r}$ or $R = \ln(r)$ (if $R' \neq 0$)

If $R' = 0$ then $R = \text{constant}$ so we take $R = 1$. Note that $R = \ln(r)$ does not work since $\lim_{r \to 0} R(r) = -\infty$.

$R_0(r) = 1$.

② $n \neq 0$. This is a Cauchy-Euler equation which cannot be solved by a characteristic equation. We know (and not prove) that the general solution is $R(r) = C_1 r^n + C_2 r^{-n}$. Now, since $R(r)$ is bounded as $r \to 0$ we must have $C_2 = 0$.

$R_n(r) = r^n$.

The fundamental solutions for this problem are

$$1, \quad r^n\cos(n\theta), \quad r^n\sin(n\theta)$$

or 1 and $A_n r^n\cos(n\theta) + B_n r^n(\sin(n\theta))$ but they are the same.

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} A_n r^n\cos(n\theta) + B_n r^n\sin(n\theta)$$

from the initial condition $v(c,\theta) = f(\theta)$ we find:

$$v(c,\theta) = a_0 + \sum_{n=1}^{\infty} A_n c^n \cos(n\theta) + B_n c^n \sin(n\theta) = f(\theta)$$

$$\text{so } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n c^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \quad \text{so} \quad A_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta$$

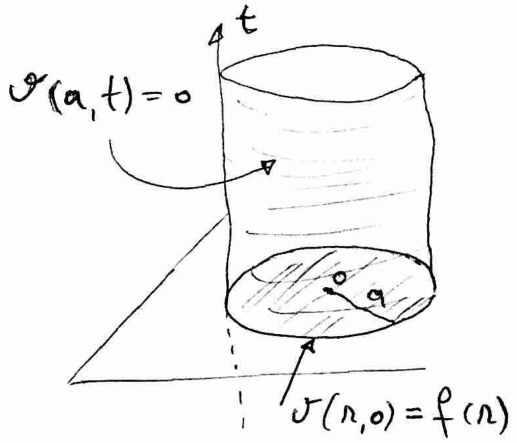
$$B_n c^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \quad \text{so} \quad B_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta.$$

using Fourier series.

5.5 - 5.7 Two dimensional heat and wave equations
Vibrations and heat of a circular membrane

Suppose the 2D heat eq. does not depend on the angular coordinate θ. Then the PDE to solve becomes

Example:



$$v_{rr} + \frac{1}{r} v_r = \frac{1}{K} v_t \quad , \; 0<r<a, \; t>0$$

$$v(a,t) = 0, \quad t>0$$

$$v(r,0) = f(r), \quad 0<r<0$$

Remark: $\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r}\right) = v_{rr} + \frac{1}{r} v_r$

Set $v(r,t) = \phi(r)\,T(t)$. Then $\phi'' T + \frac{1}{r}\phi' T = \frac{1}{K}\phi T'$ so

$$\frac{(r\phi')'}{r\phi} = \frac{T'}{KT} = -\lambda^2 \Rightarrow T' + \lambda^2 K T = 0, \text{ so } T(t) = e^{-K\lambda^2 t}$$

and

Bessel eq.

$$\begin{cases} (r\phi')' + \lambda^2 r \phi = 0, & 0<r<a \\ \phi(a) = 0 \\ \phi(r) \text{ bounded as } r \to 0^+ \end{cases}$$

We don't have two boundary conditions.

Sturm-Liouville parallel:

$$(r\phi')' + \lambda^2 r\phi = 0$$

$$\alpha_1 \phi(0) - \alpha_2 \phi'(0) = 0$$

$$\beta_1 \phi(a) + \beta_2 \phi'(a) = 0$$

$\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$

α_1, α_2 not both zero

β_1, β_2 not both zero

Bessel eq.

$$(r\phi')' - \frac{\mu^2}{r}\phi + \lambda^2 r \phi = 0 \quad , \; \phi \text{ a function of } r$$

Sol. $\phi(r) = A\, J_\mu(\lambda r) + B\, Y_\mu(\lambda r)$

J_μ: Bessel function of 1st kind

Y_μ: Bessel function of 2nd kind (unbounded as $r \to 0^+$)

$$J_\mu(\lambda r) = \left(\frac{\lambda r}{2}\right)^\mu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\mu+m+1)} \left(\frac{\lambda r}{2}\right)^{2m}$$

$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$ is Euler's Gamma function.

