	Fall 2019 **MAT 319: Foundations of Analysis**	Fall 2019 **MAT 320: Introduction to Analysis**
Schedule	TuTh 10:00-11:20 **Library E4310 (CHANGE)** (through 10/2: joint lectures in Math P-131)	TuTh 10:00-11:20 Javits Lecture 103, then Math P-131
Instructor	Yu Li	David Ebin
Office hours	Tu 11:30-12:30 in Math P-143, Tu 2:00-3:30 and Th 11:30-12:00 in Math 3-109	Tu, Th 11:30-1:00 in Math 5-107
Recitation	MW 11:00-11:53 Harriman 112	MW 11:00-11:53 P-131
TA	Apratim Chakraborty	Owen Mireles Briones
Office hours	MW 2:00-3:00, Th 4:00-5:00 in MLC	W 4:00-6:00 in MLC, W 3:00-4:00 in Math 2-105
Description	A careful study of the theory underlying topics in one-variable calculus, with an emphasis on those topics arising in high school calculus. The real number system. Limits of functions and sequences. Differentiations, integration, and the fundamental theorem. Infinite series.	A careful study of the theory underlying calculus. The real number system. Basic properties of functions of one real variable. Differentiation, integration, and the inverse theorem. Infinite sequences of functions and uniform convergence. Infinite series.
Overview	The purpose of this course is to build rigorous mathematical theory for the fundamental calculus concepts, sequences and limits, continuous functions, and derivatives. We will rely on our	An introductory course in analysis, required for math majors. It provides a closer and more rigorous look at material which most students encountered on an informal level during their first two

	intuition from calculus, but (unlike calculus) the emphasis will be not on calculations but on detailed understanding of concepts and on proofs of mathematical statements.	semesters of Calculus. Students learn how to write proofs. Students (especially those thinking of going to graduate school) should take this as early as possible.
Prerequisites	C or higher in MAT 200 or permission of instructor; C or higher in one of the following: MAT 203, 205, 211, 307, AMS 261, or A- or higher in MAT 127, 132, 142, or AMS 161. *Math majors are required to take either MAT 319 or MAT 320*	
Textbook	Robert G. Bartle and Donald R. Sherbert, *Introduction to Real Analysis*, **4th edition**	
Homework	Weekly problem sets will be assigned, and collected in *Wednesday recitation*. The emphasis of the course is on writing proofs, so please try to write legibly and explain your reasoning clearly and fully. You are encouraged to discuss the homework problems with others, but your write-up must be your own work. *Late homework will never be accepted*, but under documented extenuating circumstances the grade may be dropped.	
Grading	Homework: 20%, Midterm I: 20%, Midterm II: 20%, Final: 40%.	

Syllabus/schedule (subject to change)

All joint lectures through 10/2 meet in Math P-131.
First recitation on Wed 8/27, second recitation Wed 9/3.
During joint lectures through 10/2, students with last names starting A-O attend recitation in Harriman 112, students with last names P-Z attend recitation in Lgt Engr Lab 152
Recommendations on choosing MAT 319 vs MAT 320 will be made based upon your performance on the first midterm and homework to that date.

1.	Joint class: Introduction,	Read pages 1-19

	motivation: what are real numbers? (Ebin)	
2.	Joint class: Properties of numbers; induction; concept of a field. (Ebin)	**HW due 9/3:** 1.3, 1.4, 1.10, 1.12, 2.2, 2.5, 3.1, 3.4, 3.6
	No class: day after Labor Day	
3.	Joint class: Completeness axiom for real numbers; Archimedean property. (Ebin)	Read pages 20-27; **HW due 9/10:** parts eghimr of: 4.1,4.2,4.3,4.4; and 4.8,4.10,4.11,4.12,4.14
4.	Joint class: Infinity, unboundedness. Intro to sequences. (Ebin)	Read pages 28-38
5.	Joint class: Limit of a sequence. (Ebin)	**HW due 9/17:** 5.2, 5.6, 7.3, 7.4, 8.1ac
6.	Joint class: Limit laws for sequences. (Grushevsky)	Read pages 39-55
7.	Joint class: Divergence to infinity, more formal proofs. (Grushevsky)	**HW due 9/24:** 8.3, 8.6, 8.8, 8.10, 9.1, 9.3, 9.5, 9.12, 9.14
8.	Joint class: Monotone and Cauchy sequences.	Read pages 56-65

