Remus Radu

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MAT 341: Applied Real Analysis
Fall 2015
Course Information

Synopsis

This course is an introduction to Fourier series and to their use in solving partial differential equations (PDEs). We will discuss in detail the three fundamental types of PDEs: the heat equation, the wave equation and Laplace's equation. These equations are important in many applications from various fields (mathematics, physics, engineering, economics, etc.) and illustrate important properties of PDEs in general.

Click here to download a copy of the course syllabus. Please visit the **course website on Blackboard** to see your grades and the solutions to midterms & exams.

Lectures

Tuesdays & Thursdays 10-11:20pm in Melville Library W4525

Instructor

Remus Radu
Office hours: Wednesday 12:00-1:00pm & Thursday 11:30am-12:30pm in Math Tower 4-103;
Tuesday 11:30am-12:30pm in MLC, or by appointment

Teaching Assistant

Lilya Lyubich
Office: Math Tower 3-110
Office hours: Wednesday 1:00-2:00pm & Thursday 11:30am-12:30pm in MLC;
Wednesday 2:00-3:00pm in Math Tower 3-110

Textbook

David Powers, *Boundary Value Problems and Partial Differential Equations*, 6th ed., Elsevier (Academic Press), 2010.

Grading Policy

Grades will be computed using the following scheme:

- Homework – 20%
- Midterm 1 – 20%
- Midterm 2 – 20%
- Final – 40%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading.

Exams

There will be two midterms and a final exam, scheduled as follows:

- Midterm 1 – Thursday, October 1, 10:00-11:20am, in Library W4525.
- Midterm 2 – Thursday, November 5, 10:00-11:20am, in Library W4525.
- Final Exam – Friday, December 11, 11:15am-1:45pm, room TBA.

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About me

From 2013 to 2017 I was a Milnor Lecturer at the **Institute for Mathematical Sciences** at Stony Brook University. I got my Ph.D. in Mathematics from **Cornell University** in 2013, under the supervision of **John H. Hubbard**.

I started my undergraduate studies at the **University of Bucharest** and after one year I transfered to **Jacobs University Bremen**, where I earned my B.S. degree in Mathematics in 2007. I got a M.S. in Computer Science from Cornell University in 2012.

Research Interests

My interests are in the areas of Dynamical Systems (in one or several complex variables), Analysis, Topology and the interplay between these fields.

My research is focused on the study of complex Hénon maps, which are a special class of polynomial automorphisms of $\mathbb{C}^2$ with chaotic behavior. I am interested in understanding the global topology of the Julia sets J, J^- and J^+ of a complex Hénon map and the dynamics of maps with partially hyperbolic behavior such as holomorphic germs of diffeomorphisms of $(\mathbb{C}^n, 0)$ with semi-neutral fixed points. Some specific topics that I work on include: relative stability of semi-parabolic Hénon maps and connectivity of the Julia set J, regularity properties of the boundary of a Siegel disk of a semi-Siegel Hénon map, local structure of non-linearizable germs of diffeomorphisms of $(\mathbb{C}^n, 0)$.

Other activities

I was organizer for the **Dynamics Seminar** at Stony Brook University.
I have also developed projects for MEC (Math Explorer's Club): **Mathematics of Web Search** and **Billiards & Puzzles**.

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MAT 341: Applied Real Analysis
Fall 2015
Schedule & Homework

Schedule

The PDF version of the schedule is available for print **here**.

Date	Topic	Section	Assignments	Due date
Aug 25	Periodic functions and Fourier series	1.1	**1.1:** 1abc, 2ad, 4, 7b, 8	**HW1** Due Sept 3
Aug 27	Determining Fourier coefficients; Examples	1.2	**1.2:** 1, 7c	
Sept 1	Even & odd extensions; Examples Convergence of Fourier series	1.2, 1.3	**1.2:**10b, 11b	**HW2** Due Sept 10
Sept 3	Uniform convergence of Fourier series Gibbs phenomenon	1.3, 1.4	**1.3:**1abd, 2ad, 5	
Sept 8	*no class (Labor day)*			
Sept 10	Fourier sine & cosine series Basic operations on Fourier series	1.4, 1.5	**1.4:** 1ae, 2, 3bc, 5ab **page 120:** 19, 20	**HW3** Due Sept 17
Sept 15	Differentiation of Fourier series The heat equation	1.5, 2.1	**1.5:** 2, 5, 9 **2.1:** 2, 9	**HW4** Due Sept 24
Sept 17	Steady-state solutions Transient solutions	2.2, 2.3	**2.2:** 2, 6 **2.3:** 6	
Sept 22	Fixed-end temperatures	2.3, 2.4	**2.3:** 2, 8 [use a=pi]	**HW5** Due Oct 8
Sept 24	Insulated bar; Examples	2.4, 2.5	**2.4:** 4 [use a=pi], 5, 8	
Sept 29	Different boundary conditions Review	2.5, 2.6	**2.5:** 4, 5, 6	
Oct 1	**Midterm 1** (10:00-11:20am) Covers 1.1-1.5, 2.1-2.3 -- **Solutions** **Practice Midterms:** **Midterm SP2015** with **Solutions SP2015** **Midterm & Solutions FA2008**			
Oct 6	Convection Eigenvalues and eigenfunctions	2.6, 2.7	**2.6:** 7, 9, 10	**HW6** Due Oct 15

Oct 8	Sturm-Liouville problems Relation to Fourier series	2.7, 2.8	**2.7:** 1, 3bc, 7	**Graphs**
Oct 13	Series of eigenfunctions & examples Fourier integral	2.8, 1.9	**2.8:** 1 [use b=2] **1.9:** 1ab, 3a	**HW7** Due Oct 22
Oct 15	Fourier integral & applications to PDEs Semi-infinite rod	2.10	**2.10:** 3, 4	
Oct 20	The wave equation	3.1, 3.2	**3.2:** 3, 4, 5, 7	**HW8** Due Oct 29
Oct 22	The wave equation; Examples Solution to the vibrating-string problem	3.2	**page 255:** 18 **page 257:** 31	
Oct 27	D'Alembert's solution; Examples	3.3, 3.4	**3.3:** 1, 2, 5	**HW9** Due Nov 12 **Comments**
Oct 29	Laplace's equation Dirichlet's problem in a rectangle	4.1, 4.2	**4.1:** 2	
Nov 3	Dirichlet's problem in a rectangle Examples & Review	4.2, 4.3	**4.2:** 5 [use a=1, f(x)=sin(3pix)] **4.2:** 6	
Nov 5	**Midterm 2** (10:00-11:20am) Covers 2.4-2.8, 2.10, 1.9, 3.1-3.2 -- **Solutions** **Practice Midterms:** **Midterm SP2015** with **Solutions SP2015** **Extra practice problems**			
Nov 10	Potential in a rectangle; Examples Potential in unbounded regions	4.3, 4.4	**4.3:** 2b **4.4:** 4a, 5ab	**HW10** Due Nov 19
Nov 12	Polar coordinates Potential in a disk **Lecture notes**	4.1, 4.5	**4.1:** 6 **4.5:** 1	
Nov 17	Dirichlet problem in a disk; Examples	4.5	**4.5:** 4	**HW11** Due Dec 3
Nov 19	Two-dimensional heat equation	5.3, 5.4	**5.3:** 1, 7c [use a=b=pi]	
Nov 24	Problems in polar coordinates Bessel's equation	5.5, 5.6	**5.4:** 5	
Nov 26	*no class (Thanksgiving)*			
Dec 1	Temperature in a cylinder Applications: vibrations	5.6, 5.7	**5.6:** 3 [use a=1]	
Dec 3	Symmetric vibrations Examples & Review	5.7	**5.6:** 7 **5.7:** 2 **page 371:** 1, 2, 6	Practice problems
Dec 11	**Final Exam** (11:15am-1:45pm) -- in class, **Melville Library W4525** The final is cumulative and it covers: 1.1-1.5, 1.9, 2.1-2.8, 2.10, 3.1-3.4, 4.1-4.5, 5.3-5.7 **Practice Final FA2009** (do only problems 2, 5, 6, 8, 10) with **Solutions** **Practice Final SP2015**			