Γ is defined everywhere except negative integers

$\Gamma(\mu+1) = \mu\Gamma(\mu)$, $\Gamma(0) = 1$, $\Gamma(m+1) = m! = 1\cdot 2\cdot 3 \cdot \ldots \cdot m$

$0! = 1$ by convention

Special case $\mu = 0$:

the solution to $(r\phi')' + \lambda^2 r\phi = 0$, $0<r<a$ which is bounded as $r \to 0^+$ is $\phi(r) = A\,J_0(\lambda\mu)$,

$$J_0(\lambda\mu) = \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\lambda r}{2}\right)^{2m}$$

In our case we take $\phi(r) = J_0(\lambda\mu)$, $\phi(a) = 0$ so $J_0(\lambda a) = 0$

Let α_n be the n^{th} root of $J_0(x) = 0$ (they exist and are infinitely many by Rolle)

Orthogonality relation:

$$\int_0^a J_0(\lambda_n r) J_0(\lambda_m r)\, r\, dr = 0 \quad \text{for } n \neq m$$

$\lambda_n = \frac{\alpha_n}{a}$, $n = 1, 2 \ldots$

$\phi_n(r) = J_0(\lambda_n r)$, $n = 1, 2 \ldots$

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\lambda_n^2 k t}$$

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \Rightarrow c_n = \frac{\int_0^a f(r) J_0(\lambda r)\, r\, dr}{\int_0^a J_0^2(\lambda r)\, r\, dr}$$

Similar to Sturm - Liouville

Thm: If f is piecewise smooth on $0<r<a$ then

$$\sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = \frac{f(r+) + f(r-)}{2}$$

where λ_n are solutions to $J_0(\lambda a) = 0$

$$c_n = \frac{\int_0^a f(r) J_0(\lambda r)\, r\, dr}{\int_0^a J_0^2(\lambda r)\, r\, dr}$$

Example: Find the solution to the PDE:

$$u_{rr} + \frac{1}{r} u_r = \frac{1}{K} u_t \qquad 0 < r < a,\ t > 0$$

$$u(a,t) = 0 \quad , \ t > 0$$

$$u(r,0) = 1 \quad , \ 0 < r < a$$

We need to compute the coefficients c_n, so we have to evaluate

$$\int_0^a J_0(\lambda r)\, r\, dr \quad \text{and} \quad \int_0^a J_0^2(\lambda r)\, r\, dr.$$

Useful facts:

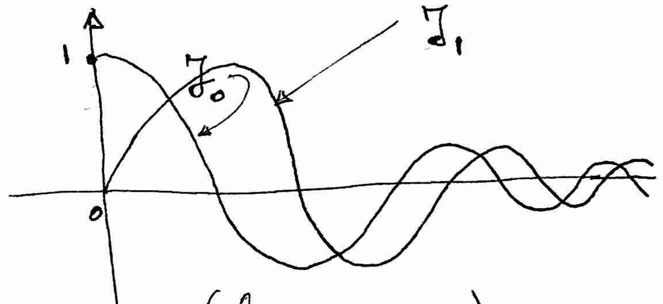
① $\frac{d}{dr} J_0(\lambda r) = -\lambda J_1(\lambda r)$