	(Grushevsky)	
9.	Joint class: Subsequences. (Grushevsky)	**No HW: prepare for the midterm**
	Joint Midterm I in Math P-131.	Practice midterm 1, Practice midterm 2, Practice midterm 2 solutions
10.	Joint class: Subsequences. (Grushevsky)	**HW due 10/8:** 10.1, 10.2, 10.5, 10.8, 10.9, 11.2, 11.4, 11.5, 11.8, 11.9
	Everything from here on is for MAT320 only	
11.	Limsup and Liminf, Bolzano-Weierstrass, Metric spaces and $\mathbf{R}^n$ as a metric space	Read pages 66-77
12.		**HW due 10/15:**12.1, 12.2, 12.4, 12.5, 12.9ab, 12.10, 12.14, 13.1, 13.3, 13.4
13.		Read pages 78-87
14.		**HW due 10/22:** nothing due this week
15.		Read pages 90-104
16.		**HW due 10/29:** 13.8b, 13.9, 13.11, 13.12, 13.14, 14.1ace, 14.3ace, 14.6, 14.12, 14.13
17.		Read pages 105-122
18.		**HW due 11/5:** 15.2, 15.3, 15.7, 16.4acd, 16.9, 17.1ac, 17.2, 17.4, 17.8, 17.14
		Second midterm on November 13 Possible topics for the exam:

19.		equivalence relations and equivalence classes; natural numbers, integers, rational numbers, algebraic numbers, real numbers (a complete ordered field) and complex numbers; absolute value;max, min sup and inf for subsets of the real numbers; Archimedian property; positive numbers have square roots; sequences and series and their properties; Bolzano Weierstrass theorem; inner product and norm for R^n; Schwartz inequality; metric spaces; R^n as a metric space; completeness for metric spaces; Compactness; Heine-Borel theorem; Is a bounded complete metric space necessarily compact; open and closed sets in a metric space; ratio test for convergence of series; harmonic series; convergence of alternating series; exponential function of a complex variable called E(z); E(z+w) = E(z) E(w); sine and cosine from E(ix); Continuity of a function from one metric space to another;
20.		Read pages 126 -143
21.		**No HW due** November 12. review for exam
22.		**HW due 11/19** 17.15, 18.3, 18.5a, 18.9, 18.12b
23.		Read pages 145-154 We did not do all of this in classs because it is rather routine, but you are responsible for it

24.		**HW due 11/24** 19.1acde, 19.4, 19.7, 20.14, 20.17
25.		Read pages 205-220 and 243-265

Final Exam: Friday December 12, 11.15AM-1.45PM

Practice final for 319

Disability Support Services: If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

Academic Integrity: Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary/.

Critical Incident Management: Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

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Professor of Mathematics

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Secretary: Mrs. Diane Williams, P-143 (631) 632-8247

LAST NAME: FIRST NAME:

STONY BROOK ID NUMBER:

Problem	1	2	3	4	5	Total
Score						

MAT 319/MAT 320 Analysis Midterm 1

October 2, 2012

NO BOOKS OR NOTES MAY BE CONSULTED DURING THIS TEST.
NO CALCULATORS MAY BE USED.
Show all your work on these pages!
Total score = 100

1. (40 points) Here **N** represents the counting numbers $\{1, 2, 3, 4, \ldots\}$, **Z** represents the integers, **Q** the rational numbers and **R** the real numbers.

   a. Explain carefully why the equation $x + 5 = 1$ has no solution in **N**.

   b. Explain carefully why the equation $3x = 2$ has no solution in **Z**.

c. Explain carefully why the equation $x^2 = 7$ has no solution in $\mathbf{Q}$.

d. Explain carefully why the least upper bound property (the Completeness Axiom) guarantees that the equation $x^2 = 7$ has a solution in $\mathbf{R}$.

2. (15 points) Prove by induction that the sum of the first n odd integers is equal to n^2, i.e. that

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

3. (15 points) For a pair (x, y) of real numbers, define $||(x, y)|| = |x| + |y|$. Prove carefully that

$$||(a + c, b + d)|| \leq ||(a, b)|| + ||(c, d)||.$$

4. (15 points) Here $\sin(x)$ is the usual sine function. Show that the sequence $a_1, a_2, a_3, \ldots$ defined by $a_n = \frac{\sin(n)}{n}$ converges, with limit 0.

5. (15 points) Suppose (s_n) is a sequence of positive numbers converging to the limit s. Prove that the sequence $(\sqrt{s_n})$ converges to $\sqrt{s}$. *Hint:* give separate proofs for $s = 0$ and $s > 0$.

END OF EXAMINATION

MAT 319 and MAT 320

Practice Midterm I, Fall 2014.