MAT 341: APPLIED REAL ANALYSIS – FALL 2015
GENERAL INFORMATION

Instructor. Remus Radu
Email: rradu@math.stonybrook.edu
Office: Math Tower 4-103, Phone: (631) 63**2-8266**
Office Hours: W 12:00-1:00pm & Th 11:30am-12:30pm in Math Tower 4-103,
Tu 11:30am-12:30pm in MLC, or by appointment

Teaching Assistant. Lilya Lyubich
Email: lilya@math.stonybrook.edu
Office Hours: W 1:00-2:00pm & Th 11:30am-12:30pm in MLC,
W 2:00-3:00 in Math Tower 3-110

Lectures. TuTh 10:00-11:20am in Library W4525.

Blackboard. Grades & course administration will take place on Blackboard. A detailed weekly schedule of the lectures and homework assignments and solutions will be posted on Blackboard. Please login using your NetID at http://blackboard.stonybrook.edu.

Course Description. This course is an introduction to Fourier series and to their use in solving partial differential equations (PDEs). We will discuss in detail the three fundamental types of PDEs: the heat equation, the wave equation and Laplace's equation. These equations are important in many applications from various fields (mathematics, physics, engineering, economics, etc.) and illustrate important properties of PDEs in general.

Prerequisites. C or higher in the following: MAT 203 or 205 or 307 or AMS 261; MAT 303 or 305 or AMS 361. Advisory Prerequisite: MAT 200. It is important to be familiar with the basic techniques in ordinary differential equations.

Textbook. The following textbook is required:

> David Powers, *Boundary Value Problems and Partial Differential Equations*, 6th ed., Elsevier (Academic Press), 2010.

Exams. There will be two midterms and a final exam, scheduled as follows:

- Midterm 1 – Thursday, October 1, 10:00-11:20am, in Library W4525.
- Midterm 2 – Thursday, November 5, 10:00-11:20am, in Library W4525.
- Final Exam – Friday, December 11, 11:15am-1:45pm, TBA.

There will be no make-up exams.

Grading policy. Grades will be computed using the following scheme:

Homework	20%
Midterm 1	20%
Midterm 2	20%
Final Exam	40%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading. It is generally useful to read the corresponding section in the book before the lecture. There will be weekly homework assignments; the lowest homework score will be dropped. You may work together on your problem sets, and you are encouraged to do so. However, all solutions must be written up independently.

Extra Help. You are welcome to attend the office hours and ask questions about the lectures and about the homework assignments. In addition, math tutors are available at the MLC: http://www.math.sunysb.edu/MLC.

Information for students with disabilities. If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631) 632-6748, or at the following website http://studentaffairs.stonybrook.edu/dss/index.shtml. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential.

Academic integrity. Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty is required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary.

Critical Incident Management. Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of University Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures. Further information about most academic matters can be found in the Undergraduate Bulletin, the Undergraduate Class Schedule, and the Faculty-Employee Handbook.

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Oct 6	Convection Eigenvalues and eigenfunctions	2.6, 2.7	**2.6:** 7, 9, 10	**HW6** Due Oct 15 Graphs
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Nov 5	**Midterm 2** (10:00-11:20am) Covers 2.4-2.8, 2.10, 1.9, 3.1-3.2 -- Solutions Midterm SP2015 with Solutions SP2015 Extra practice problems			
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Dec 11	**Final Exam** (11:15am-1:45pm) -- in class, **Melville Library W4525** The final is cumulative and it covers: 1.1-1.5, 1.9, 2.1-2.8, 2.10, 3.1-3.4, 4.1-4.5, 5.3-5.7 Practice Final FA2009 (do only problems 2, 5, 6, 8, 10) with Solutions Final SP2015			

MAT 341 – Applied Real Analysis
Fall 2015

Midterm 1 – October 1, 2015

Solutions

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
TOTAL	

Problem 1: (25 points) Consider the function

$$f(x) = \begin{cases} -x & \text{if } -2 \le x < 0 \\ x & \text{if } 0 \le x < 2, \end{cases} \qquad f(x+4) = f(x).$$

Find the Fourier series for f. Determine whether the series converges uniformly or not. To what value does the Fourier series converge at $x = 2015$?

SOLUTION. Notice that f is even, so we can use the half-formulas when computing the Fourier coefficients. The Fourier series is just a cosine series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right).$$

We have $a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = 1$ and $a_n = \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)\,dx = 4\frac{(-1)^n - 1}{n^2\pi^2}$ (using the formula at the end of the booklet). The Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} 4\frac{(-1)^n - 1}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right).$$

The function is continuous, with piecewise continuous derivative, so the Fourier series converges uniformly everywhere. This can be seen also from the coefficients as

$$\sum_{n=1}^{\infty} |a_n| = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{|(-1)^n - 1|}{n^2} \le \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges. At $x = 2015$, the Fourier series converges to $f(2015) = f(2016-1) = f(-1) = 1$. We have used the fact that f is periodic of period 4. $\square$

Problem 2: (25 points) Suppose that the Fourier series of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} e^{-341n} \cos(n\pi x)$.

a) What is the Fourier series of $1 - 2f(x)$?

SOLUTION.

$$1 - 2f(x) = 1 - 2\sum_{n=1}^{\infty} e^{-341n} \cos(n\pi x)$$

$\square$

b) What is the Fourier series of $F(x) = \int_0^x f(y)\, dy$?

SOLUTION.

$$F(x) = \int_0^x \sum_{n=1}^{\infty} e^{-341n} \cos(n\pi y)\ dy = \sum_{n=1}^{\infty} \frac{e^{-341n}}{n\pi} \sin(n\pi x)$$

$\square$

c) Find the Fourier series of $f''(x)$ if it exists. Otherwise, explain why it does not exist.

SOLUTION.

$$f''(x) = -\sum_{n=1}^{\infty} n^2\pi^2 e^{-341n} \cos(n\pi x).$$

This series converges uniformly because $\sum_{n=1}^{\infty} |n^2 a_n| = \pi^2 \sum_{n=1}^{\infty} \frac{n^2}{e^{341n}} < \infty$ (which converges by the integral test). Notice also that the denominator is a polynomial, while the nominator is an exponential, hence the series converges. $\square$

d) What is the period of f? Can f have jump discontinuities or is it a continuous function?