② $J_0(0) = 1, \quad J_1(0) = 0$

③ $\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x)$

or $\int x J_0(x)\, dx = x J_1(x)$

④ $\frac{d}{dx} J_0(x) = -J_1(x)$



(like sin, cos)

$$\Rightarrow \int_0^a J_0(\lambda_n r)\, r\, dr = \frac{1}{\lambda_n} r J_1(\lambda_n r)\Big|_0^a = \frac{a}{\lambda_n} J_1(\lambda_n a)$$

$$\int_0^a J_0^2(\lambda_n r)\, r\, dr = \frac{a^2}{2} J_1^2(\lambda_n a)$$

Fact: If $\phi(a) = 0$ then $\int_0^a \phi^2(r)\, r\, dr = \frac{1}{2\lambda^2}\left(a\phi'(a)\right)^2$

we applied this fact to $\phi(r) = J_0(\lambda r)$ to obtain the last integral.

$$c_n = \frac{\frac{a}{\lambda_n} J_1(\lambda_n a)}{\frac{a^2}{2} J_1^2(\lambda_n a)} = \frac{2}{\lambda_n a J_1(\lambda_n a)} = \frac{2}{\alpha_n J_1(\alpha_n)},$$

where α_n is the n^{th} root of $J_0(x) = 0$

So $$v(r,t) = \sum_{n=1}^{\infty} \frac{2}{\alpha_n J_1(\alpha_n)} J_0(\lambda r)\, e^{-\lambda_n^2 k t}, \quad \lambda_n = \frac{\alpha_n}{a}.$$

(α_n need to be computed numerically).

<u>Example</u>: Same as before, with $f(r) = \begin{cases} 1, & 0<r<\frac{a}{2} \\ 0, & \frac{a}{2}<r<a \end{cases}$

We have $$C_n = \frac{\int_0^a f(r) J_0(\lambda r)\, r\,dr}{\int_0^a J_0^2(\lambda r)\, r\,dr}$$ so we need to evaluate

$$\int_0^{\frac{a}{2}} J_0(\lambda_n r)\, r\,dr = \frac{1}{\lambda_n} r\, J_1(\lambda r)\Big|_0^{\frac{a}{2}} = \frac{a}{2\lambda_n} J_1\left(\lambda_n \frac{a}{2}\right)$$

and $$\int_0^a J_0^2(\lambda_n r)\, r\,dr = \frac{a^2}{2} J_1^2(\lambda_n a) \text{ as before.}$$

<u>Exercise</u>: $\phi(r) = J_0(\lambda r)$ so ϕ satisfies $(r\phi')' + \lambda^2 r\phi = 0$, $0<r<a$.
Let $0<b\le a$. Compute $\int_0^b r\phi^2\,dr$.

$$(r\phi')' + \lambda^2 r\phi = 0 \quad \Big/ \cdot 2r\phi'$$

$$\int_0^b 2(r\phi')'(r\phi')\,dr + \int_0^b \lambda^2 r^2 2\phi\phi'\,dr = 0$$

$$\Rightarrow (r\phi')^2\Big|_0^b + \lambda^2 \int_0^b r^2(\phi^2)'\,dr = (r\phi')^2\Big|_0^b + \lambda^2 r^2\phi^2\Big|_0^b - 2\lambda^2\int_0^b r\phi^2\,dr = 0$$

$$\Rightarrow \int_0^b r\phi^2\,dr = \frac{1}{2\lambda^2}\left((b\phi'(b))^2 + \lambda^2 b^2\phi^2(b)\right) = \frac{b^2(\phi'(b))^2}{2\lambda^2} + \frac{b^2\phi^2(b)}{2}$$

Special case $b=a$

Example (vibrations of a membrane, no angular coordinate)

$$v_{rr} + \frac{1}{r} v_r = \frac{1}{c^2} v_{tt} \quad 0<r<a,\ t>0$$

$$v(a,t) = 0$$
$$v(r,0) = f(r)$$
$$v_t(r,0) = g(r)$$

by setting $v(r,t) = \phi(r)\,T(t)$ as before we find

$$T_n(t) = a_n \cos(\lambda_n c t) + b_n \sin(\lambda_n c t)$$

$$\phi_n(r) = J_0(\lambda_n r), \quad \lambda_n = \frac{\alpha_n}{a}, \quad \alpha_n \text{ roots of } J_0(x)=0$$

