MAT322 - Analysis in Several Dimensions
Spring Semester 2014

- **Instructor:** Yaar Solomon
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    - **Office:** Math Tower 2-119.
    - **Office hours:** Tuesdays 10:00-11:00 am, 4:00-5:00 pm and by appointment.
- **Class:** TuTh 11:30am-12:50pm at the Library N4000.
- **Recommended books:**
    - James R. Munkres - *Analysis on Manifolds*.
    - Michael Spivak - *Calculus on Manifolds*.
    - William R. Wade - *An Introduction to Analysis, 2nd edition (Part II)*.
- **Important Information:**
    - Syllabus
    - Homework and Announcements **CHECK FOR UPDATES**
    - Exams

MAT322 - ANALYSIS IN SEVERAL DIMENSIONS

1. Course Description

The notions of a continuous, differentiable and integrable function of one variable calculus are naturally generalized to functions of several variables. In this course we will rigorously develop the theories of differentiation and integration in dimensions 2 and higher, and understand the analogous of the above notions.

2. Prerequisites

I will assume that you are familiar with the following:

- A rigorous course in one variable calculus.
- Basic notions of linear algebra, such as linear spaces (although we will mostly work with $\mathbb{R}^n$), matrices and linear transformations, determinant, etc..

3. Recommended Text

We are not following a specific textbook, but the two following books are very relevant to the material of the course, and you are encouraged to use them:

- James R. Munkres - *Analysis on Manifolds.*
- Michael Spivak - *Calculus on Manifolds.*
- William R. Wade - *An Introduction to Analysis, 2nd edition (Part II).*

4. Homework

We will have homework assignments that will be uploaded weekly to the course web-site. There will be 10-13 homework assignments during the semester, which will be due in class in most Tuesdays, starting February 4th. In these assignments you will be required to apply the definitions and theorems that were learned in class in order to prove simple claims and properties of the discussed objects. The main goal of the exercises is to let you practice the new concepts that were learned in class, and solving them will be essential for following the material in class.

5. Exams and Grading Policy

- **Final Exam:** Wednesday, May 14, 5:30-8:00pm. 40% of the course grade. Failing the exam means failing the course. There will be no makeup.
- **Midterm:** Time and place will be announced later. 30% of the course grade. There will be no makeup.
- **Homework:** The lowest two grades will be dropped. 30% of the course grade.

6. Disabilities

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at

http://studentaffairs.stonybrook.edu/dss/

or (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

http://www.sunysb.edu/facilities/ehs/fire/disabilities

7. Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another persons work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

http://www.stonybrook.edu/uaa/academicjudiciary/

8. Critical Incident Management

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students ability to learn.

MAT322 - Homework and Announcements

One can not study mathematics by watching someone else solving problems! The only way to really understand new mathematics is by sitting and solving exercises on your own. From the above reason, this web-page is for you. You are encouraged to work in groups and discuss the homework with your classmates, but the work that you hand in must be written by each student individually. In particular, in the exams I will assume that you are comfortable with all the homework assignments.

- The homework assignments and their due dates will be published below.
- You will have to submit your assignments in class, every Tuesday.
- You are encouraged to print your homework using LaTex. I'll be glad to help, and to supply a sample file for you, if you are interested.
- Please submit your homework on time. Late submissions will be accepted with a 15% penalty **for every day past the due date**.
- Homework average will take 30% of the course grade! To compute the average, the two lowest grades will be dropped.

Homework #	Due	Assignment	Remarks	Partial Solutions
1	2/4/2014	Homework 1	In question 2 draw only for n=2.	
2	2/11/2014	Homework 2		
3	2/20/2014	Homework 3	Updated notes on Limits of functions	
4	extended to 3/4/2014	Homework 4		
5	extended to 3/11/2014	Homework 5		Homework 5, Questions 4,5
6	extended to 3/27/2014	Homework 6	Notes on Lagrange Multipliers	
7	extended to 3/27/2014	Homework 7		
8	4/3/2014	Homework 8	Measure zero sets	Solutions of the midterm
9	extended to 4/15/2014	Homework 9	Null sets and Lipschitz maps	
10	4/22/2014	Homework 10		
11	4/29/2014	Homework 11		

12	5/6/2014	Homework 12		

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MAT322 - Exams

Final Exam:

- **When?** Wednesday, May 14, 5:30-8:00pm.
- **Where?** TBA.
- **Weight?** 40% of the course grade.
- There will be no makeup. Failing the exam means failing the course.

Midterm:

- **When?** 03/25/14, at 11:30-12:50.
- **Where?** Library 4000 (in class).
- **Weight?** 30% of the course grade.
- There will be no makeup.
- Solutions can be found here

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MAT322 - Analysis in Several Dimensions
Homework 1

Due in Class: February 4, 2014

1. Show that the following formulas define a norm of the linear space V:
   (a) $V = \mathbb{R}^n, x = (x_1, \ldots, x_n), \|x\|_\infty = \max_i |x_i|$.
   (b) $V = \mathbb{R}^n, x = (x_1, \ldots, x_n), \|x\|_1 = \sum_{i=1}^n |x_i|$.
   (c) $V = C[a, b]$, the space of continuous functions defined on the closed interval $[a, b]$, where $(f + g)(x) = f(x) + g(x)$, and $(\alpha f)(x) = \alpha \cdot f(x)$. $\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$ (first convince yourself that V is indeed a vector space and that the maximum is attained).
   (d) $V = C[a, b]$, $\|f\|_1 = \int_a^b |f(x)|\, dx$, as in section (d).

2. Let $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ be the Euclidean norm on $\mathbb{R}^n$, and $\|x\|_1, \|x\|_\infty$ as in exercise 1. Draw (or describe in words) the set $B(x, 1)$ with respect to these three norms, for an arbitrary point $x \in \mathbb{R}^n$.

3. Let $\|\cdot\|$ be a norm on a vector space V. Prove that:
   (a) $\forall x, y, z \in V : \|x - z\| \leq \|x - y\| + \|y - z\|$.
   (b) $\forall x, y \in V : \Big| \|x\| - \|y\| \Big| \leq \|x - y\| \leq \|x\| + \|y\|$.

4. (a) Draw the following sets in the plane and decide (without a formal proof) whether they are open, closed, or neither.
   - $\{(x, y) \in \mathbb{R}^2 : x^2 < y\}$.
   - $\{(x, y) \in \mathbb{R}^2 : x^2 \leq y\}$.
   - $\{(x, y) \in \mathbb{R}^2 : x > 0, y \geq 0\}$.
   - $\mathbb{Q}^2 = \{(x, y) \in \mathbb{R}^2 : x, y \text{ are rational}\}$ (you don't need to draw this set).

   (b) Pick one of the items from (a) and prove your claim.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that the graph of f, $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}\}$, is a closed set in $\mathbb{R}^2$.

MAT322 - Analysis in Several Dimensions
Homework 2

Due in Class: February 11, 2014

Remark. When nothing else is specified you may assume that the norm is the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

1. Prove that:
   (a) ∂A is closed for every $A \subseteq \mathbb{R}^d$.
   (b) $\partial A = \overline{A} \cap \overline{\mathbb{R}^d \setminus A}$.
   (c) $\partial A = \{x \in \mathbb{R}^d : \forall \varepsilon > 0, B(x, \varepsilon) \cap A \neq \varnothing \text{ and } B(x, \varepsilon) \cap (\mathbb{R}^d \setminus A) \neq \varnothing\}$.
   (d) Define the characteristic function of a set $A \subseteq \mathbb{R}^d$ by
   $$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$
   Prove that the ∂A is the set of points where χ_A is discontinuous.

2. Find the interior, the closure, and the boundary of the following sets (prove your statements):
   - $\{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2 + y^2} < 2\}$.
   - $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q}, y = \frac{1}{n} \text{ for } n \in \mathbb{N}\}$.
   - $\{(x, f(x)) : x \in \mathbb{R}\}$, for a continuous function $f : \mathbb{R} \to \mathbb{R}$.

3. We consider a subset $X \subseteq \mathbb{R}^d$ as a normed space with the norm induced from $\mathbb{R}^d$. Show that X is closed $\overset{\text{iff}}{\Longleftrightarrow}$ X is complete (that is, every Cauchy sequence in X converges in X).

4. (a) Let $K \subseteq \mathbb{R}^d$ be a compact set and $f : K \to \mathbb{R}^m$ a continuous one-to-one function. Prove that the inverse function $f^{-1} : f(K) \to K$ is continuous.
   (b) Show that (a) does not hold without the assumption that K is compact (Hint: map [0,1) to a circle).

5. Let $X \subseteq \mathbb{R}^d$ be a compact set. Show that there is a countable set $S \subseteq X$ such that $\overline{S} = X$ (Hint: use open covers by small balls).

MAT322 - Analysis in Several Dimensions
Homework 3

Due in Class: Thursday, February 20, 2014

1. Let $X \subseteq \mathbb{R}^d$ be closed, and $f : X \to X$ satisfies $\|f(x_1) - f(x_2)\| < \|x_1 - x_2\|$, for any $x_1, x_2 \in X$.