SOLUTION. The Fourier series is periodic of period 2, hence f is periodic of period 2. The function is continuous (by part c) we already know that f is twice differentiable, hence f is continuous). $\square$

Problem 3: (25 points) Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - S\frac{\partial u}{\partial x} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0$$

with boundary conditions

$$u(0,t) = T_0, \quad u(2,t) = 0, \qquad t > 0$$

and initial condition $u(x,0) = f(x)$, $0 \le x \le 2$. (S and T_0 are positive constants.)

a) Find the steady-state solution $v(x)$. What is the ODE that $v(x)$ satisfies?

SOLUTION. The steady-state solution verifies the equation $v''(x) - Sv'(x) = 0$, with boundary conditions $v(0) = T_0$ and $v(2) = 0$. The characteristic equation is $r^2 - Sr = 0$ and has roots $r = S$ and $r = 0$. The solution is $v(x) = C_1 + C_2 e^{Sx}$. We find the coefficients from the boundary conditions. We have $C_1 + C_2 = T_0$ and $C_1 + C_2 e^{2S} = 0$. Hence $C_1 = \frac{T_0 e^{2S}}{e^{2S}-1}$ and $C_2 = -\frac{T_0}{e^{2S}-1}$ and

$$v(x) = \frac{T_0 e^{2S}}{e^{2S} - 1} - \frac{T_0 e^{Sx}}{e^{2S} - 1}.$$

□

b) State the initial value–boundary value problem satisfied by the transient solution $w(x,t)$. You are NOT asked to solve this problem.

SOLUTION. By definition $w(x,t) = u(x,t) - v(x)$. Using the equations for u from the hypothesis and for v from part a) we find

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < 2, \quad t > 0;$$
$$w(0,t) = 0, \quad w(2,t) = 0, \qquad t > 0;$$
$$w(x,0) = f(x) - v(x), \quad 0 \le x \le 2.$$

where $v(x)$ is the steady-state solution from part a). □

Problem 4: (25 points) Solve the heat problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 1, \quad t > 0;$$
$$u(0,t) = 0, \quad u(1,t) = \beta, \quad t > 0;$$
$$u(x,0) = \beta x + \sin\left(\frac{\pi x}{2}\right), \quad 0 \le x \le 1.$$

SOLUTION. We first find the steady-state solution $v(x) = \beta x$. As shown in the lecture, the transient solution $w(x,t)$ verifies the PDE

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{4}\frac{\partial w}{\partial t}, \qquad 0 < x < 1, \quad t > 0;$$
$$w(0,t) = 0, \quad w(1,t) = 0, \quad t > 0;$$
$$w(x,0) = \sin\left(\frac{\pi x}{2}\right), \quad 0 \le x \le 1.$$

and the solution of this PDE is given by

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-4n^2\pi^2 t},$$

where

$$c_n = 2\int_0^1 \sin(n\pi x)\sin\left(\frac{\pi x}{2}\right)\, dx.$$

Note that these are not orthogonal functions! These functions have different periods: $\sin(n\pi x)$ has period 2, while $\sin\left(\frac{\pi x}{2}\right)$ has period 4. We compute the integral, using the formulas at the end of the booklet and find $c_n = \frac{(-1)^n 8n}{\pi(1-4n^2)}$. The solution to the given PDE is

$$u(x,t) = \beta x + \sum_{n=1}^{\infty} \frac{(-1)^n 8n}{\pi(1-4n^2)} \sin(n\pi x) e^{-4n^2\pi^2 t}.$$

□

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

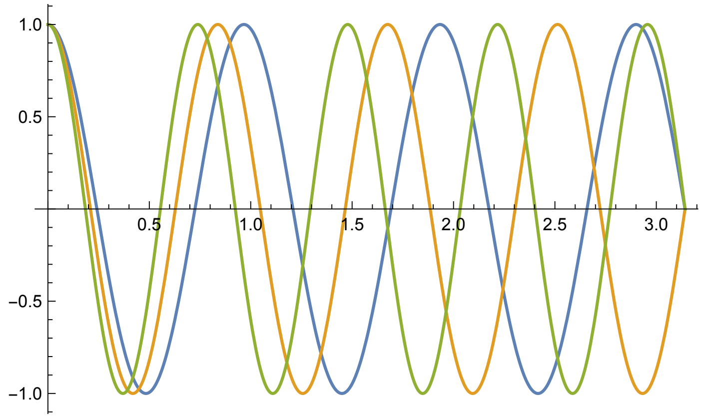
$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\sin^2(a) = \frac{1-\cos(2a)}{2} \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

```
(* Problem 2.7:3c *)
(* we can take any value for a,
say a=π. We have plotted the graphs of the 7th, 8th, and 9th eigenfunction.*)
a = π;
n = 7;
```

$$\text{Plot}\left[\left\{\text{Cos}\left[\frac{(2*n-1)\,\pi*x}{2*a}\right],\ \text{Cos}\left[\frac{(2*(n+1)-1)\,\pi*x}{2*a}\right],\ \text{Cos}\left[\frac{(2*(n+2)-1)\,\pi*x}{2*a}\right]\right\},\ \{x, 0, a\}\right]$$

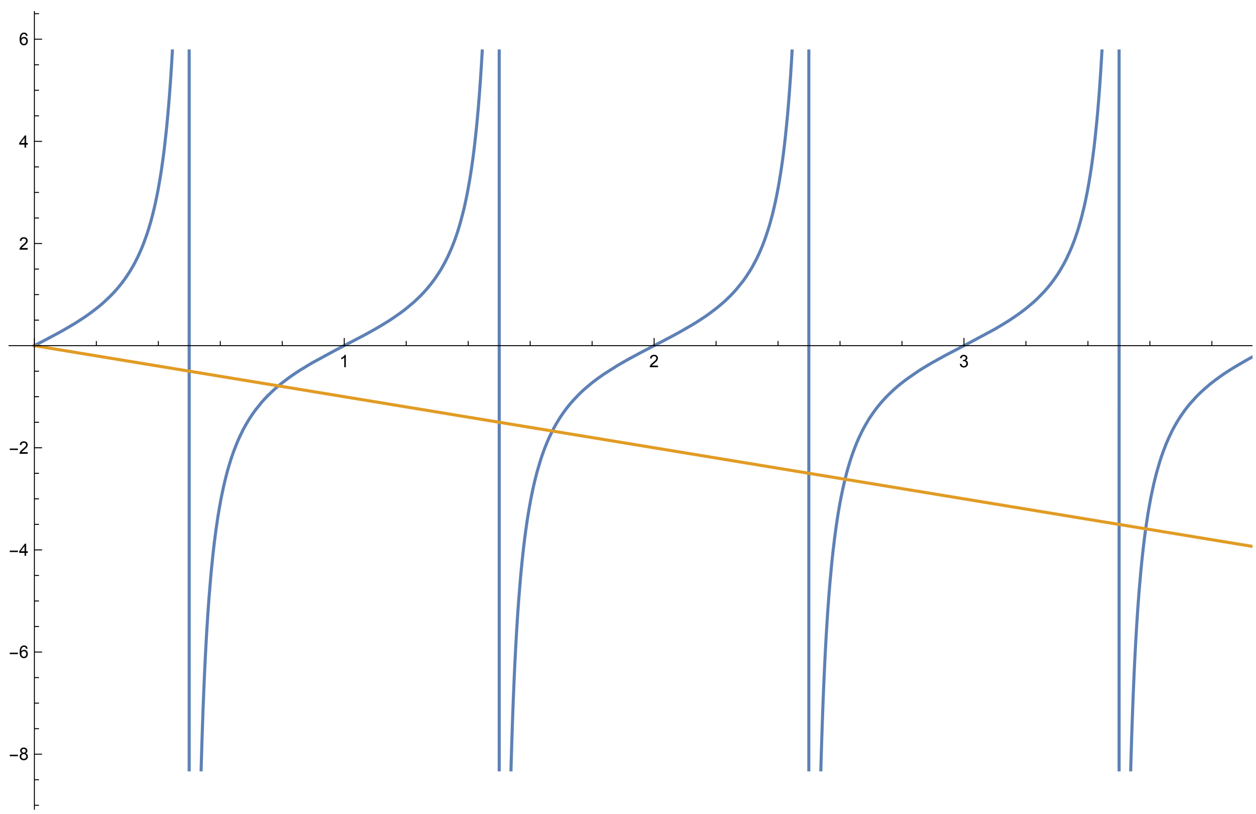
Out[57]=




```
(* Problem 2.7:3c *)
```