$$v(r,t) = \sum_{n=1}^{\infty} J_0(\lambda_n r)\left(a_n \cos(\lambda_n c t) + b_n \sin(\lambda_n c t)\right)$$

$$v(r,0) = f(r) \implies a_n = \frac{\int_0^a f(r) J_0(\lambda_n r)\, r\, dr}{\int_0^a (J_0(\lambda_n r))^2 r\, dr}$$

$$v_t(r,0) = g(r) \implies b_n = \frac{1}{\lambda_n c} \frac{\int_0^a g(r) J_0(\lambda_n r)\, r\, dr}{\int_0^a (J_0(\lambda_n r))^2 r\, dr}.$$

5.4. Heat and wave equations in polar coordinates with angular coordinate

heat eq.

$$\begin{cases} \Delta v = \frac{1}{k} v_t & 0<r<a,\ t>0,\ -\pi<\theta<\pi \\ v(a,\theta,t) = f(\theta) \\ v(r,\theta,0) = g(r,\theta) \\ v(r,\theta,t) = v(r,\theta+2\pi,t) & \text{periodic in } \theta \end{cases}$$

wave eq.

$$\begin{cases} \Delta v = \frac{1}{c^2} v_{tt} & 0<r<a,\ t>0,\ -\pi<\theta<\pi \\ v(a,\theta,t) = f(\theta) \\ v(r,\theta,0) = g(r,\theta) \\ v_t(r,\theta,0) = h(r,\theta) \\ v(r,\theta,t) = v(r,\theta+2\pi,t) \end{cases}$$

If $f(\theta) = 0$ then the eq. are homogeneous, otherwise we have to find $v^{ss}(r,\theta)$ the steady-state solution (time independent sol.) and eliminate it from the eq. (solve a homogeneous PDE for $w = v - v^{ss}$)

Steady-state sol: $v(r,\theta,t) = v^{ss}(r,\theta)$ verifies

$$v^{ss}_{rr} + v^{ss}_{\theta\theta} = 0$$
$$v^{ss}(a,\theta) = f(\theta)$$

Suppose $f(\theta)=0$. We set up separation of variables

$$v(r,\theta,t) = \phi(r,\theta)\,T(t) = R(r)\,\Theta(\theta)\,T(t)$$

we find:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = -\lambda^2\phi$$
$$\phi(a,\theta) = 0$$
$$\phi(r,\theta+2\pi) = \phi(r,\theta)$$
$$\phi \text{ bounded as } r \to 0^+$$

Separation of variables gives $\dfrac{(rR')'}{rR} + \dfrac{\Theta''}{r^2\Theta} = -\lambda^2$, $\dfrac{\Theta''}{\Theta} = -\mu^2$

with $\begin{cases} R(a) = 0 \\ \Theta(\theta) = \Theta(\theta+2\pi) \\ R(r) \text{ bounded as } r \to 0^+ \end{cases}$

eq. in Θ: $\Theta'' + \mu^2\Theta = 0$, $\Theta(\theta+2\pi) = \Theta(\theta)$ gives $\Theta_0(\theta) = 1$, $\mu_0 = 0$

$\Theta_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta)$

$\mu_m = m$, $m = 1, 2, \dots$

eq. in R: (Bessel eq)

$$(rR')' - \frac{\mu^2}{r}R + \lambda^2 rR = 0, \quad 0 < r < a$$
$$R(a) = 0$$
$$R(r) \text{ bounded as } r \to 0$$

$R(r) = J_\mu(\lambda r)$, $J_\mu(\lambda a) = 0$, $\mu_m = m$

$R_{mn}(r) = J_m(\lambda_{mn} r)$ where $J_m(\lambda_{mn} a) = 0$

$\alpha_{mn} = n^{th}$ root of $J_m(x) = 0$, $\lambda_{mn} = \dfrac{\alpha_{mn}}{a}$.

eq. in T: $T' = -\lambda^2 k T$ so $T(t) = e^{-\lambda^2 k t}$ as before.

