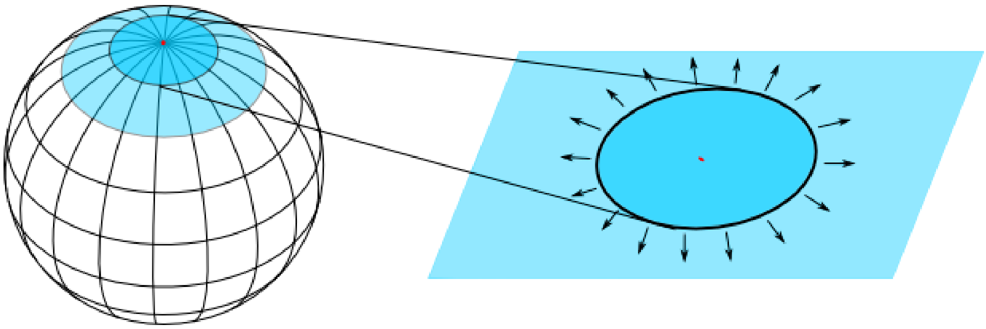
Matthew Badger | **Department of Mathematics** | **Stony Brook University**

MAT 322 - Spring 2012

Analysis in Several Dimensions



Spring 2012 Office Hours	
M	10:45 - 11:45
T	3:00 - 4:00
W	10:45 - 11:45
H	By Appointment
F	By Appointment

Office hours are held in Math Tower 4-117.

Announcements

Course Syllabus | **Guidelines on HW** | **Typing Your Homework**

Short Paper: **List of Topics**

Chosen Topics: B, D, E, I, L, N

- April 30: The **final exam** is in our classroom on Monday, May 14 from 11:15 am - 1:45 pm.
- April 30: Short papers are due in class on Friday, May 4.

[**Old Announcements**]

Homework

Homework 11
Due May 2, 2012
This is the last homework assignment!

Homework 10
Due April 25, 2012

Homework 9
Due April 18, 2012

Homework 8
Due April 11, 2012

Homework 7
Due March 21, 2012
Includes instructions on submitting corrections to the midterm.

Homework 6
Due March 7, 2012
Please read special instructions on the assignment sheet.

Homework 5
Due February 29, 2012.

Homework 4

Due February 22, 2012.

Homework 3

Due February 15, 2012.

Homework 2

Due February 8, 2012.

Homework 1

Due February 1, 2012.

Math 322, Analysis in Several Dimensions, Spring 2012 — Syllabus

Light Engineering Lab 154 — Monday, Wednesday, Friday 9:35 – 10:30

Instructor: Dr. Matthew Badger (`badger@math.sunysb.edu`)
Office: Math Tower 4-117
Office Hours: Tuesday and Wednesday 10:40 – 11:40 and by appointment

Course Description

The derivative and definite integral of the calculus of functions of a single variable generalize to operations on functions of several variables, often doing so in surprising and unexpected ways. In this course, we will rigorously develop differentiation and integration of functions in the plane (dimension $n = 2$), in space (dimension $n = 3$), and in higher dimensional Euclidean spaces $\mathbb{R}^n$. We will observe some phenomenon which behave independently of the underlying dimension and we will see other phenomenon which behave radically different depending on the dimension.

Topics to be covered (some as time permits):

- Differentiability, partial derivatives, C^1 and C^k functions, the inverse function theorem, the implicit function theorem, Lipschitz functions, Rademacher's theorem.
- Riemann integration on Euclidean domains, partitions of unity, Riemann integration on manifolds with boundary, differential forms, Stokes' theorem, harmonic functions.

Required Resources

- **Class Website:** `www.math.sunysb.edu/~badger/` → Link to MAT 322
- **Textbook:** James R. Munkres, *Analysis on Manifolds*, Westview Press.

About Attendance

Attendance is highly encouraged. **Lectures may include material not in the textbook!**

Graded Components

- **Homework**
  There will be weekly homework assignments due in class on most Wednesdays, starting on Wednesday, February 1. The homework assignments will be posted on the course webpage. The two lowest homework assignment scores for each student will be dropped.
- **Midterm Exam**
  There will be one closed book, closed notes midterm exam in class on Friday, March 9th from 9:35 am - 10:30 am.
- **Short Paper** (Optional)
  Each student may complete a 5–10 page paper on a topic to be mutually agreed upon by the student and the instructor. Suggested topics will be provided to students near the middle of the course. Papers are due in class on the last day of lecture.
- **Final Exam**
  There will be one closed book, closed notes final exam as scheduled by the university on Monday, May 14th from 11:15 am - 1:45 pm.