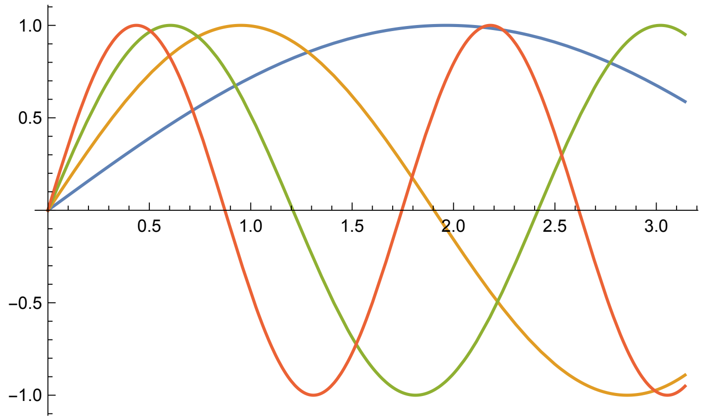
In[58]:= `Plot[{Tan[λ * a], -λ}, {λ, 0, 4}]`

Out[58]=



```
(* we observe that λ1≃.8, λ2≃.1.65, λ3≃.2.6, λ3≃.3.6,
etc. and plot the first few graphs of the eigenfunctions*)
Plot[{Sin[.8*x], Sin[1.65*x], Sin[2.6*x], Sin[3.6*x]}, {x, 0, a}]
```

Out[62]=



MAT 341: Applied Real Analysis – Fall 2015

HW9 – Comments

Sec. 3.3 – Problem 1: The problem is asking you to find some values of $u(x,t)$ such that

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \qquad t > 0;\\
&u(0,t) = 0, \quad u(a,t) = 0, \qquad t > 0;\\
&u(x,0) = f(x), \qquad t > 0;\\
&\frac{\partial u}{\partial x}(x,0) = 0, \quad 0 < x < a.
\end{aligned}$$

where $f(x)$ has the following equation:

$$f(x) = \begin{cases} \dfrac{2h}{a}x & \text{if} \quad 0 \le x \le \frac{a}{2} \\ -\dfrac{2h}{a}x + 2h & \text{if} \quad \frac{a}{2} < x \le a. \end{cases}$$

You then need to write a table with the values $u(x,t)$ at the required times, such as $u(0.25a, 0.2a/c)$. The solution $u(x,t)$ is written in Equation 13, but without the function G_e. **Note:** In the textbook, $\overline{f}_o$ means an odd periodic extension of f, while $\overline{G}_e$ means an even periodic extension of G.

Sec. 3.3 – Problem 2: You fix time $t = 0, 0.2a/c, 0.4a/c, 0.8a/c, 1.4a/c$ and you sketch 5 graphs of $u(x,t)$. For example, you need to sketch the graph of $u(x, 0.4a/c)$ as a function of x. You may assume $a = 1$ if it helps. The graphs should look like Figure 3 from Section 3.2.

Sec. 3.3 – Problem 5: The solution $u(x,t)$ verifies the PDE:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \qquad t > 0;\\
&u(0,t) = 0, \quad u(a,t) = 0, \qquad t > 0;\\
&u(x,0) = 0, \qquad 0 < x < a;\\
&\frac{\partial u}{\partial t}(x,0) = \alpha c, \quad 0 < x < a.
\end{aligned}$$

where α is just a constant, unrelated to a.

Sec. 4.1 – Problem 2: The sketch of the surfaces should look like the graphs below.

Regarding the boundary conditions: you have to evaluate $u(x,y)$, $\frac{\partial u}{\partial x}(x,y)$ and $\frac{\partial u}{\partial y}(x,y)$ at the given values. For example, if $u(x,y) = xy$ then $u(0,b) = 0$ and $u_x(0,b) = b$, $u_y(0,b) = 0$.

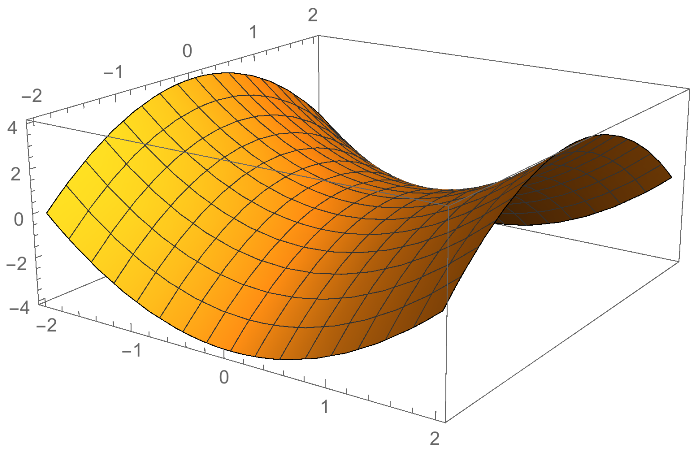


Figure 1: A sketch of the surface $z = x^2 - y^2$.