MAT 341 – Applied Real Analysis
Fall 2015

Final – December 11, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
6	
7	
TOTAL	

Problem 1: (18 points) Check TRUE or FALSE, no other explanation is necessary.

a) A periodic, continuous function can have two different Fourier series, but they both converge to $f(x)$.

TRUE | FALSE

b) Suppose the Fourier series of a function $f(x)$ converges uniformly for $0 < x < 1$. Then $f(x)$ cannot have jump discontinuities on the interval $(0, 1)$.

TRUE | FALSE

c) The general solution to the wave equation obtained by separation of variables is the same as the solution obtained by D'Alembert's method.

TRUE | FALSE

d) Suppose $\phi_n(x)$ and $\phi_m(x)$ are eigenvalues of a regular Sturm-Liouville problem on the interval $0 < x < 1$. Then $\int_0^1 \phi_n(x)\phi_m(x)\,dx = 0$ whenever $m \neq n$.

TRUE | FALSE

e) The functions $v_1(r,\theta) = r^{-n}\cos(n\theta)$ and $v_2(r,\theta) = r^n\cos(n\theta)$ are both solutions to the potential equation $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$.

TRUE | FALSE

f) If $w(x,y,t)$ is the solution of a two-dimensional wave problem with homogeneous boundary conditions, then $\lim_{t\to\infty} w(x,y,t) = 0$.

TRUE | FALSE

g) Suppose the solution of a certain two-dimensional heat problem on the square $0 < x < 1$, $0 < y < 1$ is given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin(m\pi x)\cos(n\pi y)\exp(-(m^2+n^2)\pi^2 kt).$$

If $u(x,0,0) = f(x)$, then a_{mn} are the coefficients of the Fourier sine series of $f(x)$.

TRUE | FALSE

h) The bounded solutions to the equation $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda^2 r\phi = 0$ are given by $\phi(r) = AJ_0(\lambda r)$, for some constant A.

TRUE | FALSE

i) Suppose $u(x,y) = \int_0^\infty B(\lambda)\frac{\sinh((a-x)\lambda)}{\sinh(\lambda a)}\sin(\lambda y)\,d\lambda$ is solution to a potential equation in the strip $0 < x < a$, $0 < y$. Then $B(\lambda) = \frac{2}{\pi}\int_0^\infty u(0,y)\sin(\lambda y)\,dy$.

TRUE | FALSE

Problem 2: (14 points) Consider the function

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 \leq x < 2. \end{cases}$$

Find the Fourier cosine series for f. Does the Fourier cosine series converge uniformly? Explain.

Problem 3: (12 points) The eigenvalues and eigenfunctions to the following problem

$$(e^x\phi')' + e^x\lambda^2\phi = 0, \quad 0 < x < 2$$
$$\phi(0) = 0 \quad \phi(2) = 0$$

are $\lambda_n = \dfrac{\sqrt{1+n^2\pi^2}}{2}$ and $\phi_n(x) = \exp\left(-\frac{x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)$.

a) Find the coefficients for the expansion of the function $f(x) = \exp\left(-\frac{x}{2}\right)$, $0 < x < 2$, in terms of the eigenfunctions ϕ_n.

b) To what values does the series converge at $x = 1$ and $x = 2$? Explain.

Problem 4: (12 points) Consider the two-dimensional heat problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad 0 < y < b, \quad t > 0$$

with boundary conditions

$$u(0, y, t) = \sin(5y), \qquad u(a, y, t) = 0, \qquad 0 < y < b, \quad t > 0$$

$$u(x, 0, t) = 0, \qquad u(x, b, t) = \cos(5x), \qquad 0 < x < a, \quad t > 0$$

and initial condition:

$$u(x, y, 0) = xy, \qquad 0 < x < a, \quad 0 < y < b.$$

a) State the initial value – boundary value problem satisfied by the steady-state solution $v(x, y)$. What is the PDE that $v(x, y)$ satisfies? You are **NOT** asked to solve it.

b) State the initial value – boundary value problem satisfied by the transient solution $w(x, y, t)$. You are **NOT** asked to solve it.