This is a closed notes/ closed book/ electronics off exam.

You are allowed and encouraged to motivate your reasoning, but at the end your proofs should be formal logical derivations, whether proving that something holds for all, or proving that your example works.

You can use any theorem or statement proven in the book; please refer to it in an identifiable way, eg. "by the completeness axiom", "by the definition of the limit", etc.

Please write legibly and cross out anything that you do not want us to read.

Each problem is worth 25 points (but the problems are of variable difficulty!).

Name:

Problem	1	2	3	4	Total
Grade					

Problem 1. Deduce from the Completeness Axiom that there exists a square root of a real number a if and only if $a \geq 0$.

Problem 2. Let $A \subset \mathbb{R}$ be a non-empty set, and let $c \in \mathbb{R}$.

a) Define what it means for c to be the least upper bound of A, i.e. that $c = \sup A$.

b) Prove that for all $\varepsilon > 0$ there exists $a \in A$ such that

$$\sup A - \varepsilon < a \leq \sup A.$$

c) Let (a_n) be a bounded increasing sequence. Show that (a_n) converges to $c = \sup\{a_n : n \in \mathbb{N}\}$.

Problem 3. Suppose that (a_n) and (b_n) are sequences of non-zero numbers. Define $c_n = a_n \cdot b_n$.

a) Suppose that (a_n) and (b_n) are both bounded. Prove that (c_n) is also bounded.
b) Suppose that (c_n) is bounded. Does this imply that both (a_n) and (b_n) are bounded (prove or give a counterexample)?

Problem 4. Prove rigorously that the sequence

$$a_n = \frac{n + 2^{-n}}{2n + \sqrt{n}}$$

converges, and compute its limit.

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Problem 1

The "only if" part is relatively easier. If $a \in \mathbb{R}$ has a square root, which means that $a = b^2$ for some $b \in \mathbb{R}$, and we know from the theorems in the book about the order structure of $\mathbb{R}$ that $b^2 \geq 0$.

We now prove the "if" part. For any real number a, define a subset of $\mathbb{R}$:

$$F_a = \{x \in \mathbb{R} | x^2 \leq a\}$$

Since $a \geq 0$, $0 \in F_a$, thus F_a is an nonempty subset. Since $(a+1)^2 = a^2 + 2a + 1 > a$, we claim that $a+1$ is an upper bound of F_a. To see this, if for some $x \in F_a$ we have $x > a+1$, then $x^2 > x(a+1) > (a+1)^2 > a$, contradicts to the fact that $x \in F_a$. Now, we get a nonempty subset of $\mathbb{R}$ which is bounded from above, by the *Completeness Axiom for Real Numbers*, there exists a supremum, denote $\sup F_a = b$.

We would like to prove $b^2 = a$. The proof is done by *Contradiction*. If $b^2 > a$, by the *Archimedean Property* of $\mathbb{R}$, there exists for some positive integer n, such that $n(b^2 - a) > 2b$, therefore $b^2 - \frac{2b}{n} > a$, which implies that $(b - \frac{1}{n})^2 = b^2 - \frac{2b}{n} + \frac{1}{n^2} > b^2 - \frac{2b}{n} > a$, therefore $b - \frac{1}{n}$ is also an upper bound for F_a, contradicting the fact that b is the *least upper bound* of F_a. If $b^2 < a$, also by the *Archimedean Property* of $\mathbb{R}$, for some positive integer n_1, $n_1 . \frac{1}{2}(a - b^2) > 2b$, and for some positive integer n_2, $n_2 . \frac{1}{2}(a - b^2) > 1$. Let $n = \max(n_1, n_2)$, then $\frac{2b}{n} < \frac{1}{2}(a - b^2)$ and $\frac{1}{n^2} < \frac{1}{2}(a - b^2)$, therefore $(b + \frac{1}{n})^2 = b^2 + \frac{2b}{n} + \frac{1}{n^2} < b^2 + \frac{1}{2}(a - b^2) + \frac{1}{2}(a - b^2) = a$, therefore $b + \frac{1}{n} \in F_a$, which contradicts the fact that b is an upper bound for F_a since $b < b + \frac{1}{n}$. This concludes that any nonnegative number has a square root in $\mathbb{R}$.