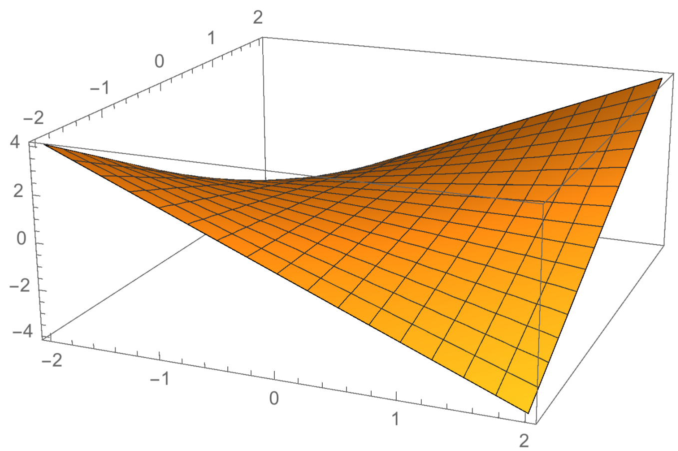


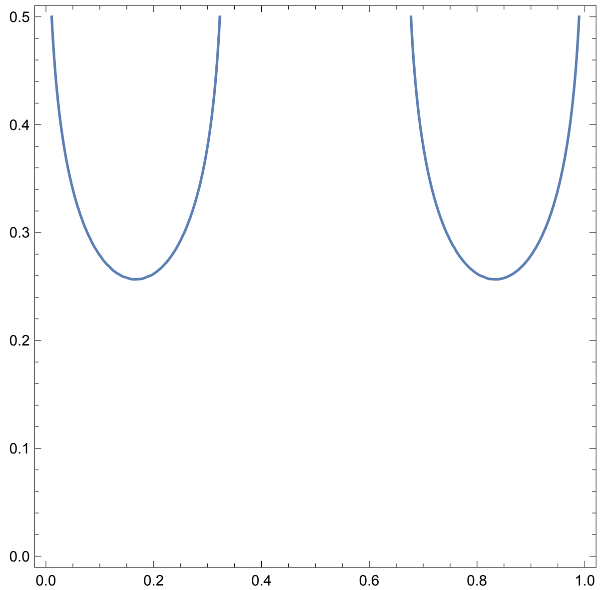
Figure 2: A sketch of the surface $z = xy$.

Sec. 4.2 – Problem 5: You are asked to solve the following PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 1, \quad 0 < y < b;$$
$$u(0, y) = 0, \quad u(1, y) = 0, \qquad 0 < y < b;$$
$$u(x, 0) = 0, \quad u(x, b) = \sin(3\pi x), \qquad 0 < x < 1;$$

You may assume that b is any constant. However, once you reach a formula for $u(x, y)$ as in Equation 9 (page 266) there is no need to compute the coefficients, simply use the fact that you already have $\sin(3\pi x)$ as a Fourier series and look for the coefficient of $n = 3$ (the rest are all zeros). To sketch the level curves, one has to do as in Figure 2, page 268 (see next page).

```
(*Problem 4.2:5 Sketch of level curve. *)
a = 1;
b = 0.5;
const = 0.1;
ContourPlot[Sinh[3*π*y]/Sinh[3*π*b] Sin[3 π*x] == const, {x, 0, a}, {y, 0, b}]
```




```
Plot3D[Sinh[3*π*y]/Sinh[3*π*b] Sin[3 π*x], {x, 0, a}, {y, 0, b}]
```

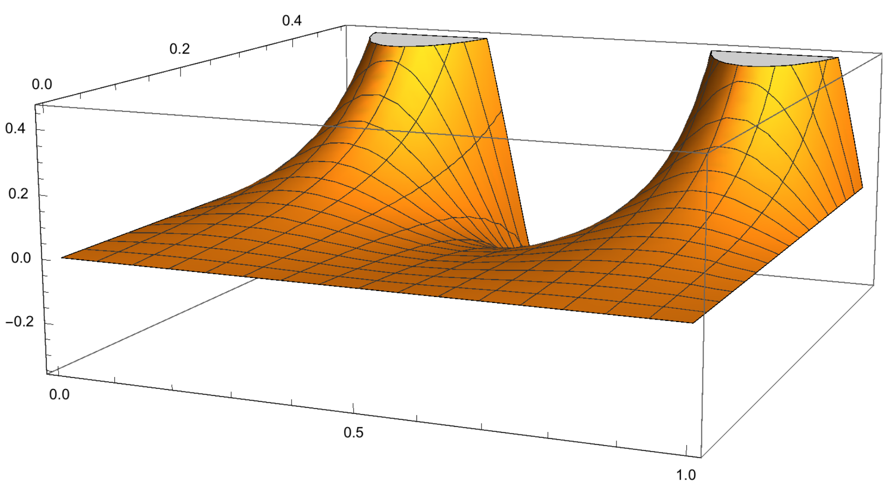


Figure 3: TOP: Level curves $u(x, y) = const$ drawn in Mathematica. BOTTOM: The surface $z = u(x, y)$. The level curves are obtained by cutting the level surface by a plane transversly.

Sec. 4.2 – Problem 6: You are asked to solve the following PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b;$$

$$u(0, y) = 0, \quad u(a, y) = 1, \qquad 0 < y < b;$$

$$u(x, 0) = 0, \quad u(x, b) = 0, \qquad 0 < x < a;$$

MAT 341 – Applied Real Analysis
Fall 2015

Midterm 2 – November 5, 2015

Solutions

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator. You **are** allowed to bring a note card to the exam (8.5 x 5.5in - front and back), but no other notes are allowed.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

Problem 1: (12 points) The *telegraph equation* governs the flow of voltage, or current, in a transmission line and has the form:

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + ku = a^2\frac{\partial^2 u}{\partial x^2} + F(x,t), \qquad 0 < x < 100, \quad t > 0.$$

The coefficients c, k, a are constants related to electrical parameters in the line. Assuming that $F(x,t) = 0$ and $u(x,t) = \phi(x)T(t)$, carry out a separation of variables and find the eigenvalue problem for ϕ. Take the boundary conditions to be

$$\frac{\partial u}{\partial x}(0,t) = 0 \text{ and } u(100,t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by $T(t)$.

Solution. If we substitute $u(x,t) = \phi(x)T(t)$ we get $\phi T'' + c\phi T' + k\phi T = a^2\phi'' T$. Separation of variables gives

$$\frac{T'' + cT' + kT}{T} = a^2\frac{\phi''}{\phi} = \lambda, \qquad \text{where } \lambda \text{ is some real number.}$$

We get $a^2\phi'' - \lambda\phi = 0$ and $T'' + cT' + (k - \lambda)T = 0$, which is an ODE satisfied by T. The first boundary condition gives $\frac{\partial u}{\partial x}(0,t) = \phi'(0)T(t) = 0$ so $\phi'(0) = 0$. The second boundary condition gives $u(100,t) = \phi(100)T(t) = 0$, so $\phi(100) = 0$. □

Problem 2: (20 points) Solve the heat problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(2,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < 2, \qquad \text{where} \quad f(x) = \begin{cases} T_0 & \text{if} \quad 0 < x < 1 \\ T_1 & \text{if} \quad 1 \le x < 2 \end{cases}$$

Solution. We identify $a = 2$ and $k = 4$. The general solution to this equation is

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t}.$$

The coefficients can be found from the initial condition $u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) = f(x)$. We have $c_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{T_0 + T_1}{2}$ and

$$\begin{aligned} c_n &= \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx = \int_0^1 T_0 \cos\left(\frac{n\pi x}{2}\right)\,dx + \int_1^2 T_1 \cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{2T_0}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{2T_1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 \\ &= \frac{2(T_0 - T_1)}{n\pi}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

The solution is

$$u(x,t) = \frac{T_0 + T_1}{2} + 2(T_0 - T_1)\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t}.$$

□

Problem 3:

a) (12 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2 \phi = 0, \quad 0 < x < 1$$

$$\phi(0) = 0, \qquad \phi'(1) - \phi(1) = 0$$

Is $\lambda = 0$ an eigenvalue?

SOLUTION. If $\lambda = 0$ then $\phi'' = 0$ so $\phi(x) = Ax + B$. From $\phi(0) = 0$ we immediately find $B = 0$. However the relation $\phi'(1) - \phi(1) = 0$ does not give other information about A. We find $\phi(x) = Ax$ for $A \neq 0$. So $\lambda = 0$ is an eigenvalue.