Problem 5: (16 points) Find the solution $u(x, y)$ of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the rectangle $0 < x < 2$, $0 < y < 3$, that satisfies the boundary conditions

$$u(0, y) = 0, \qquad u(2, y) = 0, \qquad 0 < y < 3,$$

$$\frac{\partial u}{\partial y}(x, 0) = 0, \qquad u(x, 3) = \sin\left(\frac{\pi x}{2}\right) - 17 \sin\left(\frac{5\pi x}{2}\right), \qquad 0 < x < 2.$$

Problem 6: (16 points) Consider the potential equation on a half disk:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \qquad 0 < \theta < \pi, \qquad 0 < r < 2$$

$$\frac{\partial v}{\partial \theta}(r, 0) = 0, \qquad \frac{\partial v}{\partial \theta}(r, \pi) = 0, \qquad 0 < r < 2$$

a) Set $v(r, \theta) = R(r)\Theta(\theta)$ to separate the variables and write down the associated eigenvalue problem for Θ. Write down a differential equation that is verified by $R(r)$.

b) Solve the eigenvalue problem for Θ and find the eigenfunctions $\Theta_n(\theta)$.

(Problem 6 continued)

c) Suppose the function $v(r,\theta)$ is bounded as $r \to 0^+$. Find the fundamental solutions $v_n(r,\theta) = R_n(r)\Theta_n(\theta)$.

d) Suppose the function $v(r,\theta)$ is bounded as $r \to 0^+$. Find the general solution to this PDE if the initial condition is

$$v(2,\theta) = 1 + \cos(2015\,\theta), \qquad 0 < \theta < \pi.$$

Problem 7: (12 points) Solve the following initial value – boundary value problem: :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0$$

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0$$

$$u(x,0) = 0, \qquad 0 < x < 1,$$

knowing that $\frac{\partial u}{\partial x}(0,t) = \sin\left(\frac{341\pi ct}{2}\right)$, $t > 0$. Can the solution be written as

$$u(x,t) = \phi(x+ct) - \phi(x-ct),$$

for some function ϕ?

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$

MAT 341 – Applied Real Analysis
Spring 2015

Final – May 18, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
6	
7	
TOTAL	

Problem 1: (12 points) Consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad -3 \le x < -1, \\ 1 & \text{if} \quad -1 \le x < 1, \\ 0 & \text{if} \quad 1 \le x < 3; \end{cases} \qquad f(x+6) = f(x).$$

a) Sketch the graph of f on the interval $[-7, 7]$.

b) Find the Fourier series for f. Explain why the series converges to 0.5 when $x = 7$.

Problem 2: (14 points) Suppose $u(x,t) = e^{-\lambda t} X(x)$ is a nontrivial solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad u(a,t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by $X(x)$. What initial conditions must $X(x)$ satisfy? Determine $X(x)$ and the possible values of λ.

Problem 3: (14 points) Consider the following eigenvalue problem

$$(e^{-x}\phi')' + e^{x}\gamma^2\phi = 0, \quad 0 < x < a$$
$$\phi(0) + \beta^2\phi'(0) = 0, \qquad \phi(a) + \beta^2\phi'(a) = 0$$

Check true or false, no other explanation is necessary.

a) This is a regular Sturm-Liouville problem for all values of the parameter β.

TRUE FALSE

b) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_2(x)\phi_4(x)\,dx = 0.$$

TRUE FALSE

c) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_2(x)\phi_4(x)e^{x}\,dx = 0.$$

TRUE FALSE

d) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_m(x)\phi_n(x)e^{x}\,dx = 0.$$

TRUE FALSE

e) If $\beta = 4$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_3(x)\phi_5(x)e^{x}\,dx = 0.$$

TRUE FALSE

f) $\gamma = 0$ is not an eigenvalue, regardless of the parameter β.