The *final grade* for the class will be based on your course average (see below) and participation. Grades will not be curved. Certain averages will initially guarantee the following grades:

90% guarantees an 'A', 80% guarantees a 'B', 70% guarantees a 'C'.

The average required to obtain a certain grade may be lowered throughout the quarter, but will not be raised. Any changes to the scale will be announced in lecture and on the course website. Your *course average* will be determined by the highest of the following two options:

- Option 1: 40% Homework + 15% Midterm Exam + 15% Short Paper + 30% Final Exam
- Option 2: 40% Homework + 20% Midterm Exam + 40% Final Exam

Disability Support Services

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services (631) 632-6748 or

⟨`http://studentaffairs.stonybrook.edu/dss/`⟩.

They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

⟨`http://www.sunysb.edu/facilities/ehs/fire/disabilities`⟩.

Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

⟨`http://www.stonybrook.edu/uaa/academicjudiciary/`⟩.

Critical Incident Management

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

Current as of: January 24, 2012

MAT 322 - Spring 2012

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How to Type Your Homework

To typeset mathematics, most mathematicians use some version of LaTeX. You are encouraged (but not required) to use LaTex to type your homework. To get started, you can go to **the LaTeX project**. Under the "Getting LaTeX" link, you will find several distributions for Windows, Mac and Linux. On Windows I recommend using the **MikTeX distribution** with the **WinEDT** interface.

For an example here is a **homework** (PDF) that I completed while I was in grad school. This document was produced by compiling the LaTex file **hw-sample.tex** with pdfLaTeX (from WinEDT).

Instructions: The short paper is an optional component of the course (see course syllabus). You will find a list of possible topics for the paper below, with each topic consisting of one or more theorems in analysis. The content of the paper is up to you. For example, your paper could provide a self-contained proof of the topic's theorem and present some applications of the theorem. Your paper could also discuss the history of the theorem. Topics are available on a first come, first serve basis. No topic may be repeated. Please let me know which topic you would like for your paper as early as possible. The paper should be 5 – 10 pages, typed and single spaced. Use a reasonable size for fonts (10pt, 11pt or 12pt) and margins ($1 - 1\frac{1}{4}$ inches). You are encouraged to turn in a draft of your paper to me by April 20. I will provide feedback on any drafts submitted. Your final paper must be turned in on or before the last day of class on May 4.

Topics for Short Papers

Topic A (Rank Theorem)

- **Theorem:** Let $A \subset \mathbb{R}^m$ be an open set. Let $f : A \to \mathbb{R}^n$ be a C^k function ($k \geq 1$) such that $Df(x)$ has rank r for all x in a neighborhood of $x_0 \in A$. Then there exist an open set $U \subset \mathbb{R}^m$, an open set $V \subset \mathbb{R}^m$ containing x_0, and a map $h : U \to V$ of class C^k with inverse $h^{-1} : V \to U$ of class C^k such that $f \circ h$ depends only on $x_1, \ldots, x_r$. That is, $(f \circ h)(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = \tilde{f}(x_1, \ldots, x_r)$ for some C^k map $\tilde{f}$.

Topic B (Morse Lemma)

- Let $A \subset \mathbb{R}^m$ be an open set and let $f : A \to \mathbb{R}$ be a function. The *Hessian* of f is the matrix $(Hf)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ of second partial derivatives of f (assuming they exist).
- **Theorem:** Let $f : A \to \mathbb{R}$ be class C^∞. If $Df(x_0) = 0$ and $Hf(x_0)$ is invertible, then there exist neighborhoods U of x_0 and V of 0 in $\mathbb{R}^m$ and a smooth map $g : V \to U$ with a smooth inverse such that $f \circ g = h$ has the form
$$h(y) = f(x_0) - [y_1^2 + \cdots + y_k^2] + [y_{k+1}^2 + \cdots + y_n^2]$$
where k is some fixed integer between 0 and m.