If $\lambda \neq 0$ then $\phi(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$. The condition $\phi(0) = 0$ gives $C_1 = 0$. We can take $C_2 = 1$ at this step and write $\phi(x) = \sin(\lambda x)$. The condition $\phi'(1) - \phi(1) = 0$ gives $\lambda = \tan(\lambda)$. The eigenvalues are λ_n, the n^{th} root of the equation $\lambda = \tan(\lambda)$, for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are $\phi_n(x) = \sin(\lambda_n x)$. □

(Problem 3 continued)

b) (5 points) Consider the function

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 < x < 0.5 \\ 1 - x & \text{if} \quad 0.5 \le x < 1. \end{cases}$$

Suppose $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is the expansion of the function $f(x)$ in terms of the eigenfunctions $\phi_n(x)$ from part a). Write down a formula for the coefficients c_n. You are **not** asked to compute the coefficients.

SOLUTION. We have

$$c_n = \frac{\displaystyle\int_0^1 f(x)\phi_n(x)\,dx}{\displaystyle\int_0^1 \phi_n^2(x)\,dx}.$$

□

c) (7 points) To what value does the series converge at $x = 0.5$? What about at $x = 0$ and $x = 0.3$?

SOLUTION. The function has a jump discontinuity at $x = 0.5$ so the series converges to $\dfrac{f(.5-) + f(.5+)}{2} = \dfrac{1.5}{2} = \dfrac{3}{4}$. The function is continuous at $x = 0.3$ so the series converges to $f(0.3) = 0.6$. At $x = 0$, we have $\phi_n(0) = 0$ from the hypothesis so the series converges to 0. □

Problem 4: (22 points) Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0$$

$$u(x,t) \text{ bounded as } x \to \infty$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \quad \text{where} \quad f(x) = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } \pi \le x \end{cases}$$

SOLUTION. The solution is given by

$$u(x,t) = \int_0^\infty A(\lambda)\cos(\lambda x)e^{-2\lambda^2 t}\,d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(\lambda x)\,dx \\ &= 2\int_0^\pi \cos(\lambda x)\,dx - \frac{2}{\pi}\int_0^\pi x\cos(\lambda x)\,dx \\ &= \frac{2}{\lambda}\sin(\lambda x)\Big|_0^\pi - \frac{2}{\pi}\left(\frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}\right)\Big|_0^\pi \\ &= \frac{2\sin(\lambda\pi)}{\lambda} - \frac{2}{\pi}\frac{\cos(\lambda\pi)}{\lambda^2} - \frac{2\sin(\lambda\pi)}{\lambda} + \frac{2}{\pi\lambda^2} \\ &= \frac{2 - 2\cos(\lambda\pi)}{\pi\lambda^2}. \end{aligned}$$

Therefore the solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda^2}\cos(\lambda x)e^{-2\lambda^2 t}\,d\lambda.$$

□

Problem 5: (22 points) If an elastic string is *free* at one end, the boundary condition to be satisfied there is that $\dfrac{\partial u}{\partial x} = 0$. On the other hand, if it is *fixed* at one end, the boundary condition to be satisfied there is that $u = 0$. Find the displacement $u(x,t)$ in an elastic string of length $a = 1$, fixed at $x = 0$ and free at $x = a$, set in motion with no initial velocity from the initial position $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$.

a) State the boundary value problem that $u(x,t)$ satisfies. Include the initial conditions.

Solution. The initial value-boundary value problem is the following:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0;\\
&u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0;\\
&u(x,0) = \sin\left(\frac{3\pi x}{2}\right), \qquad 0 < x < 1;\\
&\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < 1.
\end{aligned}$$

□

b) Find $u(x,t)$.

Solution. We solve the associated eigenvalue problem and find $\lambda_n = \dfrac{(2n-1)\pi}{2}$, for $n = 1, 2, \ldots$. The general solution of this PDE is therefore

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\lambda_n ct)\sin(\lambda_n x) + b_n \sin(\lambda_n ct)\sin(\lambda_n x).$$

From $\dfrac{\partial u}{\partial t}(x,0) = 0$ we find that $b_n = 0$ for all n. From the initial condition $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$ we find that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi x}{2}\right) = \sin\left(\frac{3\pi x}{2}\right)$$

The Fourier series is unique, so we just need to make the coefficients of the left-hand side equal to the coefficients of the right-hand side. This yields $a_2 = 1$ and $a_n = 0$ for all $n \neq 2$. The solution is then

$$u(x,t) = \cos\left(\frac{3\pi ct}{2}\right)\sin\left(\frac{3\pi x}{2}\right).$$

□

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

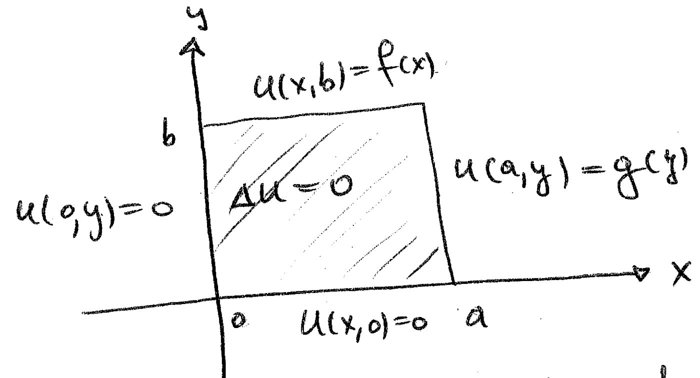
$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$

4.3. Potential in a rectangle

Example 1: $\Delta u = u_{xx} + u_{yy} = 0, \quad 0<x<a, 0<y<b$

$u(0,y) = 0, \ u(a,y) = g(y), \quad 0<y<b$

$u(x,0) = 0, \ u(x,b) = f(x), \quad 0<x<a$



Remark: We need to split this problem into two problems if we want to use separation of variables.

PDE1: $\Delta u_1 = 0$

$u_1(0,y) = 0, \ u_1(a,y) = 0$ ← homogeneous vertical boundary

$u_1(x,0) = 0, \ u_1(x,b) = f(x)$

PDE2: $\Delta u_2 = 0$

$u_2(0,y) = 0, \ u_2(a,y) = g(y)$

$u_2(x,0) = 0, \ u_2(x,b) = 0$ ← homogeneous horizontal boundary

We then have $u(x,y) = u_1(x,y) + u_2(x,y)$.

We solve PDE1 using separation of variables: $u_1(x,y) = X(x)Y(y)$ and get $X''Y + XY'' = 0$ so $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$

Also $X(0) = 0, X(a) = 0, Y(0) = 0$.

From $X'' + \lambda^2 X = 0$ we find $X(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x)$

$X(0) = 0$ so $C_2 = 0$, $X(a) = 0$ so $\sin(\lambda a) = 0$ or $\lambda = \frac{n\pi}{a}$, $n = 1, 2 \ldots$

$X_n(x) = \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{a}$.