Problem 2

a). "$c = \sup A$" means that $\forall a \in A$, $c \geq a$ and $\forall c' < c$, $\exists a \in A$ such that $a > c'$.

b). By the definition from part a), we know that $\forall \epsilon > 0$, $\exists a \in A$ such that $a > \sup A - \epsilon$, and also we have $a \leq \sup A$, therefore $\sup A - \epsilon < a \leq \sup A$.

c). By the *Complete Axiom for Real Numbers*, the bounded set $\{a_n | n \in \mathbb{N}\}$ has a supremum, denoted by $a = \sup\{a_n | n \in \mathbb{N}\}$. By part b), $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $a - \epsilon < a_N \leq a$, since the sequence is increasing, for all $n > N$, $a_n \geq a_N$, therefore $a - \epsilon < a_n \leq a$, which implies that

$$|a_n - a| < \epsilon$$

for all $n > N$, therefore $\lim_{n\to\infty} a_n = a = \sup\{a_n | n \in \mathbb{N}\}$.

Problem 3

a) Since (a_n) and (b_n) are both bounded, there exist $M_1, M_2 > 0$ such that

$$|a_n| < M_1, |b_n| < M_2$$

for all n, then $|a_n b_n| < M_1 M_2$ for all n, which means the product sequence $(a_n b_n)$ is bounded.

b) "$(a_n b_n)$ is bounded" does not imply (a_n) and (b_n) are bounded. Take for n even $a_n = n$ and $b_n = 0$, and for n odd take $a_n = 0$ and $b_n = 0$. Then for any n we have $a_n b_n = 0$, while neither (a_n) nor (b_n) is bounded.

Problem 4

There are two ways to prove the convergence of the sequence and compute the limit. The more directly computational one is by applying various limit laws for sequences, while alternatively one could verify the sequence is Cauchy and then determine the limit from the definition. The first is easier, but pedagogically we encourage you to understand both approaches.

Applying Limit Theorems:

First simplify the sequence to be

$$a_n = \frac{1 + \frac{1}{n2^n}}{2 + \frac{1}{\sqrt{n}}}.$$

Then note that

$$\frac{1}{n2^n} \leq \frac{1}{n},$$

and since $\lim \frac{1}{n} = 0$, we must also have $\lim \frac{1}{n2^n} = 0$ by the squeeze theorem between $\frac{1}{n}$ and 0. We also know that $\lim \frac{1}{\sqrt{n}} = 0$. Thus by applying the limit laws for quotients, and then the limit laws for sums we get

$$\lim_{n\to\infty} \frac{n + 2^{-n}}{2n + \sqrt{n}} = \frac{\lim\limits_{n\to\infty} \left(1 + \frac{1}{n2^n}\right)}{\lim\limits_{n\to\infty} \left(2 + \frac{1}{\sqrt{n}}\right)} = \frac{1+0}{2+0} = \frac{1}{2}.$$

Cauchy Sequence Proof:

For $n, k \in \mathbb{N}$,

$$
\begin{aligned}
|a_n - a_{n+k}| =& \left|\frac{n+2^{-n}}{2n+\sqrt{n}} - \frac{(n+k)+2^{-(n+k)}}{2(n+k)+\sqrt{n+k}}\right| \\
=& \frac{|(2(n+k)+\sqrt{n+k})(n+2^{-n}) - (2n+\sqrt{n})((n+k)+2^{-(n+k)})|}{(2n+\sqrt{n})(2(n+k)+\sqrt{n+k})} \\
\leq& \frac{2^{1-n}(n+k)}{4n(n+k)} + \frac{2^{1-(n+k)}n}{4n(n+k)} + \frac{2^{-n}\sqrt{n+k}}{4n(n+k)} + \frac{2^{-(n+k)}\sqrt{n}}{4n(n+k)} + \frac{|\sqrt{n+k}-\sqrt{n}|}{4\sqrt{n(n+k)}} \\
=& \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4\sqrt{n}} \\
\leq& \frac{2}{\sqrt{n}}
\end{aligned}
$$

Thus, $\forall \epsilon > 0$, pick $N = \lceil (\frac{2}{\epsilon})^2 \rceil$, then $\forall n > N$, $k \in \mathbb{N}$,

$$|a_n - a_{n+k}| < \epsilon$$

which implies that (a_n) is a *Cauchy sequence*.

$\forall \epsilon > 0$, pick $N = \lceil (\frac{1}{4\epsilon})^2 \rceil$, then $\forall n > N$,

$$\left|\frac{n+2^{-n}}{2n+\sqrt{n}} - \frac{1}{2}\right| = \frac{\sqrt{n}-2^{1-n}}{2(2n+\sqrt{n})} \leq \frac{\sqrt{n}}{4n} = \frac{1}{4\sqrt{n}} < \epsilon$$

which means

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$