Topic C (Blaschke's Selection Theorem)

- The *Hausdorff distance* between two non-empty bounded sets $A, B \subset \mathbb{R}^n$ is defined by
$$\mathrm{HD}[A, B] = \sup_{x \in A} \inf_{y \in B} \|x - y\| + \sup_{z \in B} \inf_{w \in A} \|z - w\|.$$
It is a fact that Hausdorff distance is a *metric* on the collection $\mathcal{C}_K$ of non-empty closed subsets of a compact set $K \subset \mathbb{R}^n$, i.e. for all $A, B, C \in \mathcal{C}_K$,
  i) $\mathrm{HD}[A, B] \geq 0$ with equality if and only if $A = B$
  ii) $\mathrm{HD}[A, B] = \mathrm{HD}[B, A]$
  iii) $\mathrm{HD}[A, B] \leq \mathrm{HD}[A, C] + \mathrm{HD}[C, B]$.
- **Theorem:** The metric space $(\mathcal{C}_K, \mathrm{HD})$ is sequentially compact, i.e. given any sequence $(F_i)_{i=1}^\infty$ of non-empty closed subsets of a compact set $K \subset \mathbb{R}^n$, there exists a closed set $F \subset K$ and a subsequence $(F_{i_j})_{j=1}^\infty$ of $(F_i)_{i=1}^\infty$ such that $\mathrm{HD}[F_{i_j}, F] \to 0$ as $j \to \infty$.

Topic D (Sard's Theorem)

- Let $A \subset \mathbb{R}^m$ be an open set and let $f : A \to \mathbb{R}^n$.
- The set of *critical points* of f is $\mathcal{C}_P(f) = \{x \in A : \operatorname{rank} Df(x) < n\}$.
- The set of *critical values* of f is $\mathcal{C}_V(f) = \{f(x) \in \mathbb{R}^n : x \in \mathcal{C}_P(f)\}$.
- **Theorem:** If f is class C^k for $k \geq \max\{m - n + 1, 1\}$, then $\mathcal{C}_V(f)$ has measure zero.

Topic E (Kirszbraun's Theorem)

- Let $A \subset \mathbb{R}^m$ (not necessarily open). A map $f : A \to \mathbb{R}^n$ is called *Lipschitz* if there exists a constant $C < \infty$ such that $\|f(x) - f(y)\| \leq C\|x - y\|$ for every $x, y \in A$.
- The *Lipschitz constant* of f, denoted by $\operatorname{Lip} f$, is the smallest constant $C \geq 0$ such that the Lipschitz inequality $\|f(x) - f(y)\| \leq C\|x - y\|$ holds for all $x, y \in A$.
- We say that $F : \mathbb{R}^m \to \mathbb{R}^n$ is a *Lipschitz extension* of f, if F is Lipschitz and $F(x) = f(x)$ for every $x \in A$.
- **Theorem:** Every Lipschitz map $f : A \to \mathbb{R}^n$ admits a Lipschitz extension F such that $\operatorname{Lip} F = \operatorname{Lip} f$.

Topic F (Rademacher's Theorem)

- **Theorem:** Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map. Then f is differentiable at every point $x \in \mathbb{R}^m$, except on a set $A \subset \mathbb{R}^m$ of measure zero. (This topic is a challenging project. A rigorous proof for $m \geq 2$ is too advanced for this course. Give a proof for $m = 1$ and give an outline of a proof for $m \geq 2$.)

Topic G (Dominated Convergence Theorem for Riemann Integrals)

- **Theorem:** Let $f_k : Q \to \mathbb{R}$ be a sequence of Riemann integrable functions defined on a closed rectangle $Q \subset \mathbb{R}^m$. Suppose that f_k converges pointwise to a Riemann integrable function $f : Q \to \mathbb{R}$. If there exists a constant $M < \infty$ such that $|f_k(x)| \leq M$ for all $x \in Q$ and for all $k \geq 1$, then $\lim_{k\to\infty} \int_Q |f_k(x) - f(x)| = 0$. In particular,
$$\lim_{k\to\infty} \int_Q f_k(x) = \int_Q \lim_{k\to\infty} f_k(x) = \int_Q f(x).$$
- W.A.J. Luxemburg, *Arzela's Dominated Convergence Theorem for the Riemann Integral*, Amer. Math. Monthly **78** (1971), no. 9, 970–979. (See for a proof when $m = 1$.)