TRUE FALSE

g) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then $\sum_{n=1}^{\infty} c_n\phi_n(x) = e^{-x}$, for $0 < x < a$, where

$$c_n = \frac{\int_0^a \phi_n(x)\,dx}{\int_0^a \phi_n^2(x)e^{x}\,dx}.$$

TRUE FALSE

Problem 4: (20 points) Consider the *dispersive wave equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \gamma^2 u, \qquad 0 < x < a, \quad t > 0$$

subject to the following boundary conditions and initial conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(a,t) = 0, \qquad t > 0; \qquad\qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < a.$$

a) Set $u(x,t) = \phi(x)T(t)$ to separate the variables and find the associated equations for ϕ and T. Solve these equations and show that the solution $u(x,t)$ can be written as

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(t\sqrt{\frac{n^2\pi^2}{a^2} + \gamma^2}\right) \cos\left(\frac{n\pi}{a}x\right).$$

(Problem 4 continued)

b) Find the formula for the coefficients c_n using $f(x)$.

c) By using trigonometric identities, rewrite the solution as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} c_n \left[\cos\left(\frac{n\pi}{a}(x - b_n t)\right) + \cos\left(\frac{n\pi}{a}(x + b_n t)\right)\right]$$

Determine b_n, the speed of wave propagation. When is b_n independent of n?

Problem 5: (10 points) Consider the potential equation in a vertical strip $0 < x < a$, $2 < y$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \qquad 2 < y;$$
$$u(0,y) = 0, \qquad u(a,y) = 0, \qquad 2 < y;$$

We know that the *bounded* solutions to this equation are given by

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\frac{n\pi y}{a}\right).$$

Find the coefficients c_n if in addition $u(x,2) = 1$, $0 < x < a$. Is the solution periodic in y? What happens to $u(x,y)$ and $\frac{\partial u}{\partial x}(x,y)$ as $y \to \infty$?

Problem 6: (12 points) We know that the following functions $v(r, \theta)$:

$$1, \quad r^n \cos(n\theta), \quad r^{-n} \cos(n\theta), \quad r^n \sin(n\theta), \quad r^{-n} \sin(n\theta), \qquad (\text{where } n = 1, 2, \ldots)$$

are all solutions to the Laplace equation in polar coordinates: $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$. Let v_1 and v_2 be any two functions from the given list.

a) Any combination $c_1 v_1 + c_2 v_2$ is a solution of the Laplace equation.

Check true or false, no other explanation is necessary: TRUE FALSE

b) Consider the Laplace equation on the disk $0 \le r < 2$. Which of the listed functions would you try for a *bounded* solution $v = c_1 v_1 + c_2 v_2$? You are asked to list possible values for v_1 and v_2, not to find c_1 and c_2.

c) Find coefficients c_1 and c_2 such that $v = c_1 v_1 + c_2 v_2$ is a solution of the Laplace equation on the disk $0 \le r < 2$, subject to the boundary condition $v(2, \theta) = \cos(3\theta)$, $-\pi \le \theta < \pi$.

Problem 7: (18 points) Find the solution $u(x,y)$ of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the rectangle $0 < x < \pi$, $0 < y < 1$, that satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(\pi,y) = 0, \qquad 0 < y < 1,$$
$$u(x,0) = 0, \qquad u(x,1) = 1 + \cos(5x), \qquad 0 < x < \pi.$$

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

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MAT 341: Applied Real Analysis
Spring 2017
Schedule & Homework

Schedule

The PDF version of the schedule is available for print **here**.