   (a) Show that if X is compact then there exist a unique point $p \in X$ with $f(p) = p$ (Hint: Consider the function $g(x) = \|x - f(x)\|$).

   (b) Show that (a) does not necessarily hold if X is not compact.

2. Check whether the following functions are continuous or not (prove your statements):

   – $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   – $f(x,y) = \begin{cases} \frac{\sin(xy)}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   – $f(x,y) = \begin{cases} \frac{\sin(x)-\sin(y)}{x-y} & x \neq y \\ \cos(x) & x = y \end{cases}$ (Hint: Use the formula for $\sin(x) - \sin(y)$).

3. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with coefficient in $\mathbb{R}$, and $GL_n(\mathbb{R})$ the subset of invertible matrices. Prove that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$ (Hint: $det : M_n(\mathbb{R}) \to \mathbb{R}$).

4. Note that a $m \times n$ matrix $A = (a_{ij}) \in M_{m\times n}(\mathbb{R})$, $i \in \{1,\ldots,m\}, j \in \{1,\ldots,n\}$, defines a function $\mathbb{R}^n \to \mathbb{R}^m$ by the ordinary matrix multiplication:

$$A(x) = A(x_1,\ldots,x_n) = \left(\sum_{j=1}^n a_{1j}x_j, \ldots, \sum_{j=1}^n a_{mj}x_j\right).$$

   (a) Prove that for any $A \in M_{m\times n}(\mathbb{R})$ and $x \in \mathbb{R}^n$ we have $\|A(x)\|_\infty \leq C\|x\|_\infty$, where $C = \max_{1\leq i\leq m}\left\{\sum_{j=1}^n |a_{ij}|\right\}$.

(b) Deduce that any $A \in M_{m\times n}(\mathbb{R})$ is a Lipschitz function, with a Lipschitz constant C as in (a) (for $\|x\| = \|x\|_\infty$).

5. Given two non-empty sets $A, B \subseteq \mathbb{R}^d$, define the distance between A and B by

$$d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}.$$

(a) Show that for any $A, B \subseteq \mathbb{R}^d$ we have $d(A, B) = d(\overline{A}, \overline{B})$.

(b) Show that if A is closed and B is compact then $d(A, B) > 0 \overset{\text{iff}}{\iff} A \cap B = \varnothing$.

(c) Find two closed, <u>disjoint</u> sets $A, B \subseteq \mathbb{R}^d$ with $d(A, B) = 0$.

Limits of Functions

Yssr Solomon

Definition 0.1. Let $U \subseteq \mathbb{R}^d$ be an open set, $x_0 \in U$, and $f : U \smallsetminus \{x_0\} \to \mathbb{R}^m$. We say that the limit of f as x approaches x_0 is y, and denote $\lim_{x\to x_0} f(x) = y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in U \smallsetminus \{x_0\}$ with $\|x - x_0\| < \delta$ we have $\|f(x) - y\| < \varepsilon$.

Claim 0.2. $\lim_{x\to x_0} f(x) = y \overset{\text{iff}}{\iff}$ *For any sequence* (x_n) *that converges to* x_0 *we have* $f(x_n) \to y$.

Proof. Left to you (similar to what we did earlier in class). □

Claim 0.3. f *is continuous at* $x_0 \overset{\text{iff}}{\iff} \lim_{x\to x_0} f(x) = f(x_0)$.

Proof. Left to you (similar to what we did earlier in class). □

Claim 0.4. *Suppose that* $\lim_{x\to x_0} f(x) = L_1$ *and* $\lim_{x\to x_0} g(x) = L_2$*, then*

- $\lim_{x\to x_0}[f(x) + g(x)] = L_1 + L_2$.
- $\lim_{x\to x_0}[f(x) - g(x)] = L_1 - L_2$.
- $\lim_{x\to x_0}[f(x) \cdot g(x)] = L_1 \cdot L_2$.
- $\lim_{x\to x_0}[f(x) + g(x)] = L_1 + L_2$*, as long as* $L_2 \neq 0$.

Proof. Left to you (the proof is similar to the one variable case). □

Corollary 0.5. *If* f, g *are continuous at* x_0 *then* $f + g, f - g, f \cdot g$ *are continuous at* x_0*, and* f/g *is continuous at* x_0 *if* $g(x_0) \neq 0$.

Example: The functions below are continuous:

$$f(x, y, z) = \left(\sin(xy), \frac{2 - x + yz^3}{5^z}\right), \quad f(x, y, z, w) = \frac{\sin(x^2 - wz)\cos(yw)}{e^{xy+z}(w^2 + 1)}.$$

While computing limits of several variables functions we can not always use the same arguments that we use for one variable functions. For instance, we can not use L'Hopital's rule, and we don't have any parallel theorem. The two following proposition presents two of our main tools.

Proposition 0.6 (Change of variable for multivariables functions)**.** *Let $x_0 \in U \subseteq \mathbb{R}^d$ open, $f : U \setminus \{x_0\} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$. Suppose that $\lim_{x\to x_0} f(x) = t_0$ and $\lim_{t\to t_0} h(t) = L$, then $\lim_{x\to x_0} h(f(x)) = L$.*

Proof. Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $\|t - t_0\| < \delta_1$ then $\|h(t) - L\| < \varepsilon$. For δ_1, there exists $\delta_2 > 0$ such that if $\|x - x_0\| < \delta_2$ then $\|f(x) - t_0\| < \delta_1$, and therefore $\|h(f(x)) - L\| < \varepsilon$. □

Proposition 0.7 (The squeeze theorem for multivariables functions)**.** *Let $x_0 \in U \subseteq \mathbb{R}^d$ open, $f, g, h : U \setminus \{x_0\} \to \mathbb{R}$. Suppose that for any $x \in U \setminus \{x_0\}$ we have $f(x) \leq g(x) \leq h(x)$, and in addition $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$, then $\lim_{x\to x_0} g(x) = L$.*

Proof. Let $\varepsilon > 0$. Let $\delta_1, \delta_2 > 0$ be such that if $\|x - x_0\| < \delta_1$ then $\|f(x) - L\| < \varepsilon$, and if $\|x - x_0\| < \delta_2$ then $\|h(x) - L\| < \varepsilon$. Set $\delta = \min\{\delta_1, \delta_2\}$. Then for any $x \in U \setminus \{x_0\}$ with $\|x - x_0\| < \delta$ we have:
If $g(x) \leq L$ then $f(x) \leq g(x) \leq L$ and therefore $\|g(x) - L\| \leq \|f(x) - L\| < \varepsilon$.
If $g(x) \geq L$ then $h(x) \geq g(x) \geq L$ and therefore $\|g(x) - L\| \leq \|h(x) - L\| < \varepsilon$. □

Examples:

1. Compute $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ $\left(f : R^2 \setminus \{(0,0)\} \to \mathbb{R}, f(x,y) = \frac{xy}{x^2+y^2}\right)$.
   Solution: We show that the limit does not exist by showing that for different sequences that approachs $(0,0)$ we obtain different limits.

   – If $x_n = y_n \to 0$ then
   $$\lim_{(x_n,y_n)\to(0,0)} \frac{x_n y_n}{x_n^2 + y_n^2} = \lim_{x_n\to 0} \frac{x_n^2}{2x_n^2} = \frac{1}{2}.$$

   – If $x_n = 0, y_n \to 0$ then
   $$\lim_{(x_n,y_n)\to(0,0)} \frac{x_n y_n}{x_n^2 + y_n^2} = \lim_{y_n\to 0} \frac{0}{y_n^2} = 0.$$

   A more general method can also be applied here. In this method we consider the limits along straight lines. Set $y_n = k \cdot x_n$ for some parameter k. Then
   $$\lim_{(x_n,y_n)\to(0,0)} \frac{x_n y_n}{x_n^2 + y_n^2} = \lim_{y_n\to 0} \frac{k y_n^2}{k^2 y_n^2 + y_n^2} = \frac{k}{1+k^2}.$$

   We see that the limit depends on the parameter k, hence it does not exist. For $k = 1$, for instance, we obtain the limit $\frac{1}{2}$, as above, and for $k = 0$ we obtain the limit 0.

Remark. Obseve that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \neq \lim_{x\to x_0}\lim_{y\to y_0} f(x,y).$$

In example 1 we see that the limit at the RHS (right hand side) exists and equal to 0, while the limit at the LHS does not exist. The limit on the LHS is called the <u>iterated limit</u>.

2. Compute $\lim\limits_{(x,y)\to(0,0)} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right)$.

<u>Solution:</u>

$$0 \leq \left|x\sin\left(\frac{1}{y}\right)\right| + \left|y\sin\left(\frac{1}{x}\right)\right| \leq |x| + |y| \to 0.$$

Then by the squeeze theorem, the limit is 0.