Topic H (Sets of Finite Length are Curves)

- Note. This project can be done without background in measure theory.
- Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuous function. The *length* of the curve γ is
$$\ell(\gamma) = \sup_{a=t_0\leq t_1\leq\cdots\leq t_k=b} \sum_{i=1}^{k} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$
- Let $\operatorname{diam} E = \sup_{x,y\in E} \|x - y\|$ denote the diameter of a bounded set E. The *length* or *1-dimensional Hausdorff measure* $\mathcal{H}^1(A)$ of a set $A \subset \mathbb{R}^n$ is defined by
$$\mathcal{H}^1(A) = \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam} E_i : A \subset \bigcup_{i=1}^{\infty} E_i \text{ and } \operatorname{diam} E_i \leq \delta \text{ for all } i \right\}.$$
- **Theorem:** If $A \subset \mathbb{R}^n$ is a compact, connected set and $\mathcal{H}^1(A) < \infty$, then there exists a continuous function $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\gamma([0, 1]) = A$ and $\ell(\gamma) < \infty$.

Topic I (Peano's Space Filling Curve)

- **Theorem:** There exists a continuous function P from $[0,1]$ *onto* $[0,1]\times[0,1]$. Moreover, P can be found which satisfies the *Hölder condition with exponent* $1/2$, i.e. there exists a constant $C<\infty$ such that $|P(t)-P(u)|\leq C|t-u|^{1/2}$ for all $t,u\in[0,1]$.

Topic J (Brunn-Minkowski Inequality)

- If $A,B\subset\mathbb{R}^n$, define $A+B=\{x\in\mathbb{R}^n : x=x'+x'' \text{ for some } x'\in A \text{ and } x''\in B\}$.
- **Theorem:** Assume that A, B and $A+B$ are bounded, open, rectifiable sets in $\mathbb{R}^n$. Then $v(A+B)^{1/n}\geq v(A)^{1/n}+v(B)^{1/n}$ where $v(A)=\int_A 1$ denotes the volume of A.

Topic K (Generalized Riemann Integral)

- Define the *generalized Riemann integral* and prove the fundamental theorem of calculus.
- **Theorem:** If $F:[a,b]\to\mathbb{R}$ is differentiable at every point of $[a,b]$, then $f=F'$ is generalized Riemann integrable on $[a,b]$ and

$$\int_a^b f = F(b)-F(a).$$

- R.G. Bartle, *Return to the Riemann Integral*, Amer. Math. Monthly **103** (1996), no. 8, 625–632.

Topic L (Harmonic Functions and the Mean Value Property)

- Let $A\subset\mathbb{R}^n$ be an open set. A function $h:A\to\mathbb{R}$ is called *harmonic* if h is class C^2 and the sum of the pure second partial derivatives of h vanish: $h_{x_1x_1}+\cdots+h_{x_nx_n}=0$ for all $x\in A$.
- A continuous function $f:A\to\mathbb{R}$ is said to have the *mean value property* if for every open ball $B(x_0,r)\subset A$,

$$\frac{1}{\omega_n r^n}\int_{B(x_0,r)} f(x) = f(x_0).$$

  Here ω_n denotes the volume of the unit ball $B(0,1)$.
- **Theorem:** Let $f:A\subset\mathbb{R}^n$ be a function of class C^2. Then f is harmonic if and only if f has the mean value property.

Topic M (De Rham Cohomology and Poincaré's Lemma)

- Give an introduction to de Rham cohomology, including a proof of Poincaré's Lemma:
- **Theorem:** Let A be a star-shaped open set in $\mathbb{R}^n$. Every closed k-form on A is exact.

Topic N (Plateau's Problem)

- Discuss the statement, history and solution of Plateau's problem. To get started, see:
- F. Almgren, *Plateau's Problem: An Invitation to Varifold Geometry*, Revised Edition, Student Mathematical Library, vol. 13, American Mathematical Society, Providence, RI, 2001.

Topic O (Honeycomb and Kepler Conjectures)

- Discuss the statement, history and solution of the honeycomb and Kepler conjectures. To get started, see:
- T.C. Hales, *Cannonballs and Honeycombs*, Notices Amer. Math. Soc. **47** (2000), no. 4, 440-449.

Topic P

- You may propose your own topic for the paper. However, please discuss your idea with me before starting on the project.