$Y'' - \lambda^2 Y = 0$ gives $Y(y) = C_1 \sinh(\lambda y) + C_2 \cosh(\lambda y)$

$Y(0) = 0$ so $C_2 = 0$ and $Y(y) = \sinh(\lambda y)$

$Y_n(y) = \sinh(\lambda_n y)$ and we have found

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

From $u_1(x,b) = f(x)$ we get $\displaystyle\sum_{n=1}^{\infty} \underbrace{c_n \sinh\left(\frac{n\pi b}{a}\right)}_{b_n} \sin\left(\frac{n\pi x}{a}\right) = f(x)$

Set $b_n = c_n \sinh\left(\frac{n\pi b}{a}\right)$ a constant.

Then $\displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) = f(x)$ so $b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

and $c_n = \dfrac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)}$. The solution of PDE1 is given by:

$$u_1(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right), \quad b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Using a similar strategy we solve PDE2 to find $u_2(x,y)$:

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n x) \sin(\lambda_n y), \quad \lambda_n = \frac{n\pi}{b}, \; n=1,2,\ldots$$

Remark: We swap $x \longleftrightarrow y$ in PDE1 to get the solution for PDE2.
$a \longleftrightarrow b$
$f \longleftrightarrow g$

So one needs to pay attention on constants.

$$u_2(a,y) = g(y) = \sum_{n=1}^{\infty} \underbrace{c_n \sinh(\lambda_n a)}_{a_n} \sin(\lambda_n y)$$

$a_n = c_n \sinh(\lambda_n a)$ so $\displaystyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{b}\right) = g(y)$ gives $a_n = \frac{2}{b}\int_0^b g(y) \sin\left(\frac{n\pi y}{b}\right) dy$

and $c_n = \dfrac{a_n}{\sinh\left(\frac{n\pi a}{b}\right)}$

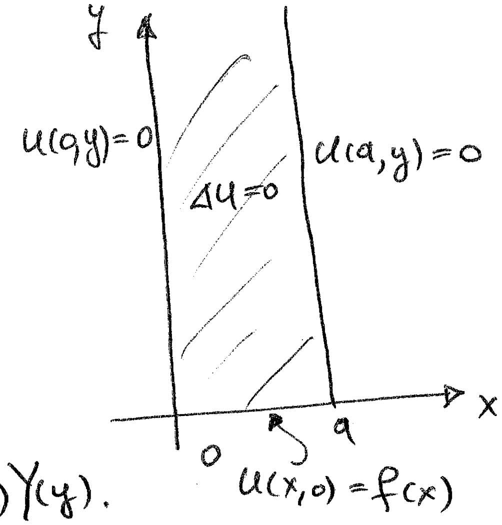
(by Fourier series)

The general solution of PDE2 is:

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \sin\left(\frac{n\pi y}{b}\right), \text{ where}$$

$$a_n = \frac{2}{b}\int_0^b g(y)\sin\left(\frac{n\pi y}{b}\right)dy.$$

4.4. Potential in unbounded regions



Example 2: $\Delta u = 0, \quad 0<x<a, y>0$

$u(x,0) = f(x), \quad 0<x<a$

$u(0,y) = 0, u(a,y) = 0, \quad 0<y$

$u(x,y)$ bounded as $y \to \infty$

we set up separation of variables $u(x,y) = X(x)Y(y)$.

Then $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$ (we need to set it to $-\lambda^2$, not λ^2, because otherwise $X(x)$ will become ~~unbounded~~ zero.)

$X(0) = X(a) = 0$

$X'' + \lambda^2 X = 0$ and $Y'' - \lambda^2 Y = 0$

We find $X_n(x) = \sin(\lambda_n x)$, $\lambda_n = \frac{n\pi}{a}$, $n=1,2\ldots$

and $Y(y) = C_1 e^{-\lambda y} + C_2 e^{\lambda y}$ (It is more convenient to use this notation instead of $C_1\sinh(\lambda y) + C_2\cosh(\lambda y)$.)

$Y(y)$ bounded as $y \to \infty$ means that $C_2 = 0$ so $Y(y) = C_1 e^{-\lambda y}$. We found $Y_n(y) = e^{-\lambda_n y}$, $\lambda_n = \frac{n\pi}{a}$.

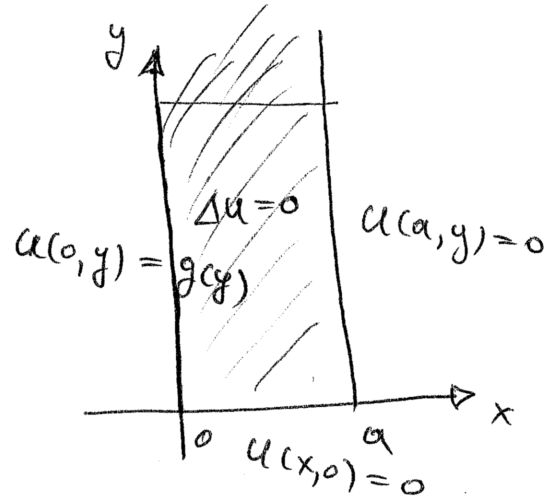
Putting all together we find:

$$u(x,y) = \sum_{n=1}^{\infty} C_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{n\pi}{a}y}$$

From $u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)$ we get $C_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$

Example 3:

$$\Delta u = 0, \quad 0<x<a, \; 0<y$$
$$u(x,0) = 0, \quad 0<x<a$$
$$u(0,y) = g(y), \quad 0<y$$
$$u(a,y) = 0, \quad 0<y$$
$u(x,y)$ bounded as $y \to \infty$



$u(x,y) = X(x)\,Y(y)$ so $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = \lambda^2$

and:
$X(a) = 0$
$Y(0) = 0$

(here we set it equal to λ^2 because if $-\dfrac{Y''}{Y} = -\lambda^2$ then $Y'' - \lambda^2 Y = 0$ and $Y(0) = 0$ will give
$Y(y) = c_1 e^{\lambda y} + c_2 e^{-\lambda y}$, $c_2 = -c_1$
$Y(y) = c_1(e^{\lambda y} - e^{-\lambda y})$ but $Y(y)$ is bounded as $y \to \infty$ so $c_1 = 0$ and $Y \equiv 0$)

we find $\begin{cases} Y'' + \lambda^2 Y = 0 \\ X'' - \lambda^2 X = 0. \end{cases}$

So $Y(y) = c_1 \cos(\lambda y) + c_2 \sin(\lambda y)$, $Y(0) = 0$ gives $c_1 = 0$

$Y(y) = \sin(\lambda y)$ (can take $c_1 = 1$ at this step)

$X(x) = c_1 e^{-\lambda x} + c_2 e^{\lambda x}$ or

$X(x) = A \sinh(\lambda x) + B \cosh(\lambda x)$ ← this is a more convenient notation in this problem.

$X(a) = 0$ gives $A \sinh(\lambda a) + B \cosh(\lambda a) = 0$ and $A = -B \dfrac{\cosh(\lambda a)}{\sinh(\lambda a)}$

$$\text{so } X(x) = B\left(-\frac{\cosh(\lambda a)\sinh(\lambda x)}{\sinh(\lambda a)} + \cosh(\lambda x)\right)$$
$$= B\left(\frac{\cosh(\lambda x)\sinh(\lambda a) - \sinh(\lambda x)\cosh(\lambda a)}{\sinh(\lambda a)}\right)$$
$$= B\left(\frac{\sinh((a-x)\lambda)}{\sinh(\lambda a)}\right)$$

we have used the identity:
$\sinh(a \pm b) = \sinh a \cosh b \pm \sinh b \cosh a$

The constant B can depend on λ.