Observe that with regard to the remark above, the limit at the LHS does not exist in this example (even when changing the roles of x and y).

3. Compute $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$.

<u>Solution:</u> Set $t = x^2+y^2$, then $(x,y)\to(0,0) \overset{\text{iff}}{\Longleftrightarrow} t\to 0^+$, and therefore

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{t\to 0^+}\frac{\sin t}{t} = 1.$$

4. $f(x,y) = \begin{cases} 1, & x^2 = y \\ 0, & x^2 \neq y \end{cases}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$.

<u>Solution:</u> We'll see that the limit does not exist here, but first observe that for any fixed $k\in\mathbb{R}$, along the orbit $y_n = kx_n$ we have

$$\lim_{(x_n,kx_n)\to(0,0)} f(x_n,y_n) = 0.$$

Indeed, when k is fixed, $\frac{x^2}{k|x|} \xrightarrow{x\to 0} 0$, and in particular $x^2 < k|x|$ for small enough x's. So for small enough x's the value of $f(x,y)$ is 0.

The limit does not exist since along a different orbit, $y_n = x_n^2$, the limit is obviously 1.

5. Is the function $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2-xy+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$ continuous on $\mathbb{R}^2$?

<u>Solution:</u> We first show that $x^2-xy+y^2 \geq |xy|$: if $xy<0$ then there is nothing to prove. Otherwise:

$$0 \leq (x-y)^2 = x^2-2xy+y^2 \Longrightarrow x^2-xy+y^2 \geq xy = |xy|.$$

This in particular implies that $x^2 - xy + y^2 > 0$ when $(x, y) \neq (0, 0)$, and therefore f is well defined on $\mathbb{R}^2$. It is clearly continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$ as the sum/difference/product/qoutient of continuous functions. Secondly,

$$0 \leq |f(x, y)| = \frac{x^2 y^2}{x^2 - xy + y^2} \leq \frac{x^2 y^2}{|xy|} = |xy| \xrightarrow{(x,y)\to(0,0)} 0.$$

So by the squeeze theorem, the limit at $(0, 0)$ is $0 = f(0, 0)$. Then f is continuous on $\mathbb{R}^2$.

MAT322 - Analysis in Several Dimensions
Homework 4

Due in Class: Thursday, February 27, 2014

1. For the following functions $f : \mathbb{R}^d \to \mathbb{R}^m$ write the martix representation of $D_f(x_0)$ with respect to the standard bases of $\mathbb{R}^d$ and $\mathbb{R}^m$:

   (a) $d = 3, m = 2, f(x, y, z) = \begin{pmatrix} 2e^{xy}z \\ x^3y - z^2x \end{pmatrix}, x_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (arbirary point).

   (b) $d = 2, m = 3, f(x, y) = \begin{pmatrix} x^3y \\ y(x-2) \\ x^2 - 2xy + 3y \end{pmatrix}, x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

2. $g : \mathbb{R}^2 \to \mathbb{R}$ is given by the following formula:

$$g(x, y) = \begin{cases} 1, & x^2 + (y-1)^2 \leq 1 \text{ or } y \leq 0 \\ 0, & \text{otherwise} \end{cases}.$$

   Prove that the limit $T(v) = \lim_{t\to 0} \frac{g(0+tv)-g(0)}{t}$ exists for every $v \in \mathbb{R}^2$, and the function T is a linear transformation, and that g is not continuous at $(0, 0)$.

3. Check differentiability at $(0, 0)$:

   (a) $f(x, y) = (x^3 + y^3)^{1/3}$.

   (b) $f(x, y) = \begin{cases} e^{-\frac{1}{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$.

   (c) $f(x, y) = \begin{cases} y, & x^2 = y \\ 0, & x^2 \neq y \end{cases}$.

4. (a) Let $x_0 \in U \subseteq \mathbb{R}^d$, $U \subseteq \mathbb{R}^d$ open, and $f : U \to \mathbb{R}^m$ differentiable at x_0. Show that for any $v \in \mathbb{R}^d$ we have

$$D_v f(x_0) = D_f(x_0)(v).$$

(b) Deduce that for a differentiable function $f : U \to \mathbb{R}$, for any $v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \in \mathbb{R}^d$ we have

$$D_v f(x_0) = \sum_{j=1}^{d} \alpha_j \frac{\partial f}{\partial x_j}(x_0).$$

(c) Find $D_v f(x_0)$ for $f(x, y, z) = 3\sin^2(xy) + e^z x$, $v = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$ and $x_0 = \begin{pmatrix} \pi \\ 1/4 \\ 0 \end{pmatrix}$.

5. We say that a function $g : \mathbb{R}^d \to \mathbb{R}^m$ is homogeneous if for any $t \in \mathbb{R}$ and $v \in \mathbb{R}^d$ we have $g(tv) = tg(v)$.

(a) Show that if $f : \mathbb{R}^d \to \mathbb{R}^m$ is homogeneous and differentiable at 0 then it is a linear transofrmation.

(b) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is homogeneous but not linear.

MAT322 - Analysis in Several Dimensions
Homework 5

Due in Class: Thursday, March 6, 2014

1. Let $a \in U \subseteq \mathbb{R}^d$, $U \subseteq \mathbb{R}^d$ open, $\alpha \in \mathbb{R}$, and $f, g : U \to \mathbb{R}^m$ differentiable at a. Show that:

   (a) $D_{\alpha f}(a) = \alpha D_f(a)$.

   (b) $D_{f+g}(a) = D_f(a) + D_g(a)$.

   (c) $D_{f \cdot g}(a) = f(a)D_g(a) + g(a)D_f(a)$ (<u>Hint:</u> Add and subtract $f(a) \cdot g(a+h) + D_f(a)(h) \cdot g(a+h)$).

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3, g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the following formulas:

$$f(x) := f(x_1, x_2) = (x_2 e^{x_1^2}, x_1 x_2 - \cos(x_1), \sin(x_1 x_2)).$$

$$g(y) := g(y_1, y_2, y_3) = (y_2^3 y_3 \sin(y_1), y_1 + y_2^2 y_3^3).$$

   Define $F(x) = g(f(x)), \quad G(y) = f(g(y))$. Compute

   (a) $D_F \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   (b) $D_G \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

3. For parameters $a, b, c > 0$ consider the surface

$$\Gamma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : ax^2 + by^2 + cz^2 = 1, z > 0 \right\}.$$

   Show that the tangent space at a general point (x_0, y_0, z_0) is given by

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : axx_0 + byy_0 + czz_0 = 1 \right\}.$$

4. **Definition.** We say that a set $A \subseteq \mathbb{R}^d$ is connected if for any two points $a, b \in A$ there exists a *path* in A between a and b. That is, there exists a continuous function $\gamma : [0, 1] \to A$ with $\gamma(0) = a$ and $\gamma(1) = b$ (Remark: this notion is sometimes called path connected).
   Prove that if $A \subseteq \mathbb{R}^d$ is open and connected then for any two points $a, b \in A$ there exists a **polygonal path** between a and b. That is a path $\gamma : [0, 1] \to A$, and a partition $\{0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1\}$ of $[0, 1]$, such that for every $1 \leq i \leq m$, if $t \in [t_{i-1}, t_i]$ then
   $$\gamma(t) = \frac{t_i - t}{t_i - t_{i-1}}\gamma(t_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}}\gamma(t_i).$$

5. Let $f : \mathbb{R}^d \to \mathbb{R}^m$ satisfy $D_f(a) = 0$ for any point a in an open, connected set $U \subseteq \mathbb{R}^d$. Prove that f is constant on U.

MAT322 - Analysis in Several Dimensions
Homework 5 - Partial Solutions

Yaar Solomon

4. **Definition.** We say that a set $A \subseteq \mathbb{R}^d$ is connected if for any two points $a, b \in A$ there exists a *path* in A between a and b. That is, there exists a continuous function $\gamma : [0,1] \to A$ with $\gamma(0) = a$ and $\gamma(1) = b$ (Remark: this notion is sometimes called path connected).
Prove that if $A \subseteq \mathbb{R}^d$ is open and connected then for any two points $a, b \in A$ there exists a **polygonal path** between a and b. That is a path $\gamma : [0,1] \to A$, and a partition $\{0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1\}$ of $[0,1]$, such that for every $1 \le i \le m$, if $t \in [t_{i-1}, t_i]$ then
$$\gamma(t) = \frac{t_i - t}{t_i - t_{i-1}}\gamma(t_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}}\gamma(t_i).$$
Solution: Let $a, b \in A$, by connectness of A there exists a continuous path $\gamma : [0,1] \to A$ such that $\gamma(0) = a, \gamma(1) = b$. γ is continuous and $[0,1]$ is compact, then the image $K = \gamma([0,1])$ is a compact subset of $\mathbb{R}^d$. $\gamma(t) \in A$ for every $t \in [0,1]$, so for every t there is some $r_t > 0$ such that $B_t := B(\gamma(t), r_t) \subseteq A$. That gives an open cover $\mathcal{U} = \{B_t\}_{t\in[0,1]}$ of K. Let $\varepsilon > 0$ be the Lebesgue number of the cover $\mathcal{U}$ (i.e. for every $p \in K$ there is some $t \in [0,1]$ such that $B(p,\varepsilon) \subseteq B_t$).
By continuity of γ on $[0,1]$ we also deduce that γ is uniformly continuous. That is, there exists some $\delta > 0$ such that for every $t, t' \in [0,1]$ with $|t - t'| < \delta$ we have $\|\gamma(t) - \gamma(t')\| < \varepsilon$. Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ and divide the interval $[0,1]$ into n equally spaced subintervals, using the partition $0 = t_0 < t_1 < \ldots < t_n = 1$. Then for any $i \in \{1, \ldots, n\}$ the points $\gamma(t_{i-1}), \gamma(t_i) \in B(\gamma(t_i), \varepsilon)$, and by convexity of the balls (that you will prove in ex6) the straight line between them is contained in A.