Date	Topic	Section	Assignments	Due date
Jan 24	An introduction to Fourier series	1.1	**1.1:** 1abc, 2ad, 4, 7b, 8	**HW1** Due Jan 31
Jan 26	Determining Fourier coefficients; Examples	1.2	**1.2:** 1, 7c	
Jan 31	Even & odd extensions Convergence of Fourier series	1.2, 1.3	**1.2:**10b, 11b	**HW2** Due Feb 7
Feb 2	Uniform convergence of Fourier series	1.3, 1.4	**1.3:**1abd, 2ad, 6	
Feb 7	Fourier sine & cosine series Basic operations on Fourier series	1.4, 1.5	**1.4:** 1ae, 2, 3ab, 5bc **page 120:** 19, 20 [use a=3]	**HW3** Due Feb 14
Feb 9	*no class (snow storm)*			
Feb 14	Differentiation of Fourier series The heat equation	1.5, 2.1	**1.5:** 2, 5, 9 **2.1:** 2, 9	**HW4** Due Feb 23
Feb 16	The heat equation Steady-state & transient solutions	2.1, 2.2	**2.2:** 2, 6	
Feb 21	Fixed-end temperatures	2.3	**2.3:** 8 [use a=pi]	
Feb 23	Insulated bar; Examples Review	2.4	**2.3:** 6 **2.4:** 4 [use a=pi], 5, 8	**HW5** Due Mar 9
Feb 28	**Midterm 1** (2:30-3:50pm) Covers 1.1-1.5, 2.1-2.3 -- **Solutions** Practice exams: **Fall 2015** (**Solutions**) and **Spring 2015** (**Solutions**)			
Mar 2	Different boundary conditions	2.5	**2.5:** 4, 5 [use a=pi], 6	
Mar 7	Eigenvalues and eigenfunctions Convection	2.6, 2.7 **Notes**	**2.6:** 7, 9, 10	**HW6** Due Mar 23 **Problem 3c**
Mar 9	Sturm-Liouville problems	2.7	**2.7:** 1, 3abc, 7	

Mar 14	*no class (Spring break)*			
Mar 16	*no class (Spring break)*			
Mar 21	Series of eigenfunctions & examples Fourier integral & applications to PDEs	2.8, 1.9	**2.8:** 1 [use b=2] **1.9:** 1ab, 3a	**HW7** Due Mar 30
Mar 23	Semi-infinite rod The wave equation	2.10, 3.1	**2.10:** 3, 4	
Mar 28	The wave equation	3.2	**3.2:** 3, 4, 5, 7	**HW8** Due Apr 6 **Comments**
Mar 30	D'Alembert's solution; Examples	3.3, 3.4	**3.3:** 1, 2, 5	
Apr 4	The wave equation: generalizations Laplace's equation	3.4, 4.1	**page 255:** 18 **page 257:** 31	**HW9** Due Apr 20 **Comments**
Apr 6	Dirichlet's problem in a rectangle Examples & Review	4.2, 4.3	**4.1:** 2 **4.2:** 5 [use a=1, f(x)=sin(3pix)] **4.2:** 6	
Apr 11	**Midterm 2** (2:30-3:50pm) Covers 2.4-2.8, 2.10, 1.9, 3.1-3.4 -- **Solutions** Practice exams: **Fall 2015** (**Solutions**) and **Spring 2015** (**Solutions**) **Extra practice problems**			
Apr 13	Potential in a rectangle; Examples Potential in unbounded regions	4.3, 4.4	**4.3:** 2b **4.4:** 4a, 5ab	**HW10** Due Apr 27
Apr 18	Polar coordinates Potential in a disk	4.1, 4.5 **Notes**	**4.1:** 6 **4.5:** 1	
Apr 20	Dirichlet problem in a disk; Examples	4.5	**4.5:** 4	
Apr 25	Two-dimensional heat equation	5.3, 5.4 **Notes**	**5.3:** 1, 7c [use a=b=pi]	**HW11** Due May 4
Apr 27	Problems in polar coordinates Bessel's equation	5.5, 5.6	**5.4:** 5	
May 2	Temperature in a cylinder Applications: symmetric vibrations	5.6, 5.7	**5.6:** 3 [use a=1] **page 371:** 1	
May 4	Examples & Review	5.7		
May 15	**Final Exam** (11:15am-1:45pm) -- in class, **Melville Library E4315** The final is cumulative and covers: 1.1-1.5, 1.9, 2.1-2.8, 2.10, 3.1-3.4, 4.1-4.5, 5.3-5.6 Practice exams: **Fall 2015** and **Spring 2015**.			

Last updated January 2017