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Announcements

- April 30: The final exam is in our classroom on Monday, May 14 from 11:15 am - 1:45 pm.
- April 30: Short papers are due in class on Friday, May 4.
- April 10: If you are writing the short paper, please remember to tell me your topic (if you haven't done so already). Aim to turn in a draft of the paper by April 20.
- March 27: Start Reading Chapter 5
- February 24: In-class midterm on Friday, March 9 will cover Chapters 2 and 3.
- February 20: Suggested topics for the short paper are now posted!
- February 15: Office hours on Tuesdays have been changed to 3pm - 4pm.
- February 15: NEW COURSE POLICY: You may turn in each homework assignment once. Late homework will be accepted, but will be penalized for being turned in late.
- February 13: Start reading Chapter 3.
- February 3: Solutions to Problems C and F from HW 1 are available on **BlackBoard**.
- January 30: HW 2 Updated (The first version had 7 questions, the new version has 6 questions.)
- January 30: Notice the change to office hours (right).
- January 25: **Notes** for Lecture 1 (and part of Lecture 2).

Due In Class: May 2, 2012

Reading: Finish reading Chapter 6. Start reading Chapter 7.

Turn in the following problems.

Problem A. Exercise 1 on page 251.

Problem B. Exercise 2 on page 251.

Problem C. Exercise 2 on page 260.

Problem D. Exercise 3 on page 260.

Problem E. Exercise 4 on page 260.

Problem F. Exercise 4 on page 265.

Due In Class: April 25, 2012

Reading: Continue reading Chapter 6.

Turn in the following problems.

Problem A. Exercise 4 on page 226.

Problem B. Exercise 8 on page 226.

Problem C. Exercise 1 on page 236.

Problem D. Exercise 3 on page 236.

Problem E. Exercise 2 on page 243.

Problem F. Exercise 3 on page 243.

Due In Class: April 18, 2012

Reading: Finish Chapter 5. Start Chapter 6.

Turn in the following problems.

Problem A. Exercise 6 on page 202.

Problem B. Exercise 4 on page 209.

Problem C. Exercise 2 on page 217.

Problem D. Exercise 4 on page 217.

Due In Class: April 11, 2012

Reading: Start Chapter 5.

Turn in the following problems.

Problem A. Exercise 1 on page 187.

Problem B. Exercise 3 on page 187.

Problem C. Exercise 1 on page 193.

Problem D. Exercise 2 on page 193.

Problem E. Exercise 1 on page 202.

Problem F. Exercise 3 on page 202.

Due In Class: March 21, 2012

Corrections to Midterm: For any problem on the midterm on which you lost more than a few points, you may turn in a correction in order to receive additional partial credit on the problem. A correction consists of a complete, correct solution to the question. Corrections are due no later than in-class on March 21. Please return your midterm with any corrections submitted.

Reading: Start Chapter 4.

Turn in the following problems.

Problem A. Exercise 3 on page 144.

Problem B. Exercise 2 on page 151.

Problem C. Exercise 5 on page 151.

Problem D. Exercise 6 on page 151.

Due In Class: March 7, 2012

This homework assignment has a different format than previous assignments. Please read the instructions carefully. It is not possible to complete this assignment the day before it is due.

Step 1. Complete Exercise 7 on page 111 and write down a draft of the proof.

Step 2. Set aside your draft of the proof for at least 1 day.

Step 3. Reread your draft, make corrections and then rewrite the proof.

Step 4. Bring your second draft (not your first draft) to class on March 7.

Step 5. During class on March 7 we will do a workshop on writing. Students will divide into pairs (and maybe one triple) and read their partner's proof. Each student will make comments on the proof that he or she reads and then discuss it with their partner.

Step 6. Near the end of the class, we will wrap up with a group discussion.

Rubric: Your grade on this assignment will be primarily based on participation.

Due In Class: February 29, 2012

Reading: Finish Chapter 3.

Turn in the following problems.

Solutions to Problems A through C only require 1–3 sentences each.

Problem A. Exercise 3 on page 97.

Problem B. Exercise 4 on page 97.

Problem C. Exercise 5 on page 97.

Problem D. Exercise 6 on page 97.

Problem E. Exercise 1 on page 103.

Problem F. Exercise 3 on page 103 (a) and (b). *Bonus points for (c).*

Due In Class: February 22, 2012

Start reading Chapter 3.

Turn in the following problems.

Problem A. Exercise 1 on page 90.

Problem B. Exercise 2 on page 90.

Problem C. Exercise 4 on page 90.

Problem D. Exercise 7 on page 90. (You do not need to prove Exercise 6.)

Problem E. Exercise 1 on page 97.

Problem F. Exercise 2 on page 97.

Due In Class: February 15, 2012

Turn in the following problems.