5. Let $f : \mathbb{R}^d \to \mathbb{R}^m$ satisfy $D_f(a) = 0$ for any point a in an open, connected set $U \subseteq \mathbb{R}^d$. Prove that f is constant on U.
Solution: Let $x, y \in U$, we show that $f(x) = f(y)$. By connectness of U there exists a continuous path $\gamma : [0,1] \to U$ such that $\gamma(0) = x, \gamma(1) = y$. By

question (4), we may assume that γ is a polygonal path. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be the partition of $[0,1]$ that forms the polygonal path γ. Then for every $j \in \{1, \ldots, n\}$ the straight line between $\gamma(t_{j-1})$ and $\gamma(t_j)$ is contained in U. Apply the Mean Value Theorem (that we learned in class) on f with $I_j = \{\gamma(s) : s \in [t_{j-1}, t_j]\}$ to obtain that

$$\|f(\gamma(t_j)) - f(\gamma(t_{j-1}))\| \leq \sup_{z \in I_j} \{D_f(z)\} \|\gamma(t_j) - \gamma(t_{j-1})\| .$$

But $D_f(z) = 0$ for every $z \in U$, and $z \in U$ for every $z \in I_j$, for every j. Then for every j we have $\|f(\gamma(t_j)) - f(\gamma(t_{j-1}))\| = 0$, and therefore $f(\gamma(t_j)) = f(\gamma(t_{j-1}))$. Hence $f(x) = f(\gamma(t_0)) = f(\gamma(t_1)) = \ldots = f(\gamma(t_n)) = f(y)$, as required.

MAT322 - Analysis in Several Dimensions
Homework 6

Due in Class: Tuesday, March 25, 2014

Remark. *Questions 1 and 2 deal with facts that we used in the proof of the Inverse Function Theorem:*

1. Let $\|\cdot\|$ be a norm on a vector space V over $\mathbb{R}$. Denote by $\hat{B}(a,r) = \{x \in V : \|x-a\| \leq r\}$. Prove that $\hat{B}(a,r) = \overline{B(a,r)}$.

2. **Definition.** A set $B \subseteq \mathbb{R}^n$ is called <u>convex</u> if for any $x, y \in B$ the (open) straight line between x and y is contained in B. That is $I_{xy} = \{sy+(1-s)x : s \in (0,1)\} \subseteq B$. B is <u>strictly convex</u> if for every $x, y \in B$ we have $I_{xy} \subseteq int(B)$.

   (a) Let $\|\cdot\|$ be a norm on a vector space V over $\mathbb{R}$. Show that for any $r > 0$ the set $\overline{B(0,r)}$ is convex.

   (b) Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. Show that for any $r > 0$ the set $\overline{B(0,r)}$ is strictly convex (<u>Hint:</u> We may want to use the formula $\cos(\theta_{xy}) = \frac{\langle x,y\rangle}{\|x\|\|y\|}$, where θ_{xy} is the angle between x and y).

3. For each of the following functions prove that f^{-1} exists and differentiable in a neighborhood of the given point t_0, and compute $D_{f^{-1}}(f(t_0))$:

   (a) $f(x,y) = (3x+2y, 5x+4y)$ at $t_0 = (a,b)$ (arbitrary point).

   (b) $f(x,y) = (x^2y^3, (x+y)^2+x)$ at $t_0 = (2,1)$.

   (c) $f(x,y,z) = (x\sin y, yz^2+x, \cos(2z))$ at $t_0 = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4})$.

4. Prove that each of the following equations can be solved for z in a neighborhood of $(0,0,0)$. Is the solution differentiable near $(0,0)$?

   (a) $x^2+y^2+z^2+\sqrt[3]{2xy+3z+1} = 3$.

   (b) $(x+y+z+1)^3 + \sin(-2z) = 4$.

5. Prove that there is a C^1 function $g(x, y) = (u(x, y), v(x, y), w(x, y))$, defined on a neighborhood A of $(1, 1)$, such that $g(1, 1) = (1, 1, -1)$, and

$$u^5 + xv^2 - y + w = 0$$

$$v^5 + yu^2 - x + w = 0$$

$$w^4 + y^5 - x^4 = 1$$

on A.

Lagrange Multipliers - Examples

Yaar Solomon

Theorem (Lagrange Multipliers). *Suppose that $B \subseteq \mathbb{R}^n$ open, $f, g_1, \ldots, g_k : B \to \mathbb{R}$ C^1, $A = \{x \in B : \forall i \ \ g_i(x) = 0\}$, and $a \in A$ a local minimum (maximum) of f on A [i.e. $\exists U \subseteq B$ open, $a \in U$ such that for any $z \in A \cap U$ we have $f(z) \geq f(a)$ $(f(z) \leq f(a))$]. Suppose additionally that $\nabla g_1(a), \ldots, \nabla g_k(a)$ are linearly independent, then there are scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that*

$$\nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a).$$

[$\lambda_1, \ldots, \lambda_k$ are called the *<u>Lagrange Multipliers</u>*.*]*

One can use this theorem to find a local and global minimum and maximum of a function on the domain A, which is given by the constrains $g_1, \ldots, g_1$ (in most of the examples we will only have one constrain g), or more precisely to find the candidates for the extremum points. According to the theorem, if $a \in A$ is a local minimum (maximum) of f then one of the following must happen:

- f or one of the g_i's are not C^1 at a.
- $\nabla g_1(a), \ldots, \nabla g_k(a)$ are linearly dependent.
- a is in the boundary of A.
- $\exists \lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such taht $\nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a)$..

(if one of the first three holds then we can not apply the theorem, and therefore it is also a candidate for an extremum).

<u>Examples:</u>

1. Find global minimum and maximum of $f(x, y) = x^2 - 4xy + 4y^2$ subject to the constrain $x^2 + y^2 = 1$.
   <u>Solution:</u> Let $g(x, y) = x^2 + y^2 - 1$, then $A = \{(x, y) : x^2 + y^2 = 1\} = g^{-1}(\{0\})$. A is compact and f is continuous then f attain min and max on A. From the above list of candidates, only the 4'th is relevant here (and this is how it is in most exercises). Since $\nabla f\binom{x}{y} = \binom{2x-4y}{-4x+8y}$, $\nabla g\binom{x}{y} = \binom{2x}{2y}$, we are looking for

points where there exists a scalar λ such that $\binom{2x-4y}{-4x+8y} = \lambda\binom{2x}{2y}$. From the first equation we get $2y = x(1-\lambda)$, and by plug it in the second equation we obtain $-4x + 4x(1-\lambda) = \lambda x(1-\lambda) \Rightarrow \lambda x(\lambda - 5) = 0$. So we get the following list of suspicious points:

- If $x = 0$, then $y = 0 \Longrightarrow \binom{0}{0} \notin A$.
- If $\lambda = 0$ then $x = 2y$. Using the constrain g we obtain that $y = \pm 1/\sqrt{5} \Rightarrow x = \pm 2/\sqrt{5} \Longrightarrow \binom{2/\sqrt{5}}{1/\sqrt{5}}, \binom{-2/\sqrt{5}}{-1/\sqrt{5}}$.
- If $\lambda = 5$ then $y = -2x$. Using the constrain g we obtain that $x = \pm 1/\sqrt{5} \Rightarrow y = \mp 2/\sqrt{5} \Longrightarrow \binom{1/\sqrt{5}}{-2/\sqrt{5}}, \binom{-1/\sqrt{5}}{2/\sqrt{5}}$.

So the global min and max values of f must be obtained in one of the 4 points that we found. Checking the values of f at these points we obtain:

$$f\begin{pmatrix}2/\sqrt{5}\\1/\sqrt{5}\end{pmatrix} = f\begin{pmatrix}-2/\sqrt{5}\\-1/\sqrt{5}\end{pmatrix} = 0, \qquad f\begin{pmatrix}1/\sqrt{5}\\-2/\sqrt{5}\end{pmatrix} = f\begin{pmatrix}-1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix} = 5.$$

Then the global min of f is obtained at $\binom{2/\sqrt{5}}{1/\sqrt{5}}$ and $\binom{-2/\sqrt{5}}{-1/\sqrt{5}}$, and it is 0, and the global max is obtained at $\binom{1/\sqrt{5}}{-2/\sqrt{5}}$ and $\binom{-1/\sqrt{5}}{2/\sqrt{5}}$, and it is 5.