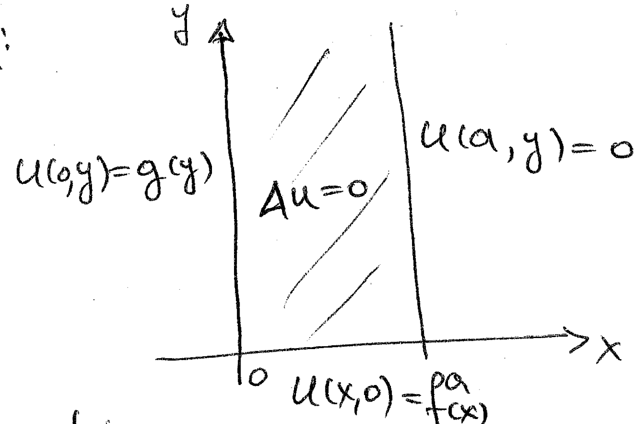
So

$$u(x,y) = \int_0^\infty B(\lambda) \frac{\sinh(a-x)\lambda}{\sinh(\lambda a)} \sin(\lambda y)\, d\lambda$$

From $u(0,y) = g(y)$ we find $u(0,y) = \int_0^\infty B(\lambda) \sin(\lambda y)\, d\lambda$

so $B(\lambda) = \frac{2}{\pi}\int_0^\infty g(y) \sin(\lambda y)\, dy$. (Review ch. 2.10 and 1.9).

Example 4:



the solution to this PDE is Example 2 + Example 3.

4.5. Potential in a disk

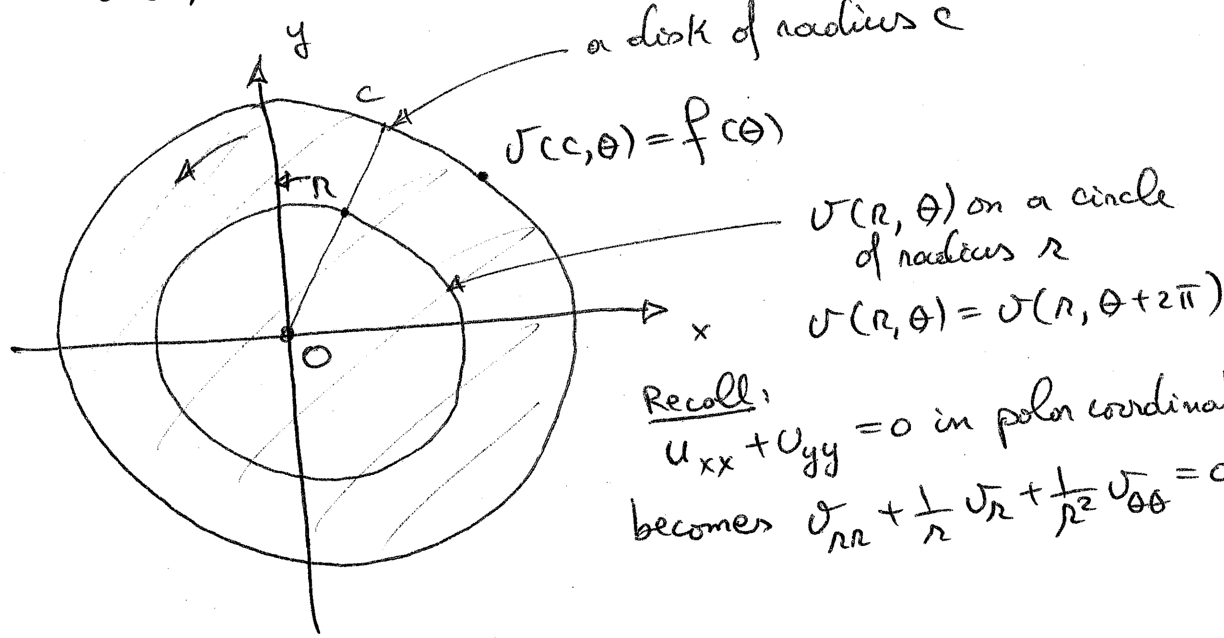
Example 5:

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0, \quad 0 \le r < c$$

$v(c,\theta) = f(\theta)$, $-\pi < \theta < \pi$

$v(r, \theta + 2\pi) = v(r,\theta)$, $0 < r < c$

$v(r,\theta)$ is bounded as $r \to 0$.



Recall: $u_{xx} + u_{yy} = 0$ in polar coordinates becomes $v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0$.

Set $v(r,\theta) = R(r)\Theta(\theta)$ so

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \text{ gives}$$

$$\left(R'' + \frac{1}{r}R'\right)\Theta = -\frac{1}{r^2}R\Theta'' \quad \text{or} \quad \frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{\Theta''}{\Theta}$$

$$\text{or} \quad \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

(if we set $= -\lambda^2$ then Θ would be exponential so not periodic; we need Θ to be periodic)

$\Theta'' + \lambda^2\Theta = 0$, $\Theta(\theta + 2\pi) = \Theta(\theta)$

so $\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$ if this is periodic of period 2π then λ is an integer, so $\lambda = n$, $n = 0, 1, \ldots$

if $\lambda = 0$ we get $\Theta_0(\theta) = \text{constant}$ so we pick $\Theta_0(\theta) = 1$.

if $\lambda = n$, for $n = 1, 2\ldots$ $\quad \Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$, $n = 1, 2\ldots$

The equation for R is $\quad r^2R'' + rR' - n^2R = 0$, $n = 0, 1, 2\ldots$

There are 2 cases:

① if $n = 0$ then $r^2R'' + rR' = 0$ so $\frac{R''}{R'} = -\frac{1}{r}$ (if $R' \neq 0$) and so $R' = \frac{1}{r}$ or $R = \ln(r)$

If $R' = 0$ then $R = \text{constant}$ so we take $R = 1$. Note that $R = \ln(r)$ does not work since $\lim_{r \to 0} R(r) = -\infty$.

$R_0(r) = 1$.

② $n \neq 0$. This is a Cauchy-Euler equation which cannot be solved by a characteristic equation. We know (and not prove) that the general solution is $R(r) = C_1 r^n + C_2 r^{-n}$. Now, since $R(r)$ is bounded as $r \to 0$ we must have $C_2 = 0$.

$R_n(r) = r^n$.

The fundamental solutions for this problem are

$$1, \quad r^n\cos(n\theta), \quad r^n\sin(n\theta)$$

or 1 and $A_n r^n\cos(n\theta) + B_n r^n(\sin(n\theta))$ but they are the same.

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} A_n r^n\cos(n\theta) + B_n r^n\sin(n\theta)$$

from the initial condition $u(c,\theta)=f(\theta)$ we find:

$$u(c,\theta) = a_0 + \sum_{n=1}^{\infty} A_n c^n \cos(n\theta) + B_n c^n \sin(n\theta) = f(\theta)$$

$$\text{so } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n c^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \quad \text{so} \quad A_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta$$

$$B_n c^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \quad \text{so} \quad B_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta.$$

using Fourier series.