Problem A. Let $Q = [-\frac{1}{2}, \frac{1}{2}]^n$ denote the closed n-dimensional unit cube, and let $Q^0 = (-\frac{1}{2}, \frac{1}{2})^n$ be its interior. Let $f : Q \to \mathbb{R}^n$ be a continuous map and assume that f is differentiable on Q^0. Prove that if f is one-to-one and $Df(x)$ is invertible for each $x \in Q^0$, then $f|_{Q^0}$ is an open map, i.e. prove that $f(U)$ is an open set for each open set $U \subset Q^0$.

[*Remark:* Do not assume that $f|_{Q^0}$ is C^1. *Hint:* Repeat 'Step 2' in the proof of Theorem 8.2.]

Problem B. Exercise 1 on page 70.

Problem C. Exercise 3 on page 70.

Problem D. Exercise 4 on page 70.

Problem E. Exercise 3 on page 79.

Problem F. Use the Implicit Function Theorem to prove the Inverse Function Theorem.

[*Hint:* Consider the function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(x, y) = f(y) - x$.]

Due In Class: February 8, 2012

Turn in the following problems.

Problem A. Exercise 4 on page 54.

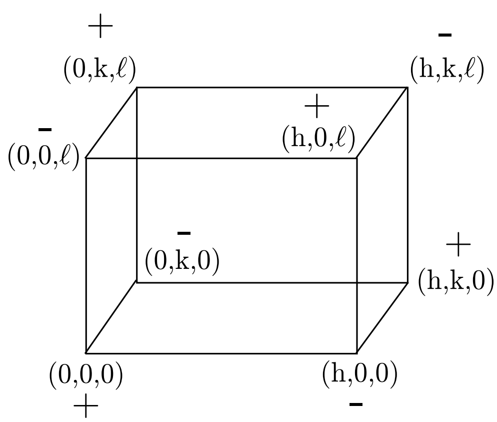
Problem B. Exercise 5 on page 54.

Problem C. Exercise 10 on page 54.

Problem D. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\tau = (\tau_1, \tau_2, \tau_3)$ be permutations of $(1, 2, 3)$. Prove that if $f : \mathbb{R}^3 \to \mathbb{R}$ is a function of class C^3, then

$$D_{\sigma_3} D_{\sigma_2} D_{\sigma_1} f(0, 0, 0) = D_{\tau_3} D_{\tau_2} D_{\tau_1} f(0, 0, 0).$$

Hint: Modify the proof of Theorem 6.3. The following diagram might be useful.



Problem E. Exercise 2 on page 63.

Problem F. Exercise 3 on page 63.

Math 322, Analysis in Several Dimensions, Spring 2012 — Homework #1

Due In Class: February 1, 2012

Warm-up: Read Chapter 1. Do some exercises.

Convention:

- If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denotes the Euclidean norm of x and $|x| = \max\{|x_1|, \ldots, |x_n|\}$ denotes the supremum norm of x.

Turn in the following problems.

Problem A.

(i) Prove the inequality $|x| \leq \|x\| \leq \sqrt{n}\,|x|$ for all $x \in \mathbb{R}^n$.
(ii) Describe the sets $S = \{x \in \mathbb{R}^n : |x| = \|x\|\}$ and $T = \{x \in \mathbb{R}^n : \|x\| = \sqrt{n}|x|\}$.

Problem B. Let $x \in \mathbb{R}^n$ and $r > 0$. Describe the open sets $U_2(x,r) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$ and $U_\infty(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. Prove that there exist constants $c = c(n)$ and $C = C(n)$ such that $U_\infty(x, cr) \subset U_2(x,r) \subset U_\infty(x, Cr)$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Problem C. A map $a : \mathbb{R}^m \to \mathbb{R}^n$ is *affine* if $a(x) = z + Ax$ for some $z \in \mathbb{R}^n$ and A is an $n \times m$ matrix. Prove that a map $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $x \in \mathbb{R}^m$ (as defined in the book) if and only if there exists an affine map $a : \mathbb{R}^m \to \mathbb{R}^n$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|y - x| < \delta$ implies that $|f(y) - a(y)| < \epsilon|y - x|$ (this says f is *locally well-approximated at x by a unique affine map.*)

Problem D. Exercise 1 on page 48 of the book.

Problem E. Exercises 2–5 on pages 48–49 of the book. (You do not need to turn in your work for this problem. Just record answers to the questions for each function.)

Problem F. Exercise 1 on page 54 of the book.