2. Prove the Inequality of Geometric and Harmonic Means: $\forall x_1, \ldots, x_n > 0$ we have

$$(*) \qquad \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \le \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Proof. First observe that if we replace x_i by tx_i for every i then both sides will be multiplied by t, and therefore $(*)$ holds if and only if it holds with the tx_i's, for some $t > 0$. For the right $t > 0$ we have $\sqrt[n]{tx_1 \cdots tx_n} = 1$, so we may prove $(*)$ under the assumption $\sqrt[n]{x_1 \cdots x_n} = 1$. That is, we need to show that $n \le \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}$, under the constrain $\sqrt[n]{x_1 \cdots x_n} = 1$.
So let

$$A = \{x \in \mathbb{R}^n : x_1 \cdots x_n = 1, \forall i \ x_i > 0\}.$$

We look for a global minimum for $f(x) = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}$ on A.
Remark: Here A is not compact, so it is not guaranteed to have a global minimum. To deal with it we may define $A_\varepsilon = \{x \in \mathbb{R}^n : x_1 \cdots x_n = 1, \forall i \ x_i \ge \varepsilon\}$ for $\varepsilon > 0$, and analyse our results at the end.

Using the Lagrange Multipliers method, define $g(x) = x_1 \cdots x_n - 1$, then

$$\nabla f\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}-1/x_1^2\\-1/x_2^2\\\vdots\\-1/x_n^2\end{pmatrix}, \qquad \nabla g\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}x_2x_3\cdots x_n\\x_1x_3\cdots x_n\\\vdots\\x_1x_2\cdots x_{n-1}\end{pmatrix} \overset{g}{=} \begin{pmatrix}1/x_1\\1/x_2\\\vdots\\1/x_n\end{pmatrix}$$

Then $\exists \lambda : \nabla f(x) = \lambda \nabla g(x) \implies \forall i : -\frac{1}{x_i^2} = \frac{\lambda}{x_i} \implies \forall i : x_i = -\frac{1}{\lambda} \implies$ $x_1 = x_2 = \cdots = x_n$. Since $x_1 \cdots x_n = 1$ we have $x_1 = x_2 = \cdots = x_n = 1$, a unique suspicious point, and observe that $f(1, 1, \ldots, 1) = n$.

Now, for any $\varepsilon > 0$ A_ε is compact, and therefore there are three options:

(i) The minimum is $f(1, 1, \ldots, 1) = n$ and the maximum is on ∂A_ε.

(ii) The maximum is $f(1, 1, \ldots, 1) = n$ and the minimum is on ∂A_ε.

(iii) Both the minimum and the maximum are on ∂A_ε.

On the boundary of A_ε at least one of the x_i's is equal to ε. Then $f(x_1, x_2, \ldots, x_n) = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \geq \frac{1}{\varepsilon} \xrightarrow{\varepsilon \to 0} \infty$. In particular, if $\varepsilon < \frac{1}{n}$, all the values on the boundary are greater than n, so (i) is the correct option. Since $A = \bigcup A_\varepsilon$, and $(1, 1, \ldots, 1)$ is a global minimum in every A_ε, for ε small enough, it is also a global minimum of A. $\square$

MAT322 - Analysis in Several Dimensions
Homework 7

Due in Class: Tuesday, March 25, 2014

1. Find global minimum and maximum of $f(x, y) = x + y^2$ subject to the constrain $x^2 + y^2 = 4$.

2. Consider the following ellipsoid, for parameters $a, b, c > 0$

$$E = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}.$$

   Find the axes parallel box of maximal volume that is inscribed in E.

3. Let $L \subseteq \mathbb{R}^3$ be the line that is defined by the following equations

$$L = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \begin{matrix} x - y + 2z = 4 \\ 2x + y - z = 1 \end{matrix} \right\}.$$

   Find the point on L which is closest to $0 \in \mathbb{R}^3$.

4. (a) Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = x_1 x_2 \cdots x_n$, where $x = (x_1, \ldots, x_n)$. Find the global maximum of f on $A = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = n, \forall i \;\; x_i \geq 0\}$.

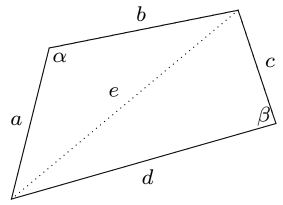
   (b) Deduce the Inequality of Arithmetic and Geometric Means: $\forall x_1, \ldots, x_n \geq 0$ we have

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

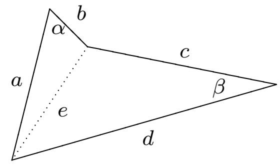
   (Hint: Show that the inequality is invariant under multiplying every x_i by a positive parameter t, and deduce that it suffices to prove it under the constrain $x_1 + \ldots + x_n = n$).

5. (a) Let Q be a quadrilateral with edges of lengths $a, b, c, d > 0$, in that order,

and two opposing angles $0 < \alpha, \beta < \pi$ (see the picture)



or



if Q is
not convex

Find a condition on $a, b, c, d, \alpha, \beta$ for when such a quadrilateral exists.

(b) Prove that for given $a, b, c, d > 0$ such a quadrilateral Q has maximal area if it can be inscribed in a circle. (Recall: A quadrilateral can be inscribed in a circle if and only if the sum of its opposite angles is π).

MAT322 - Analysis in Several Dimensions
Homework 8

Due in Class: Thursday, April 3, 2014

1. Let $Q \subseteq \mathbb{R}^n$ be a rectangle, $f : Q \to \mathbb{R}$ bounded. Prove that f is integrable on $Q \overset{\text{iff}}{\Longleftrightarrow}$ for every $\varepsilon > 0$ there exists a partition P of Q such that $U(f, P) - L(f, P) < \varepsilon$.

2. Define $f, g : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}, \qquad g(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \text{ irreducible fraction} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

   (a) Prove that f is not integrable on $[0, 1]$.

   (b) Prove that g is continuous at $x \overset{\text{iff}}{\Longleftrightarrow} x \notin \mathbb{Q}$.

   (c) Prove that g is integrable on $[0, 1]$ and find $\int_{[0,1]} g$ (without using Lebesgue Theorem).

3. Prove that if $f : Q \to \mathbb{R}$ is continuous in the box $Q \subseteq \mathbb{R}^n$ then it is integrable on Q (<u>**Hint:**</u> Uniform continuity).

4. Let $f_1, f_2 : [0, 1] \to \mathbb{R}$ be monotonically increasing (i.e. $x_1 \leq x_2 \Rightarrow f_i(x_1) \leq f_i(x_2)$), non-negative functions. Prove that $g(x, y) := f_1(x) \cdot f_2(y)$ is integrable on $Q := [0, 1]^2$.

5. **Definition.** Let $A \subseteq \mathbb{R}^n$. The <u>oscillation</u> of a function $f : A \to \mathbb{R}$ at a point $x_0 \in A$ is

$$\omega_f(x_0) = \lim_{\delta \to 0} diam\left(f(B(x_0, \delta))\right).$$

   Prove that the limit exists (in the sense that it may also be ∞) for any point $x_0 \in A$, and that f is continuous at $x_0 \overset{\text{iff}}{\Longleftrightarrow} \omega_f(x_0) = 0$.

Properties of Measure Zero Sets

Yaar Solomon

Definition. We say that a set $E \subseteq \mathbb{R}^n$ <u>has measure zero</u> if for every $\varepsilon > 0$ there is a countable collection of closed boxes $Q_1, Q_2, \ldots$ such that

$$(*) \qquad E \subseteq \bigcup_{i=1}^{\infty} Q_i, \quad \text{and} \quad \sum_{i=1}^{\infty} v(Q_i) < \varepsilon.$$

Proposition. *(1) If $B \subseteq A$ and A has measure zero then B has measure zero.*

(2) E has measure zero $\overset{\text{iff}}{\Longleftrightarrow}$ there is a countable collection of open boxes $\tilde{Q}_1, \tilde{Q}_2, \ldots$ such that $(*)$ holds with the $\tilde{Q}_i$'s.

(3) If $E_1, E_2, \ldots$ are of measure zero then $\bigcup_{i=1}^{\infty} E_i$ has measure zero.

(4) If E is compact and has measure zero then for every $\varepsilon > 0$ there is a finite collection of closed (or open) boxes $Q_1, \ldots, Q_k$ such that $(*)$ holds.

(5) If $Q \subseteq \mathbb{R}^n$ is a box then Q does not have measure zero, but ∂Q does.

Proof. (1) Clear. If $Q_1, Q_2, \ldots$ satisfy $(*)$ for A then they also satisfy $(*)$ for B.

(2) <u>$\Longleftarrow$:</u> If the collection $\tilde{Q}_1, \tilde{Q}_2, \ldots$ satisfies $(*)$ then so does $Q_1, Q_2, \ldots$, where Q_i is the closure of $\tilde{Q}_i$.
<u>$\Longrightarrow$:</u> Follows from the fact that for any closed box Q there exists an open box $\tilde{Q}$ such that $Q \subseteq \tilde{Q}$ and $v(\tilde{Q}) \leq 2v(Q)$. To see it, denote $Q = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and take $Q_\delta = (a_1 - \delta, b_1 + \delta) \times \ldots \times (a_n - \delta, b_n + \delta)$. For every $\delta > 0$ the set Q_δ is an open box containing Q, and

$$v(Q_\delta) = \prod_{i=1}^{n} (b_i - a_i + 2\delta) \xrightarrow{\delta \to 0} \prod_{i=1}^{n} (b_i - a_i) = v(Q).$$

So for δ small enough $v(Q_\delta) \leq 2v(Q)$.

The claim follows since given an $\varepsilon > 0$, let $Q_1, Q_2, \ldots$ be a collection of closed boxes that covers E and $\sum_{i=1}^{\infty} v(Q_i) < \frac{\varepsilon}{2}$. Now let $\tilde{Q}_i$ be an open box containing Q_i with $v(\tilde{Q}_i) \leq 2v(Q_i)$. Then $\tilde{Q}_1, \tilde{Q}_2, \ldots$ is again a cover of E and

$$\sum_{i=1}^{\infty} v(\tilde{Q}_i) \leq \sum_{i=1}^{\infty} 2v(Q_i) < 2\frac{\varepsilon}{2} = \varepsilon.$$

(3) Given $\varepsilon > 0$, for every given $j \in \mathbb{N}$ let $Q_1^{(j)}, Q_2^{(j)}, \ldots$ be a countable collection of boxes that cover E_j, and with $\sum_{i=1}^{\infty} v(Q_i^{(j)}) < \frac{\varepsilon}{2^j}$. Then the collection $\{Q_i^{(j)} : i, j \in \mathbb{N}\}$ is again countable, it covers $\bigcup_{i=1}^{\infty} E_j$, and

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} v(Q_i^{(j)}) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

(4) By (2), let $B_1, B_2, \ldots$ be a countable collection of open boxes that satisfies $(*)$. Then we have an open cover of E, and by compactness we get a finite sub-cover $B_{i_1}, \ldots, B_{i_k}$ with property $(*)$. We may also get a finite collection of closed boxes by taking Q_j to be the closure of B_{i_j}.

(5) Note that ∂Q is a finite union of sets of the form $F = [a_1, b_1] \times \ldots \times a_i \times \ldots \times [a_n, b_n]$, with perhaps b_i instead of a_i at the i'th place. Every such set F can be covered by one box $Q_\delta^F = [a_1, b_1] \times \ldots \times [a_i, a_i + \delta] \times \ldots \times [a_n, b_n]$, and $v(Q_\delta^F)$ depends on δ and approaches 0 with $\delta \to 0$. This proves that every such F has measure zero, so it follows from (3) that ∂Q has measure zero.

For Q, assume that Q has measure zero. Let $\varepsilon < v(Q)$, then by (4) there is a finite collection of closed boxes $A_1, \ldots, A_k$ that cover Q and with $\sum_{i=1}^{k} v(A_i) < \varepsilon$. Let P be a partition of Q that contains all the vertices of all the A_i's, and denote by $Q_1, \ldots, Q_\ell$ the sub-rectangles of the partiion P. So in particular, for any $1 \leq j \leq \ell$ the sub-rectangle Q_j is contained in one of the A_i's (at least one, but the A_i's may overlap). Then

$$v(Q) = \sum_{j=1}^{\ell} v(Q_j) \leq \sum_{i=1}^{k} v(A_i) < \varepsilon < v(Q).$$

A contradiction.

$\square$

MAT322 - Analysis in Several Dimensions
Midterm

Yaar Solomon

Tuesday, March 25, 2014

Name (First - Last): ______________________________

Stony Brook ID: ______________________________

Signature: ______________________________

Instructions

1. Start when told to; stop when told to.
2. No notes, books, etc...
3. Turn OFF your cell phone and all other unauthorized electronic devices.
4. Write coherent mathematical statements and show your work on all problems.
5. If you use a theorem from class, you must fully state it.
6. If you give an example/construction then you must prove it is such.
7. Please write clearly.

(1) (26pts)	(2) (26pts)	(a) (16pts)	(b) (16pts)	(c) (16pts)	TOTAL

1. (26 points)
Let $K \subseteq \mathbb{R}^n$ be a compact set, $\{U_i\}_{i \in I}$ an open cover of K. Prove that there exists $\varepsilon > 0$ such that for any $x \in K$ there is some $i \in I$ such that $B(x, \varepsilon) \subseteq U_i$.
Remark. You may **not** use here the criterion of compactness saying that for any cover there is a finite subcover (we used the above lemma to prove this criterion).

Solution: Assume otherwise, then $\forall \varepsilon > 0 \exists x \in K \forall i \in I$ we have $B(x, \varepsilon) \nsubseteq U_i$. Then for $\varepsilon = \frac{1}{m}$ let $x_m \in K$ be such that $B(x_m, \frac{1}{m}) \nsubseteq U_i$ for every $i \in I$. K is compact, then (x_m) has a converging subsequence $x_{m_j} \to x_0 \in K$. Since $\{U_i\}_{i \in I}$ is a cover of K, there is some i such that $x_0 \in U_i$. U_i is open, so there is some $\delta > 0$ such that $B(x_0, \delta) \subseteq U_i$. To derive a contradiction, let $j \in \mathbb{N}$ be large enough such that $\frac{1}{m_j} < \frac{\delta}{2}$, and $\|x_{m_j} - x_0\| < \frac{\delta}{2}$. Then for every $z \in B(x_{m_j}, \frac{1}{m_j})$ we have

$$\|z - x_0\| \leq \|z - x_{m_j}\| + \|x_{m_j} - x_0\| < \frac{1}{m_j} + \|x_{m_j} - x_0\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Then $B(x_{m_j}, \frac{1}{m_j}) \subseteq B(x_0, \delta) \subseteq U_i$, which contradicts our initial assumption.

2. (26 points)
Explain why global minimum and maximum of $f(x,y,z) = xy + xz + yz$ exists on the domain $A = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 1, x,y,z \geq 0\}$, and find them.

Solution: For $(x,y,z) \in A$ we have $x,y,z \geq 0$ and $x+y+z=1$ then $x,y,z \leq 1$. So $A \subseteq [0,1]^3$ and therefore bounded. To see that A is closed (you may also write that this is clear), let $g(x,y,z) = x+y+z-1$, and P_x, P_y, P_z the projections on the x,y,z coordinates respectively. These functions are continuous, and

$$A = g^{-1}(\{0\}) \cap P_x^{-1}([0,\infty)) \cap P_y^{-1}([0,\infty)) \cap P_z^{-1}([0,\infty)),$$

so A is closed as the intersection of 4 closed sets. So A is compact. Since f is continuous, it attains global minimum and maximum on A. We use Lagrange Multipliers to find them.

The suspicious points are either in the boundary of A

$$\partial A = \{(x,y,z) \in A : x = 0 \text{ or } y = 0 \text{ or } z = 0\},$$

or at points $a \in A$ where $\nabla f(a) = \lambda \nabla g(a)$, for some $\lambda \in \mathbb{R}$. That is

$$\begin{array}{c} (i) \\ (ii) \\ (iii) \end{array} \qquad \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix} = \nabla f(a) = \lambda \nabla g(a) = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix}.$$

By (ii) and (iii) we have $x = \lambda - y = \lambda - z \Rightarrow y = z$. By (i) $y = z = \lambda/2$. Using (ii) we obtain that $x = \lambda/2$. Since their sum is 1 we have $1 = \lambda/2 + \lambda/2 + \lambda/2 = 3\lambda/2 \Rightarrow \lambda = \frac{2}{3} \Rightarrow x = y = z = 1/3$. Then $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the unique suspicious point which is not on the boundary of A.

On the boundary of A either x, y or z are 0. Then $f(x,y,z)$ is given by the equation, yz, or xz, or xy respectively. In each of these cases the sum of the other two parameters in 1, and therefore we can write the formula for $f(x,y,z)$ as a funcion of one variable. Where $x = 0$ we have $f(x,y,z) = h(y) = y(1-y) = y - y^2$. $h'(y) = 1 - 2y = 0 \overset{\text{iff}}{\iff} y = \frac{1}{2}$. So $(0, \frac{1}{2}, \frac{1}{2})$ is a suspicious point on the boundary of A. By symmetry we also get the points $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$. In addition we have to check the endpoints of these intervals $(1,0,0), (0,1,0), (0,0,1)$.

Comparing the values of f at these points we obtain

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3\frac{1}{3^2} = \frac{1}{3}.$$

$$f(1,0,0) = f(0,1,0) = f(0,0,1) = 0.$$

$$f\left(0, \frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{1}{2}, 0, \frac{1}{2}\right) = f\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \frac{1}{4}.$$

Then the global maximum is obtained at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and it is $\frac{1}{3}$, and the global minimum is 0 and it is attained at $(1,0,0), (0,1,0)$ and $(0,0,1)$.

Short Questions:

a. (16 points)
Let $U \subseteq \mathbb{R}^n$ open, $a \in U$. Prove that if $f : U \to \mathbb{R}^m$ is differentiable at a then it is continuous at a.

Solution: f is differentiable at a and therefore there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{\|h\|} = 0.$$

In particular $\lim\limits_{h \to 0}[f(a+h) - f(a) - T(h)] = 0$. Since T is linear, it is continuous, and therefore $\lim\limits_{h \to 0} T(h) = 0$. Then $0 = \lim\limits_{h \to 0}[f(a+h) - f(a)] = \lim\limits_{h \to 0}[f(a+h)] - f(a)$, and hence $f(a) = \lim\limits_{h \to 0}[f(a+h)]$. So f is continuous at a.

b. (16 points)
$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$. Decide whether f is differentiable at $(0, 0)$ or not. Show your work and prove your answer.

Solution: We first prove the existence and compute the partial derivatives:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} \frac{\frac{t^3}{t^2}}{t} = 1.$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \to 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \to 0} \frac{\frac{0}{t^2}}{t} = 0.$$

Then our candidate for the differential of f is $D_f(0, 0) = (1, 0)$.

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - f(0, 0) - (1, 0)\binom{x}{y}}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{\frac{x^3 - xy^2}{x^2 + y^2} - x}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{-2xy^2}{(x^2 + y^2)^{3/2}}.$$

Consider the above limit along straight lines, namely where $y = kx$ for some $k \in \mathbb{R}$ we obtain

$$\lim_{x \to 0} \frac{-2x(kx)^2}{(x^2 + (kx)^2)^{3/2}} = \lim_{x \to 0} \frac{-2x^3 k^2}{x^3(1 + k^2)^{3/2}} = \frac{-2k^2}{(1 + k^2)^{3/2}}.$$

We see that the limit depends on k, and in particular for $k = 0$ and $k = 1$ the limits are 0 and $-\frac{1}{\sqrt{2}}$ respectively. Then f is not differentiable at $(0, 0)$.

c. (16 points)
Prove that f^{-1} exists and differentiable in a neighborhood of $(1,2)$, for $f(x,y) = (xy^2, x^2+y)$, and compute $D_{f^{-1}}(1,2)$.

Solution: We rely on the Inverse Function Theorem: Let $A \subseteq \mathbb{R}^n$ open, $f : A \to \mathbb{R}^n$ C^1, and $a \in A$ such that $J_f(a) \neq 0$, then there open neighborhoods $U \subseteq A, V = f(U) \subseteq \mathbb{R}^n$ such that $f|_U : U \to V$ is $1-1$, and the inverse function $f^{-1} : V \to U$ is also C^1 and we have $D_{f^{-1}}(f(a)) = [D_f(a)]^{-1}$.

Here $(1,2)$ is in the image of f, and observe that $f(1,1) = (1,2)$. Set $a = (1,1)$, then

$$D_f(a) = \det \begin{pmatrix} y^2 & 2xy \\ 2x & 1 \end{pmatrix} \Big|_{(x,y)=(1,1)} = [y^2 - 4x^2y]|_{(x,y)=(1,1)} = 1 - 4 = -3 \neq 0.$$

Then by the Inverse Function Theorem there is an open neighborhood V of $(1,2)$ such that f^{-1} exists and C^1 on V, and we have

$$D_{f^{-1}}(1,2) = [D_f(1,1)]^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = -\frac{1}{3} \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

MAT322 - Analysis in Several Dimensions
Homework 9

Due in Class: Thursday, April 10, 2014

1. Define the following sequence of sets: $I_0 = [0,1], I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], I_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$... Generally I_{n+1} is a union of 2^{n+1} closed intervals, and it is obtained from I_n by dividing every interval of I_n into three equal subintervals and throwing the middle open third. Now define $\mathcal{C} = \bigcap_{n=0}^{\infty} I_n$. $\mathcal{C}$ is called the (standard) Cantor Set.
   (a) Prove that $\mathcal{C}$ has measure zero.
   (b) Prove that $|\mathcal{C}| = \aleph$ [in particular $\mathcal{C}$ is uncountable] (**Hint:** Find a $1-1$ map from $\{0,1\}^{\mathbb{N}}$).
2. Prove or disprove by a counter example:
   (a) If $A \subseteq \mathbb{R}^n$ has measure zero then ∂A has measure zero.
   (b) If $A \subseteq \mathbb{R}^n$ is open then it does not have measure zero.
3. Let $Q \subseteq \mathbb{R}^n$ be a (closed) box, $f : Q \to \mathbb{R}$ continuous. Prove that the graph of f

$$\Gamma_f := \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} : x \in Q\}$$

   has measure zero in $\mathbb{R}^{n+1}$ (**Hint:** Uniform continuity).
4. Let $Q \subseteq \mathbb{R}^n$ be a box, $f : Q \to \mathbb{R}$ integrable on Q. Show that if $f(x) > 0$ for any $x \in Q$ then $\int_Q f > 0$.
5. Let $Q \subseteq \mathbb{R}^n$ be a box, $f : Q \to \mathbb{R}$ (bounded) integrable on Q.
   (a) Show that if $g : Q \to \mathbb{R}$ is another bounded function, such that $f(x) = g(x)$ for any $x \in Q \setminus E$ for some closed set E of measure zero, then g is integrable on Q and $\int_Q g = \int_Q f$.
   (b) Does (a) hold without the assumption that E is closed?

Measure Zero Sets Under Lipschitz Maps

Yaar Solomon

Proposition. *Let $S \subseteq \mathbb{R}^n$ be an open set, $E \subseteq S$ of measure zero, $g : S \to \mathbb{R}^n$ Lipschitz, then $g(E)$ has measure zero.*

Proof. Let K be the Lipschitz constant of g. First note that for a cube $D \subseteq S$ of edge length δ we have $diam(D) = \delta\sqrt{n}$, and therefore $diam(g(D)) \leq \delta K \sqrt{n}$ (Remark: by *cube* I mean a box that all of its n edges has the same length). Then for any $x \in g(D)$ we have $g(D) \subseteq \overline{B(x, \delta K\sqrt{n})} \subseteq \underbrace{\prod_{i=1}^{n} [x_i - \delta K\sqrt{n}, x_i + \delta K\sqrt{n}]}_{F}$. So $g(D)$ is contained in a cube F of edge length $2\delta K\sqrt{n}$, which implies that

$$(*) \quad v(F) = (2\delta K\sqrt{n})^n = \delta^n \underbrace{(2K\sqrt{n})^n}_{C} = v(D) \cdot C.$$

Let $\varepsilon > 0$. Since E has measure zero there exists boxes $Q_1, Q_2, ... \subseteq S$ such taht $E \subseteq \bigcup_{i=1}^{\infty} Q_i$ and $\sum_{i=1}^{\infty} v(Q_i) < \frac{\varepsilon}{2C}$. For each i cover Q_i with cubes $D_1^{(i)}, \ldots, D_{m_i}^{(i)}$ such that $Q_i \subseteq \bigcup_{j=1}^{m_i} D_j^{(i)}$, and $\sum_{j=1}^{m_i} v(D_j^{(i)}) \leq 2v(Q_i)$. For each of these cubes $D_j^{(i)}$ let $F_j^{(i)}$ be a cube containing $g(D_j^{(i)})$ with the property as in $(*)$. Then

$$g(E) \subseteq \bigcup_{i=1}^{\infty} g(Q_i) \subseteq \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{m_i} g(D_j^{(i)}) \subseteq \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{m_i} F_j^{(i)},$$

and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{m_i} v(F_j^{(i)}) = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} v(D_j^{(i)}) \cdot C \leq \sum_{i=1}^{\infty} 2Cv(Q_i) < 2C \cdot \frac{\varepsilon}{2C} = \varepsilon.$$

□

MAT322 - Analysis in Several Dimensions
Homework 10

Due in Class: Tuesday, April 22, 2014

1. Let $\mathcal{C}$ be the standard Cantor set (that was defined in question 1 of ex.9).
   (a) Explain why $\mathcal{C} = \left\{\sum_{i=1}^{\infty} \frac{a_i}{3^i} : a_i \in \{0, 2\}\right\}$ (this is the set of numbers in $[0, 1]$ that their ternary expansions contain only the digits 0 and 2).
   (b) Prove that the function $f : \mathcal{C} \to [0, 1]$ defined by $\sum_{i=1}^{\infty} \frac{a_i}{3^i} \overset{f}{\mapsto} \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1}}$ is continuous.
   (c) Deduce that the image of a measure zero set under a continuous function does not necessarily have measure zero.
2. Let $\varnothing \neq S \subseteq \mathbb{R}^n$ be Jordan measurable. Prove that for every $\varepsilon > 0$ there are boxes $Q_1, \dots, Q_k, \dots, Q_n$ $(0 \leq k < n)$ such that $\bigcup_{i=1}^{k} Q_i \subseteq S \subseteq \bigcup_{i=1}^{n} Q_i$ and $\sum_{i=k+1}^{n} v(Q_i) < \varepsilon$ (**<u>Hint:</u>** Use the definitions of the upper and lower integrals).
3. Find an example for two integrable functions $f, g : [0, 1] \to \mathbb{R}$ such that the composition $f \circ g$ is not integrable.
4. Let $f : [0, 1]^2 \to \mathbb{R}$ be the function $f(x, y) = \begin{cases} 1 & x \in \mathbb{Q} \\ 2y & x \notin \mathbb{Q} \end{cases}$.
   (a) Prove that the integral $\int_0^1 \overline{\int_0^1} f(x, y)dxdy$ exists, and compute its value.
   (b) Prove that the integral $\int_0^1 \int_0^1 f(x, y)dydx$ exists, and compute its value.
   (c) Deduce that f is not integrable on $[0, 1]^2$.
5. For each of the following pairs of a set $D \subseteq \mathbb{R}^n$ and a function $f : D \to \mathbb{R}$, draw the domain D, justify the existence of the integral $\int_D f$ and compute it.
   <u>Remark:</u> For those of you how write in LaTex, leave some room for the drawings and draw by hand.
   (a) $f(x, y) = x^2y + xy^2$, $D = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, 0 \leq y \leq |x|\}$.
   (b) $f(x, y, z) = x + y$, $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x^2 \leq y \leq x, 0 \leq z \leq x\}$.

(c) $f(x, y, z) = x$, $D \subseteq \mathbb{R}^3$ is enclosed by the planes $x = 0, y = 0, z = 0, x + y + z = 1$.

MAT322 - Analysis in Several Dimensions
Homework 11

Due in Class: Tuesday, April 29, 2014

Remark: For those of you how write in LaTex, leave some room for the drawings and draw by hand. When it is difficult to draw, like in 1(b), explain in words.

1. For each of the following pairs of a set $D \subseteq \mathbb{R}^n$ and a function $f : D \to \mathbb{R}$, draw the domain D, justify the existence of the integral $\int_D f$ and compute it.

   (a) $f(x, y) = x^2$, $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, 1 \leq x^2 + y^2 \leq 2\}$.

   (b) $f(x, y, z) = z^2$, $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 + z^2 \leq 4\}$.

   (c) $f(x, y, z) = (x^2 + y^2)(x^2 + y^2 + z^2)$, $D = B(0, 1) \subseteq \mathbb{R}^3$.

   (d) $f(x, y) = (x + y)^2$, $D \subseteq \mathbb{R}^2$ is the set that is enclosed between the lines $x = y, x + y = 2, x + y = 4, x^2 - y^2 = 4$ (**Hint:** You may consider the change of variables $u = x + y, v = x - y$).

   (e) $f(x, y) = e^{x^2+y^2}$, $D = B(0, 1) \subseteq \mathbb{R}^2$.

2. Let $S_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \forall i \ \ x_i \geq 0, x_1 + x_2 + \ldots + x_n \leq 1\}$. Draw S_n for $n = 1, 2, 3$, and compute the volume of S_n for every n (**Hint:** You may consider the change of variables $y_i = \sum_{j=1}^{i} x_j$).

3. Draw the following sets S and compute their volumes.

   (a) $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 \leq 1\}$ where $0 < a \leq b \leq c$.

   (b) $S = B(0, 1) \cap \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 \leq \frac{1}{4}\}$.

   (c) $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z^2 \leq 4\}$.

4. Define $f : [0, \infty)^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 0, & x = 0 \\ \arctan\left(\frac{y}{x}\right), & 0 \leq \arctan\left(\frac{y}{x}\right) \leq \frac{\pi}{4} \\ -\arctan\left(\frac{y}{x}\right) + \frac{\pi}{4}, & \frac{\pi}{4} \leq \arctan\left(\frac{y}{x}\right) \leq \frac{\pi}{2} \end{cases}$$

Prove that $\lim_{t\to\infty} \int_{B(0,t)} f(z)dz$ exists, but $\lim_{t\to\infty} \int_{[0,t]^2} f(z)dz$ does not (in the first one I mean that $f = 0$ outside of $[0,\infty)^2$).

5. Compute the following improper integrals $\int_S f(z)dz$, where:

   (a) $S = [0,\infty)^2$, $f(x,y) = (x+y)e^{-x-y}$.

   (b) $S = B(0,1) \subseteq \mathbb{R}^2$, $f(x,y) = \ln\sqrt{x^2+y^2}$.

   (c) $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 1\}$, $f(x,y) = (x^2+y^2)^{-(2+\varepsilon)}$, where $\varepsilon > 0$ is some fixed parameter.

MAT322 - Analysis in Several Dimensions
Homework 12

Due in Class: Thursday, May 8, 2014

1. Compute the following line integrals:
   (a) $\int_\gamma (x+y)\,dx + (x-y)\,dy$, where $\gamma(t) = \binom{4t^3}{t^2+2t-4}$, and $t \in [0,1]$.
   (b) $\int_\gamma \frac{(3x^2-y^2)(x^2+y^2)}{x^2y} dx + \frac{(3y^2-x^2)(x^2+y^2)}{xy^2} dy$, where $\gamma(t) = \binom{t+\cos^2 t}{1+\sin^2 t}$, and $t \in [0, \pi/2]$.
   (c) $\int_\gamma e^x \sin y dx + e^x \cos y dy$, where γ is one loop, counterclockwise, on the ellips $x^2 + xy + y^2 = 1$.
   (d) $\int_\gamma (3x^2y^2 + \cos(xy) - xy\sin(xy))\,dx + (2x^3y - x^2\sin(xy) + 10y)\,dy$, where $\gamma(t) = \binom{-t+\cos^2 t}{1+\sin^2 t}$, and $t \in [0, \pi/2]$.

2. Decide whether the following vector fields f are conservative or not, and prove it (the domain A is $\mathbb{R}^n$ unless indicated otherwise). If f is conservative, find the potential function.
   (a) $f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ on $A = \{(x,y) \in \mathbb{R}^2 : x > 0\}$.
   (b) $f(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $A = \{(x,y) \in \mathbb{R}^2 : x > 0\}$.
   (c) $f(x,y) = \left(\frac{x}{(x^2+y^2)^2}, \frac{y}{(x^2+y^2)^2}\right)$ on $A = \{(x,y) \in \mathbb{R}^2 : x > 0\}$.
   (d) $f(x,y,z) = (y, z, x)$.
   (e) $f(x,y,z) = (y + 2xy^2, x + 2x^2y, z^3 + \sin z)$.

3. In each of the following sections $\|x\|$ denotes the Euclidean norm on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field.
   (a) Let $\phi : [0,\infty) \to \mathbb{R}^n$ be a continuous map, which is differentiable at $(0,\infty)$, show that $f(x) = \phi(\|x\|)$ is conservative on $\mathbb{R}^n \overset{\text{iff}}{\Longleftrightarrow} \phi$ is constant.
   (b) Suppose that $\psi : [0,\infty) \to \mathbb{R}$ and let $f(x) = \psi(\|x\|)x$. Show that if ψ is differentiable at $(0,\infty)$ then for every $i \neq j$ we have $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ on $\mathbb{R}^n \setminus \{0\}$.

(c) Let ψ, f be as in (b). Given only that ψ is continuous, prove that f is conservative on $\mathbb{R}^n$. **Hint:** Fundamental Theorem of Calculus (from one variable calculus).

4. Let $a, b, c, d \in \mathbb{R}$ be such that the vectors $\binom{a}{b}, \binom{c}{d} \in \mathbb{R}^2$ are linearly independent, and let γ be the boundary of the parallelogram with vertices $\binom{0}{0}, \binom{a}{b}, \binom{c}{d}$ and $\binom{a+c}{b+d}$, oriented counterclockwise. Compute $\int_\gamma xdy - ydx$.

5. Compute the following line integrals, where γ is the boundary of D, oriented counterclockwise:

   (a) $\int_\gamma (4x^2 + x - 2y)dx + (4y^2 + 2y - x)dy$ where $D \subseteq \mathbb{R}^2$ is the domain bounded by $x = 3, y = 2x$ and $y = 3x$.

   (b) $\int_\gamma (4x^3y + sin^2(x))dx + (x^4 + x^2y)dy$ where $D \subseteq \mathbb{R}^2$ is the domain bounded by $x = 0, y = 0$ and $y = \sqrt{4 - x^2}$.